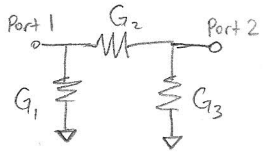
ECE 2C HW #3 Solutions Spring 2012

1.



a) Compute the 2-port Y parameters

b) Compute the 2-port Z parameters

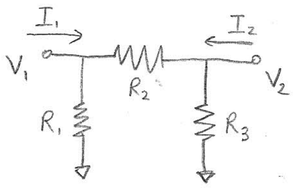
$G_1 = 1\ nS$
$G_2 = 4\ mS$
$G_3 = 3\ mS$

$\Updownarrow$

$R_1 = \frac{1}{G_1} = 1 \times 10^9\ \Omega$

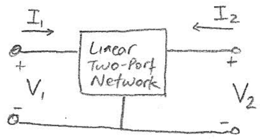
$R_2 = \frac{1}{G_2} = 250\ \Omega$

$R_3 = \frac{1}{G_3} = 333\ \Omega$



a) G is the electrical conductance of a circuit element. For resistors, like in this problem, $G = \frac{1}{R} = \frac{I}{V}$.

For a linear 2-port network, we want to relate V_1, I_1, V_2, and I_2.



$$\begin{cases} I_1 = Y_{11} V_1 + Y_{12} V_2 \\ I_2 = Y_{21} V_1 + Y_{22} V_2 \end{cases}$$

or, in matrix form, $\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$

Now we want to find the Y parameters.

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = \frac{\frac{V_1}{R_1 \| R_2}}{V_1} = \frac{1}{R_1 \| R_2} = G_1 + G_2 \approx \underline{4\ mS}$$

$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = \frac{\frac{-V_2}{R_2}}{V_2} = \frac{-1}{R_2} = -G_2 = \underline{-4\ mS}$$

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = \frac{\frac{-V_1}{R_2}}{V_1} = -\frac{1}{R_2} = -G_2 = \underline{-4\ mS}$$

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = \frac{\frac{V_2}{R_2 \| R_3}}{V_2} = \frac{1}{R_2 \| R_3} = G_2 + G_3 = \underline{7\ mS}$$

$$\Rightarrow \boxed{Y = \begin{bmatrix} 4 & -4 \\ -4 & 7 \end{bmatrix} mS}$$

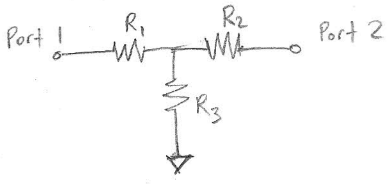
b) To find the Z parameters, we could repeat the process in part (a) using

$$\begin{cases} V_1 = Z_{11} I_1 + Z_{12} I_2 \\ V_2 = Z_{21} I_1 + Z_{22} I_2 \end{cases} \Leftrightarrow \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix},$$

or, we could note that $\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}^{-1}$.

$$\Rightarrow \boxed{Z = \begin{bmatrix} 7/12 & 1/3 \\ 1/3 & 1/3 \end{bmatrix} k\Omega}$$

(2.)

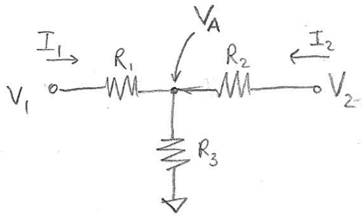


a) Compute the 2-port Y parameters.
b) Compute the 2-port Z parameters.

$R_1 = 1\,k\Omega$
$R_2 = 2\,k\Omega$
$R_3 = 6\,k\Omega$

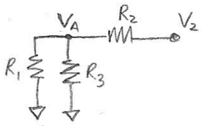
a) Same approach as problem 1.

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$



$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = \frac{1}{R_1 + (R_2 \| R_3)} = \frac{1}{2.5\,k\Omega} = \underline{0.4\,mS}$$

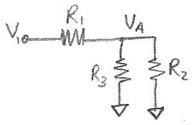
$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = \left.\frac{\frac{-V_A}{R_1}}{V_2}\right|_{V_1=0}$$



$$V_A = V_2\left(\frac{R_1 \| R_3}{R_2 + (R_1 \| R_3)}\right)$$

$$= -\frac{1}{R_1}\left(\frac{R_1 \| R_3}{R_2 + (R_1 \| R_3)}\right) = \underline{-0.3\,mS}$$

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = \left.\frac{\frac{-V_A}{R_2}}{V_1}\right|_{V_2=0}$$



$$V_A = V_1\left(\frac{R_2 \| R_3}{R_1 + (R_2 \| R_3)}\right)$$

$$= -\frac{1}{R_2}\left(\frac{R_2 \| R_3}{R_1 + (R_2 \| R_3)}\right) = \underline{-0.3\,mS}$$

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = \frac{1}{R_2 + (R_1 \| R_3)} = \frac{1}{2.86\,k\Omega} = \underline{0.35\,mS}$$

$$\boxed{Y = \begin{bmatrix} 0.4 & -0.3 \\ -0.3 & 0.35 \end{bmatrix} mS}$$

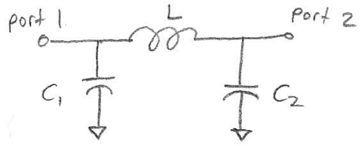
b) Same approach as problem 1.

$$Z = Y^{-1} = \begin{bmatrix} 0.4 & -0.3 \\ -0.3 & 0.35 \end{bmatrix}^{-1}$$

$$\boxed{Z = \begin{bmatrix} 7 & 6 \\ 6 & 8 \end{bmatrix} k\Omega}$$

(3.) At a signal frequency of 1 GHz, find the Y parameters.

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$



$C_1 = 1\ pF$
$C_2 = 2\ pF$
$L = 1\ nH$

Note that this circuit is identical to the one in problem 1, except with capacitors and inductor instead of resistors. So, we can use the same equations, just replacing G (conductance) with Y (admittance).

$$Z_{C_1} = \frac{1}{j\omega C_1} = \frac{1}{j2\pi(1\,GHz)(1\,pF)} = \frac{1}{j2\pi}\ k\Omega$$

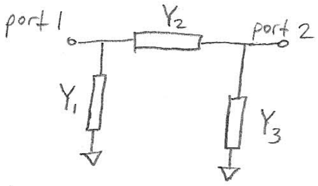
$$Z_{C_2} = \frac{1}{j\omega C_2} = \frac{1}{j2\pi(1\,GHz)(2\,pF)} = \frac{1}{j4\pi}\ k\Omega$$

$$Z_L = j\omega L = j2\pi(1\,GHz)(1\,nH) = j2\pi\ \Omega$$

$$Y_1 = Y_{C_1} = \frac{1}{Z_{C_1}} = j2\pi\ mS$$

$$Y_3 = Y_{C_2} = \frac{1}{Z_{C_2}} = j4\pi\ mS$$

$$Y_2 = Y_L = \frac{1}{Z_L} = \frac{1}{j2\pi}\ S = \frac{1000}{j2\pi}\ mS$$



same as problem 1!

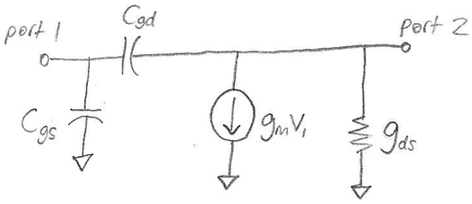
$$Y_{11} = Y_1 + Y_2 = Y_{C_1} + Y_L$$
$$= j2\pi\ mS + \frac{1}{j2\pi}\ S$$
$$\approx j(6.28) - j(159.16)\ mS \approx \underline{-j152.9\ mS}$$

$$Y_{12} = Y_{21} = -Y_2 = -Y_L$$
$$= -\frac{1000}{j2\pi}\ mS \approx \underline{-j159.16\ mS}$$

$$Y_{22} = Y_2 + Y_3 = Y_L + Y_{C_2}$$
$$= \frac{1000}{j2\pi}\ mS + j4\pi\ mS$$
$$\approx \underline{-j146.6\ mS}$$

$$\boxed{Y \approx \begin{bmatrix} -j(153) & -j(159) \\ -j(159) & -j(147) \end{bmatrix} mS}$$

4. Simple FET model. (a) As a function of frequency, find algebraic expressions for the Y parameters. If $C_{gs} = 1$ pF, $C_{gd} = 0.1$ pF, $g_m = 100$ mS, $g_{ds} = 1$ mS, make plots of the real and imaginary parts as a function of frequency. (b) Now setting $C_{gd} = 0$ fF, compute an algebraic expression of H_{21} as a function of frequency.

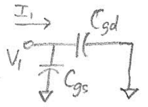


a)

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0}$$

Ground port 2 $\Rightarrow$



C_{gs} & C_{gd} in parallel. $\Rightarrow$ $Z_{eq} = \frac{1}{j\omega(C_{gs}+C_{gd})}$

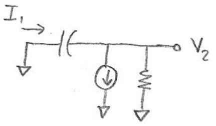
$Y_{eq} = j\omega(C_{gs} + C_{gd})$

$$= \frac{1}{V_1}\left(\frac{V_1}{Z_{C_{gs}||C_{gd}}}\right) = \frac{1}{Z_{C_{gs}||C_{gd}}} = Y_{C_{gs}||C_{gd}}$$

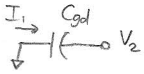
$$= \underline{j\omega(C_{gs} + C_{gd})}$$

$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0}$$

Ground port 1 $\Rightarrow$



$\Rightarrow$



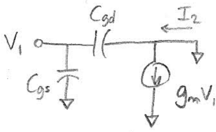
$$= \frac{1}{V_2}\left(\frac{-V_2}{Z_{C_{gd}}}\right) = -Y_{C_{gd}}$$

$Y_{C_{gd}} = j\omega C_{gd}$

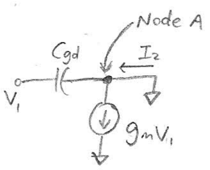
$$= \underline{-j\omega C_{gd}}$$

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0}$$

Ground port 2 $\Rightarrow$



$\Rightarrow$



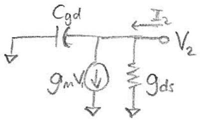
$= \underline{g_m - j\omega C_{gd}}$ b/c KCL at Node A:

$$I_2 + \frac{V_1 - 0}{Z_{C_{gd}}} - g_m V_1 = 0$$

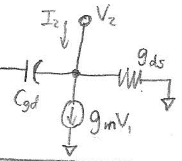
$$I_2 = g_m V_1 - V_1 Y_{C_{gd}} = (g_m - j\omega C_{gd}) V_1$$

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0}$$

Ground port 1 $\Rightarrow$



$$= \underline{g_{ds} + j\omega C_{gd}}$$



KCL: $I_2 + \frac{0 - V_2}{Z_{C_{gd}}} + \frac{0 - V_2}{R_{ds}} - \cancel{g_m V_1} = 0$

$$I_2 = Y_{C_{gd}} V_2 + g_{ds} V_2$$

$$= (g_{ds} + j\omega C_{gd}) V_2$$

$$\boxed{Y = \begin{bmatrix} j\omega(C_{gs} + C_{gd}) & -j\omega C_{gd} \\ g_m - j\omega C_{gd} & g_{ds} + j\omega C_{gd} \end{bmatrix} \text{ S}}$$

where $\omega = 2\pi f$

$\mathrm{real}(Y_{11}) = 0$ $\qquad$ $\mathrm{imag}(Y_{11}) = \omega(C_{gs} + C_{gd})$

$\mathrm{real}(Y_{12}) = 0$ $\qquad$ $\mathrm{imag}(Y_{12}) = -\omega C_{gd}$

$\mathrm{real}(Y_{21}) =$ $\qquad$ $\mathrm{imag}(Y_{21}) = -\omega C_{gd}$

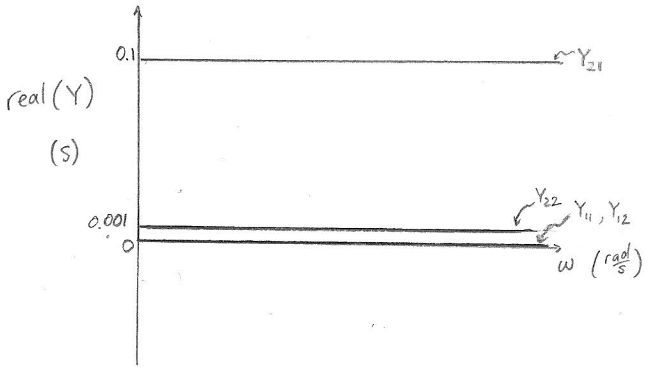
$\mathrm{real}(Y_{22}) =$ $\qquad$ $\mathrm{imag}(Y_{22}) = \omega C_{gd}$

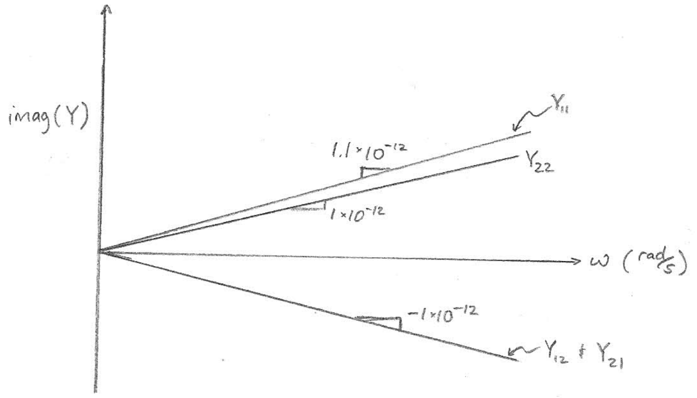
$C_{gs} = 1 \times 10^{-12}\ F$

$C_{gd} = 0.1 \times 10^{-12}\ F$

$G_m = 0.1\ S$

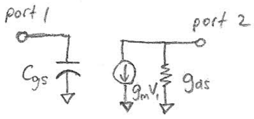
$G_{ds} = 0.001\ S$





b) If $C_{gd} = 0$; $Z_{C_{gd}} = \frac{1}{j\omega C_{gd}} \to \infty$,

C_{gd} acts as open circuit. $\Rightarrow$



Hybrid Parameters: $\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$

$$H_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = Z_{C_{gs}} = \frac{1}{j\omega C_{gs}}$$

$$H_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} = 0 \quad \text{since } I_1 = 0 \Rightarrow V_1 = 0$$

$$H_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = \frac{g_m V_1}{V_1 Y_{C_{gs}}} = \frac{g_m}{j\omega C_{gs}} \Rightarrow \boxed{H_{21} = \frac{g_m}{j\omega C_{gs}}}$$

$$H_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} = \frac{V_2 g_{ds}}{V_2} = g_{ds} \quad \text{since } I_1 = 0 \Rightarrow V_1 = 0 \Rightarrow V_1 = 0$$

$$H = \begin{bmatrix} \frac{1}{j\omega C_{gs}} & 0 \\ \frac{g_m}{j\omega C_{gs}} & g_{ds} \end{bmatrix}$$

5. $I_a = k_p\left(V_g + \frac{V_a}{\mu}\right)^{3/2}$

a) $g_{out} \triangleq \frac{\partial I_a}{\partial V_a} = \frac{\partial}{\partial V_a}\left[k_p\left(V_g + \frac{V_a}{\mu}\right)^{3/2}\right]$

$= \frac{3}{2}\left[k_p\left(V_g + \frac{V_a}{\mu}\right)^{1/2}\right]\frac{1}{\mu}$ by chain rule.

$g_{out} = \boxed{\frac{3k_p}{2\mu}\left(V_g + \frac{V_a}{\mu}\right)^{1/2}}$

$g_m \triangleq \frac{\partial I_a}{\partial V_g} = \frac{\partial}{\partial V_g}\left[k_p\left(V_g + \frac{V_a}{\mu}\right)^{3/2}\right]$

$= \frac{3}{2}k_p\left(V_g + \frac{V_a}{\mu}\right)^{1/2}(1)$ by chain rule.

$g_m = \boxed{\frac{3k_p}{2}\left(V_g + \frac{V_a}{\mu}\right)^{1/2}}$

b) We don't know k_p and μ, so we must estimate g_{out} and g_m from the IV curves.
At $V_a = 200$ V and $V_g = -0.5$ V, $I_a \approx 3.3$ mA.

$g_{out} = \frac{\partial I_a}{\partial V_a} \approx \frac{\Delta I_a}{\Delta V_a}$ is approximately the slope of the IV curve.

At $V_a = 200$ V, $V_g = -0.5$V, $g_{out} \approx \frac{3.8 - 2.75 \text{ mA}}{225 - 175 \text{ V}} = 0.021$ mS

$g_m = \frac{\partial I_a}{\partial V_g} \approx \frac{\Delta I_a}{\Delta V_g}$

At $V_a = 200$ V, $g_m \approx \frac{4.5 - 2.25 \text{ mA}}{0-(-1) \text{ V}} = 2.25$ mS

$$\boxed{\begin{array}{l} g_{out} \approx 0.021 \text{ mS} \\ g_m \approx 2.3 \text{ mS} \end{array}}$$