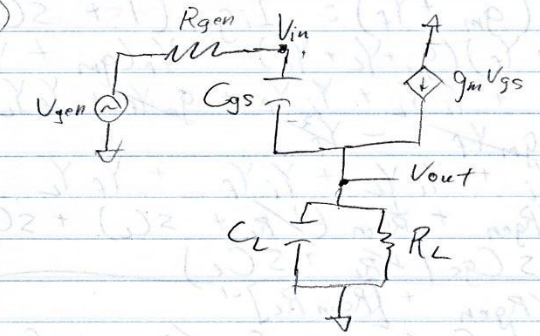
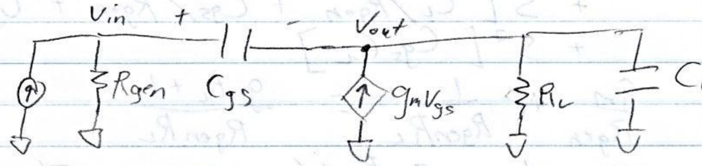
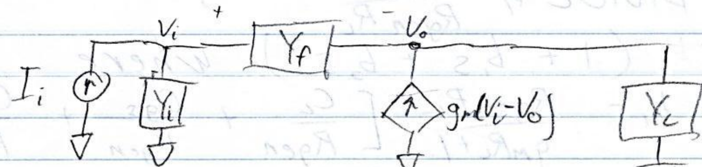
Home Work 8

1. a.)

$R_{ds} = \infty$







$$Y_i = \frac{1}{R_{gen}} \qquad Y_f = sC_{gs}$$

$$Y_L = 1/R_L + sC_L$$

$$-I_i + V_i Y_i + (V_i - V_o) Y_f = 0$$

① $$(Y_i + Y_f) V_i - Y_f V_o = I_i$$

$$-g_m (V_i - V_o) + V_o Y_L + (V_o - V_i) Y_f = 0$$

② $$-(g_m + Y_f) V_i + (g_m + Y_L + Y_f) V_o = 0$$

1. and 2 as a matrix:

$$\begin{bmatrix} Y_i + Y_f & -Y_f \\ -(g_m + Y_f) & g_m + Y_L + Y_f \end{bmatrix} \begin{bmatrix} V_i \\ V_o \end{bmatrix} = \begin{bmatrix} I_i \\ 0 \end{bmatrix}$$

By Cramer's Rule

$$V_o = \frac{\begin{vmatrix} Y_i + Y_f & I_i \\ -(g_m + Y_f) & 0 \end{vmatrix}}{\begin{vmatrix} Y_i + Y_f & -Y_f \\ -(g_m + Y_f) & g_m + Y_L + Y_f \end{vmatrix}} = \frac{N}{D}$$

$$N = I_i(g_m + Y_f) = I_i g_m (1 + sC_{gs}/g_m)$$

$$D = (Y_i + Y_f)(g_m + Y_L + Y_f) - Y_f(g_m + Y_f)$$

$$= g_m Y_i + Y_i Y_L + Y_i Y_f + \cancel{g_m Y_f} + Y_f Y_L + \cancel{Y_f^2}$$

$$- \cancel{g_m Y_f} - \cancel{Y_f^2}$$

$$= g_m Y_i + Y_i Y_L + Y_i Y_f + Y_f Y_L$$

$$= g_m/R_{gen} + \tfrac{1}{R_{gen}}(1/R_L + sC_L) + sC_{gs}/R_{gen}$$

$$+ sC_{gs}(1/R_L + sC_L)$$

$$= g_m/R_{gen} + [R_{gen}R_L]^{-1}$$

$$+ s[C_L/R_{gen} + C_{gs}/R_{gen} + C_{gs}/R_L]$$

$$+ s^2[C_{gs}C_L]$$

$$\frac{g_m}{R_{gen}} + \frac{1}{R_{gen}R_L} = \frac{g_m R_L + 1}{R_{gen}R_L}$$

Divide by $\frac{g_m R_L + 1}{R_{gen}R_L}$ to get standard form

$(1 + b_1 s + b_2 s^2)$ where

$$b_1 = \frac{R_{gen}R_L}{g_m R_L + 1}\left[\frac{C_L}{R_{gen}} + \frac{C_{gs}}{R_{gen}} + \frac{C_{gs}}{R_L}\right]$$

$$b_2 = \frac{R_{gen}R_L}{g_m R_L + 1}\left[C_{gs}C_L\right]$$

From N $a_1 = C_{gs}/g_m$

$$V_o = \frac{N}{D} = \frac{I_i g_m R_{gen} R_L}{g_m R_L + 1}\left(\frac{1 + a_1 s}{1 + b_1 s + b_2 s^2}\right)$$

$$V_{in} = R_i I_i = R_{gen} I_i$$

$$\frac{V_o}{V_i} = \frac{g_m R_L}{g_m R_L + 1}\left(\frac{1 + a_1 s}{1 + b_1 s + b_2 s^2}\right)$$

where α_1, b_1, b_2 are as defined above

c.) zero at $s = -\frac{1}{a_1} = -5\times10^9$ rad/s ≈ 0.8 GHz

$g_m = 10$ mS, $C_{gs} = 2$ pF

the poles can be approximated

$b_1 = 1.6 \times 10^{-9}\,s, \quad b_2 \approx 1 \times 10^{-18}\,s^2$

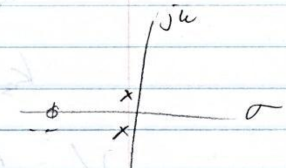
by evaluating with $g_m = 10\,mS$, $C_{gs} = 2pF$, $C_L = 20pF$; $R_L = 100\,\Omega$; $R_{gen} = 500\,\Omega$

$$s = \frac{-b_1}{2b_2} \pm \frac{\sqrt{b_1^2 - 4b_2}}{2b_2}$$

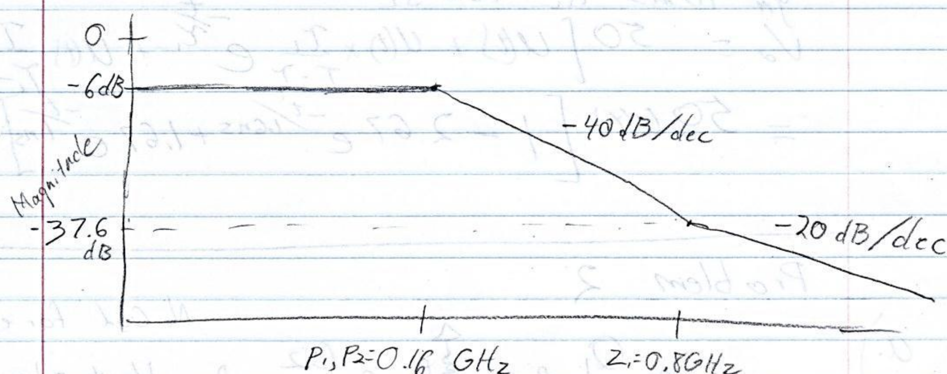
$$s = (-8 \pm j6) \times 10^{8}$$

$$= -0.8 \pm j0.6 \times 10^{9}\ rad/s$$

$$\approx -0.13 + j0.1\ GHz$$



$$Gain = 20\log_{10}\left[g_m R_L / (g_m R_L + 1)\right] \approx -6\,dB$$



$\log_{10}(0.8/0.16) = 0.79\ dec \rightarrow 0.79 \times 40\,dB/dec$

$-6 - 0.79 \cdot 40 = -37.6\ dB$

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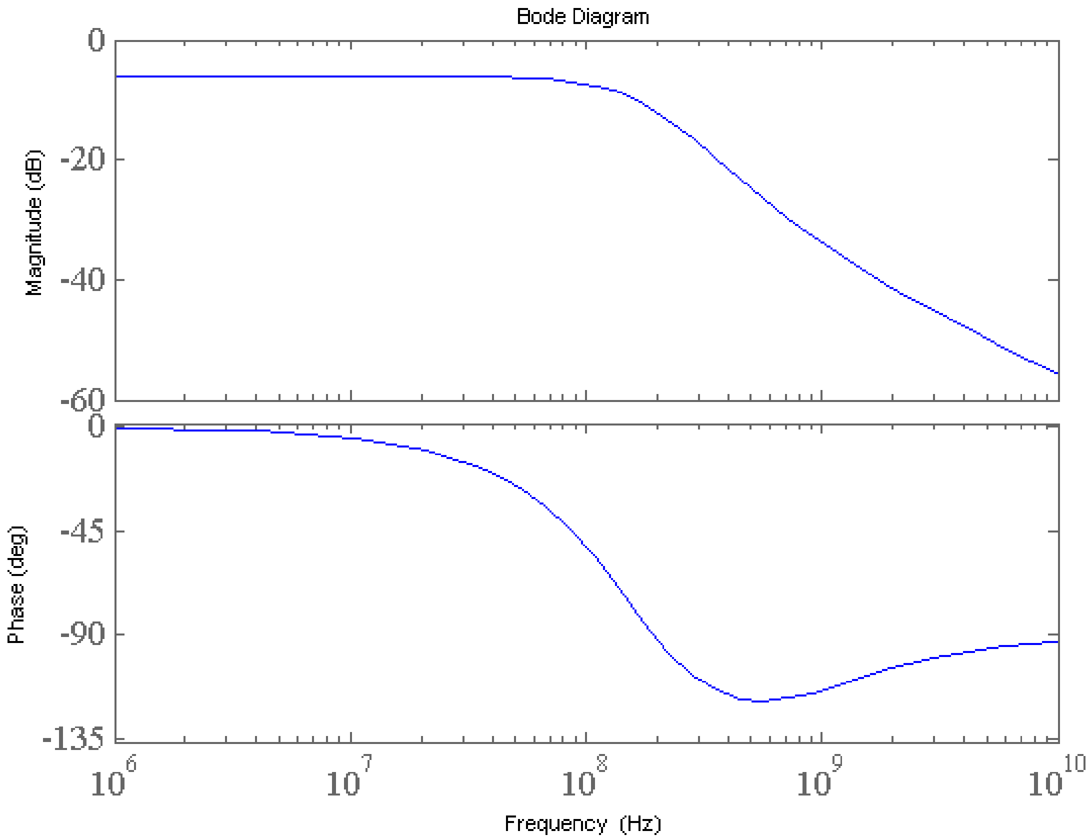
d.) $v_i(t) = 100\,u(t)\ mV \qquad V_i(s) = 100/s\ mV$

$$V_o = \frac{g_m R_L}{g_m R_L + 1}\left(\frac{1 + a_1 s}{1 + b_1 s + b_2 s^2}\right) \times \frac{100}{s}\ mV$$

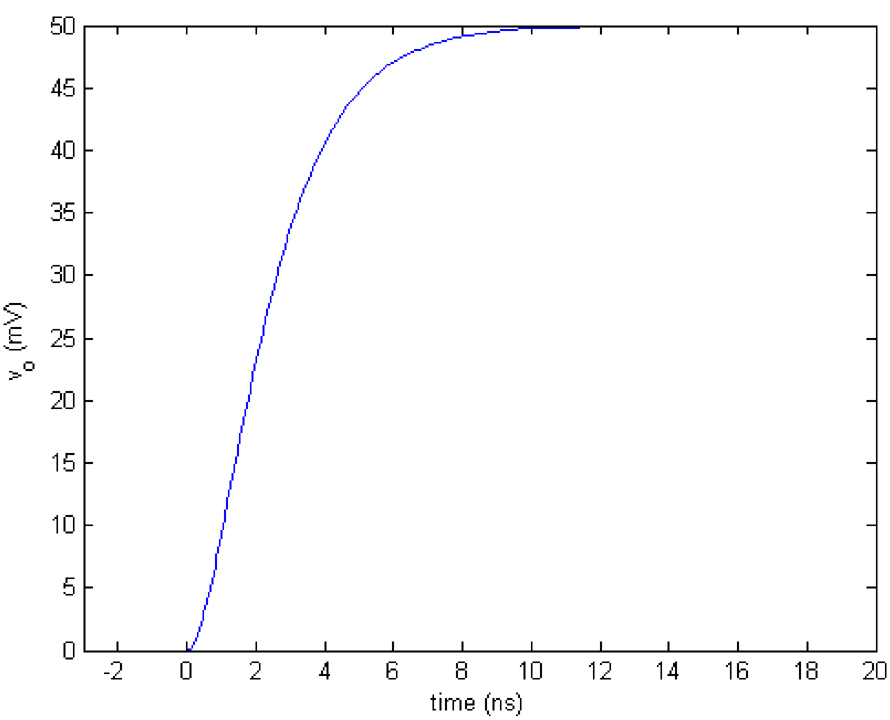
approximate by letting $a_1 = 0$

$\tau_1 = b_1 = 1.6\,ns \qquad \tau_2 = \sqrt{b_2} = 1\,ns$

Both are included for accuracy since they are comparable magnitude



Bode Diagram
Magnitude (dB)
0
-20
-40
-60
Phase (deg)
0
-45
-90
-135
10^6
10^7
10^8
10^9
10^{10}
Frequency (Hz)



v_o (mV)
50
45
40
35
30
25
20
15
10
5
0
-2
0
2
4
6
8
10
12
14
16
18
20
time (ns)

$$V_o = \frac{100\, g_m R_L}{g_m R_L + 1} \cdot \frac{1}{s(1+s\tau_1)(1+s\tau_2)}$$

$$\left(\frac{k_1}{s} + \frac{k_2}{1+s\tau_1} + \frac{k_3}{1+s\tau_2} \right) \times \frac{100\, g_m R_L}{g_m R_L + 1}$$

$$k_1 = 1 \qquad k_2 = \frac{1}{-\frac{1}{\tau_1}\left(1 - \frac{\tau_2}{\tau_1}\right)} = \frac{\tau_1^2}{(\tau_2 - \tau_1)}$$

$$k_3 = \frac{\tau_2^2}{(\tau_1 - \tau_2)}$$

$$V_o = \frac{100\, g_m R_L}{g_m R_L + 1} \left[\frac{1}{s} + \frac{\tau_1}{(1+s\tau_1)} \times \frac{\tau_1}{(\tau_2 - \tau_1)} + \frac{\tau_2}{(1+s\tau_2)} \times \frac{\tau_2}{(\tau_1 - \tau_2)} \right]$$

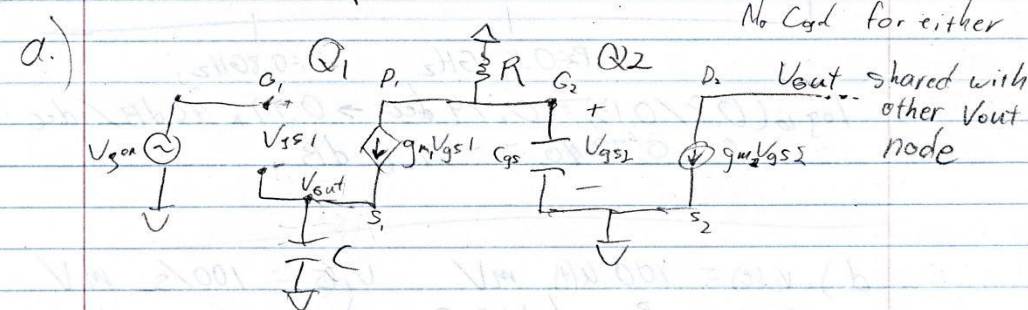
$g_m = 10\,mS \quad R_L = 100\ \Omega$

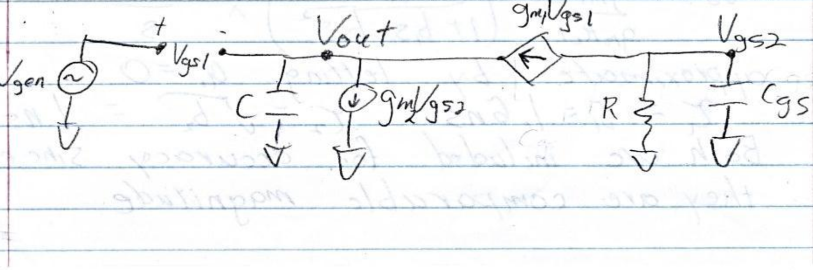
$$V_o = 50\left[u(t) + u(t) \times \frac{\tau_1}{\tau_2 - \tau_1} e^{-\frac{t}{\tau_1}} + u(t) \frac{\tau_2}{\tau_1 - \tau_2} e^{-\frac{t}{\tau_2}} \right]$$

$$= 50\, u(t) \left[1 - 2.67 e^{-t/1.6ns} + 1.67 e^{-t/1ns} \right] mV$$

Problem 2

a.)

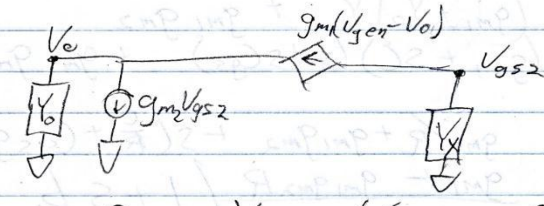




$V_{in} = V_{gen}$, $V_{gs1} = V_{gen} - V_{out}$

Perform nodal analysis using nodes V_{out} and V_{gs2} with the above subsitutions

$V_{out} = V_o$



$Y_o = sC \qquad Y_x = 1/R + sC_{gs}$

b.) $-g_{m1}(V_{gen} - V_o) + V_o Y_o + g_{m2}V_{gs2} = 0$

① $(g_{m1} + Y_o)V_o + g_{m2} V_{gs2} = g_{m1} V_{gen}$

$g_{m1}(V_{gen} - V_o) + Y_x V_{gs2} = 0$

② $-g_{m1} V_o + Y_x V_{gs2} = -g_{m1} V_{gen}$

Combine ① and ② to form a matrix

$$\begin{bmatrix} g_{m1} + Y_o & g_{m2} \\ -g_{m1} & Y_x \end{bmatrix} \begin{bmatrix} V_o \\ V_{gs2} \end{bmatrix} = \begin{bmatrix} g_{m1} V_{gen} \\ -g_{m1} V_{gen} \end{bmatrix}$$

Apply Cramer's Rule to find V_o

$$V_o = \frac{\begin{vmatrix} g_{m1} V_{gen} & g_{m2} \\ -g_{m1} V_{gen} & Y_x \end{vmatrix}}{\begin{vmatrix} g_{m1} + Y_o & g_{m2} \\ -g_{m1} & Y_x \end{vmatrix}} = \frac{N}{D}$$

$N = g_{m1} V_{gen} Y_x + g_{m1} g_{m2} V_{gen}$

$= V_{gen} \left(g_{m1} g_{m2} + g_{m1}\left(\frac{1}{R} + sC_{gs}\right)\right)$

$= V_{gen} g_{m1} \left(g_{m2} + \frac{1}{R} + sC_{gs} \right)$

$$N = V_{gen}\, g_{m1}\left[(g_{m2}R+1)/R \quad + s\,C_{gs}\right]$$

$$N = V_{gen}\, g_{m1}\,\frac{(g_{m2}R+1)}{R}\left(1 + s\,\frac{C_{gs}R}{(g_{m2}R+1)}\right)$$

$$D = (g_{m1}+Y_o)\,Y_x \quad + g_{m1}g_{m2}$$
$$= (g_{m1}+sC)\left(\tfrac{1}{R}+sC_{gs}\right) \quad + g_{m1}g_{m2}$$

$$D = g_{m1}/R + g_{m1}g_{m2} + s\left(\tfrac{C}{R}+C_{gs}g_{m1}\right) + s^2(C_{gs}\cdot C)$$

$$D = \frac{g_{m1}+g_{m1}g_{m2}R}{R}\left(1+s\,b_1 + s^2 b_2\right)$$

$$b_1 = \frac{C + C_{gs}g_{m1}R}{g_{m1}+g_{m1}g_{m2}R}$$ By dividing the outside term through the s and s^2 coefficients

$$b_2 = \frac{RC_{gs}\cdot C}{g_{m1}+g_{m1}g_{m2}R}$$

$$a_1 = \frac{C_{gs}R}{g_{m2}R+1}$$

$$V_o = \frac{N}{D} = \frac{V_{gen}\,g_{m1}\,(g_{m2}R+1)\cancel{R}}{(g_{m1}+g_{m1}g_{m2}R)\,\cancel{R}}\left(\frac{1+sa_1}{1+sb_1+s^2b_2}\right)$$

$(g_{m1}+g_{m1}g_{m2}R) = g_{m1}(1+g_{m2}R)$
so that the other terms cancel as well leaving only

$$V_o = V_{gen}\left(\frac{1+sa_1}{1+sb_1+s^2b_2}\right)$$

$$\frac{V_o}{V_{gen}} = \left(\frac{1+sa_1}{1+sb_1+s^2b_2}\right)$$

Where a_1, b_1, b_2 are as defined above

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c.) Divide top and bottom by b_2

$$\frac{V_o}{V_{gen}} = \frac{1/b_2 + s\,a_1/b_2}{s^2 + s\,b_1/b_2 + 1/b_2}$$

From this we can read $\omega_n^2 = b_2^{-1}$

$2\zeta\omega_n = b_1/b_2$

$f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{b_2}}$

Plugging in $C_{gs}=100\,fF$, $g_{m1}=50\,mS$, $g_{m2}=100\,mS$, $R=1000\,\Omega$, $C=200\,fF$

$b_2 = 4.0\times10^{-24}\ s^2$

$f_n = 8\times10^{10} = 80\ GHz$

$2\zeta\omega_n = b_1\omega_n^2$

$2\zeta = b_1\omega_n$

$\zeta = b_1\pi f_n$

$b_1 = 1.03\times10^{-12}\ s$

$\zeta = 0.259$

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d.) zero at $s = \frac{-1}{a_1}$

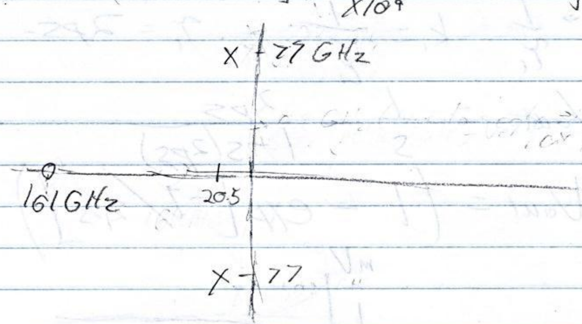
$a_1 = 9.9\times10^{-13}\ s$

$s = -1\times10^{12}$ rad/s $= -161\ GHz$

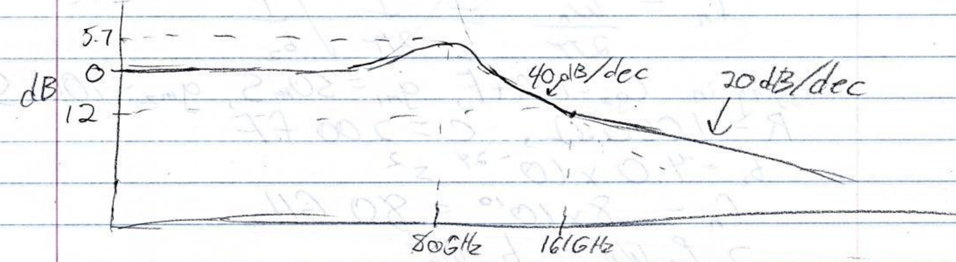
$1 + sb_1 + s^2b_2$

$s = \frac{-b_1}{2b_2} \pm \frac{\sqrt{b_1^2-4b_2}}{2b_2}$ rad/s

$s = (-129 \pm j483)\times10^9 = -20.5 \pm j77\ GHz$



e) Resonance peak $\approx 20\log_{10}\left(\frac{1}{2\zeta}\right) = 5.7\,dB$



$\log_{10}(161/80) \times 40\,dB/dec \approx 12\,dB$

f) $V_{gen}(t) = 1\,u(t)\ mV \qquad V_{gen}(s) = 1/s\ mV$

$\tau_2 = \sqrt{b_2}\ ,\quad \tau_1 = b_1$

$= 2\,ps \qquad = 1\,ps$

For this problem we will ignore τ_1 and ignore a_1 for simplicity

$$\frac{V_{out}}{V_{gen}} \approx \frac{1}{1+s(2ps)}$$

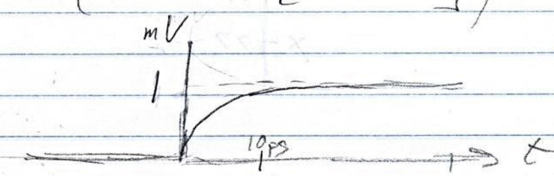
$$V_{out} \approx \frac{1mV}{s}\left(\frac{1}{1+s(2ps)}\right)$$

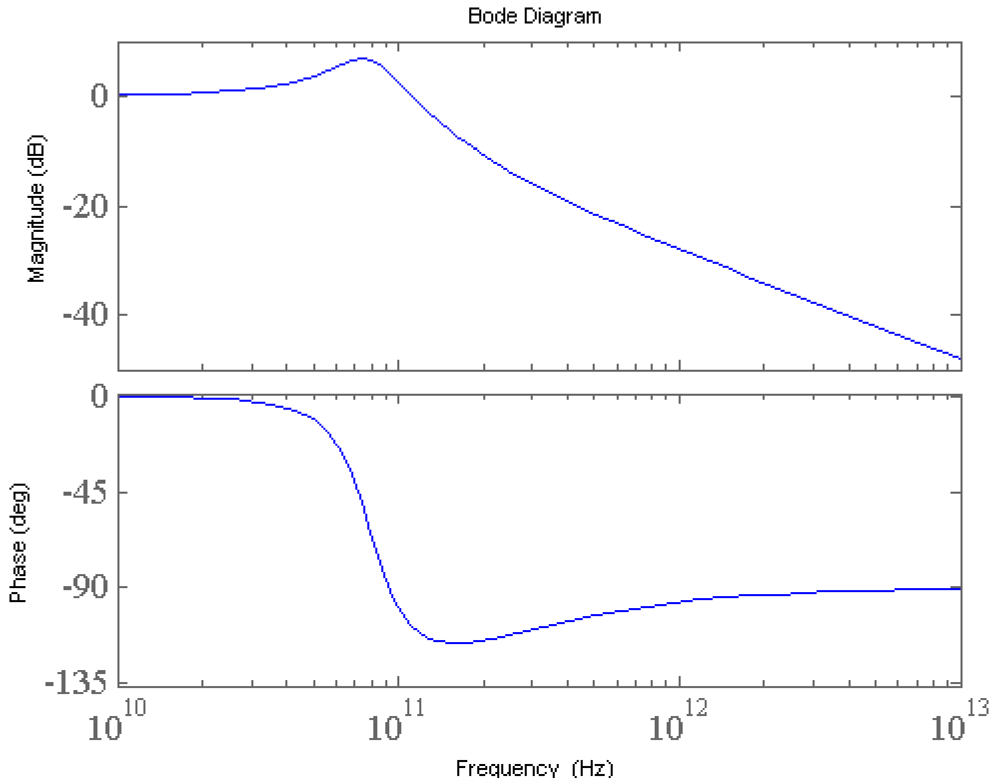
$$V_{out} \approx \frac{1}{s} + \frac{k_1}{(1+s(2ps))}$$

$$s \Rightarrow \frac{1}{\tau_1} \quad k_1 = \frac{1}{1/\tau_1} \Rightarrow \tau_1 = 2ps$$

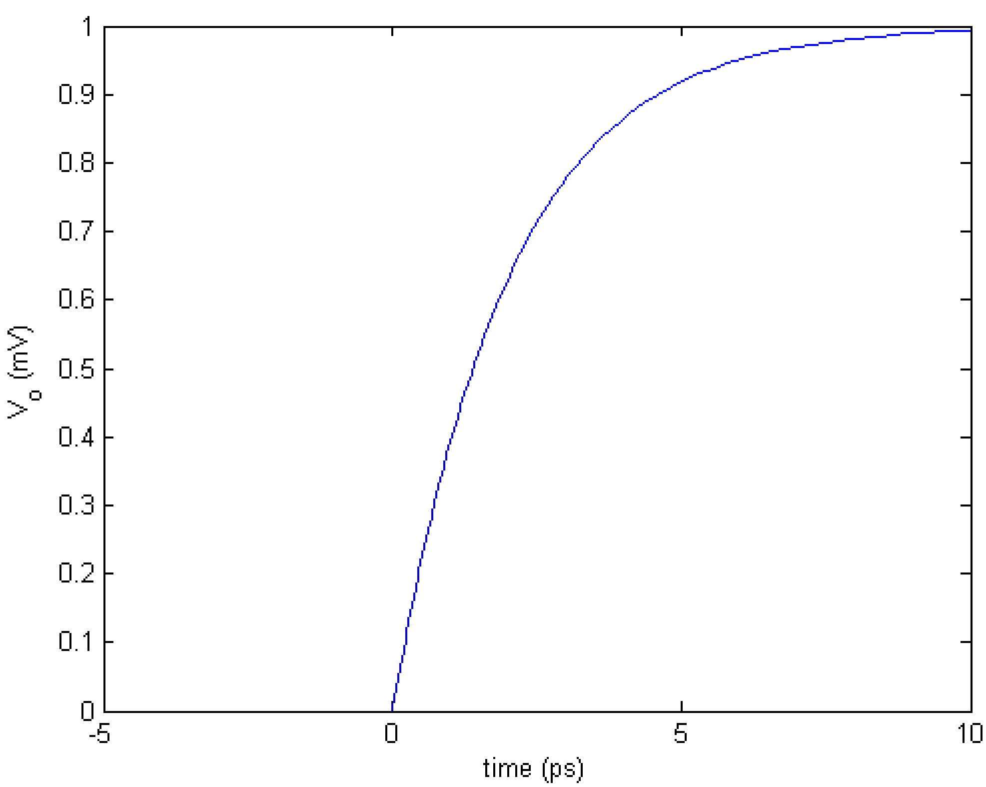
$$V_{out} = \frac{1}{s} - \frac{2ps}{1+s(2ps)}$$

$$V_{out} = \left(1 - \exp\left[-t/2ps\right]\right) u(t)\ mV$$





Bode Diagram
Magnitude (dB)
0
-20
-40
Phase (deg)
0
-45
-90
-135
10^{10}
10^{11}
10^{12}
10^{13}
Frequency (Hz)



V_o (mV)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-5
0
5
10
time (ps)

If you want the precise resonant response instead of the time delay response all terms (a_1, τ_1, τ_2) must be included.

$$V_{out}(s) = \frac{1}{s}\,\frac{1 + s(1ps)}{1 + s(1ps) + s^2((2ps)^2)}\ mV$$

Perform partial fraction expansion

$$= \frac{A}{s} + \frac{Bs + C}{1 + s(1ps) + s^2(2ps)^2}$$

$$1 + s(1ps) = A(1 + s(1ps) + s^2(2ps)^2) + s(Bs + C)$$

$$s(1ps) + 1 = s^2(A \cdot 4ps^2 + B) + s(C + A \cdot 1ps) + A$$

$$A = 1$$

$$1ps = C + 1ps \qquad C = 0$$

$$4ps^2 + B = 0 \qquad B = -4ps^2$$

Equate s, s^2 coefficients

$$\frac{1}{s} + \frac{-4ps^2}{1 + s \cdot 1 + s^2 \cdot 4ps^2} = \frac{-s}{\frac{1}{4} + s\frac{1}{4} + s^2}$$

$$V_{out}(s) = \frac{1}{s} - \frac{s}{\frac{1}{(2ps)^2} + s \cdot \frac{1}{4ps} + s^2}$$

Complete the square for denominator

$$\left[s^2 + s \cdot \frac{1}{4ps} + \frac{1}{(8ps)^2}\right] - \frac{1}{(8ps)^2} + \frac{1}{(2ps)^2}$$

$$\left(s + \frac{1}{8ps}\right)^2 - (8ps)^{-2} + (2ps)^{-2} \approx \left(s + \frac{1}{8ps}\right) + 2.34 \times 10^{23}\, s^{-2}$$

$$V_{out}(s) = \frac{1}{s} + \frac{-(s-}{(s+\frac{1}{8ps})^2 + 2.34\times10^{23}\,sec^{-2}}$$

we want the second term in the form

$$\frac{s+\alpha}{(s+\alpha)^2+\omega^2} \quad \text{or} \quad \frac{\omega}{(s+\alpha)^2+\omega^2} \quad \text{(Laplace table)}$$

$$-\left(s + \frac{1}{8ps} - \frac{1}{8ps}\right) = -\left(s+\frac{1}{8ps}\right) + \frac{1}{8ps}$$

$$1/8_{ps} = 1.75\times10^{11}\,s^{-1}$$

$$\omega = \sqrt{2.34\times10^{23}} \approx 4.84\times10^{11}\,s^{-1}$$

$$1/8_{ps} = 4.84\times10^{11} \times \frac{1.75}{4.84} = 0.362\cdot(4.84\times10^{11}s^{-1})$$

$$V_{out}(s) = \frac{1}{s} - \frac{s+\frac{1}{8ps}}{(s+\frac{1}{8ps})^2+2.34\times10^{23}} + 0.362\times\frac{4.84\times10^{11}s^{-1}}{(s+\frac{1}{8ps})^2+2.34\times10^{23}s^{-2}}$$

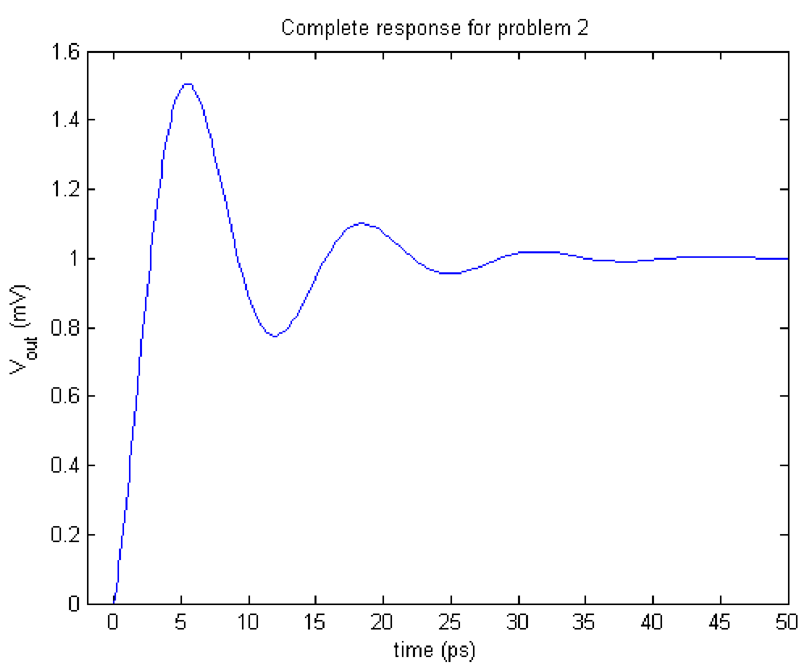
From this we can read the inverse Laplace transform from the table

$$V_{out}(t) = u(t) - \exp[-t/8ps]\cos(4.84\times10^{11}t)\,u(t) + 0.362\cdot\exp[-t/8ps]\sin(4.84\times10^{11}t)\,u(t)$$

$$V_{out}(t) = u(t)\left[1 - \exp\left(\frac{-t}{8ps}\right)\left[\cos(4.84\times10^{11}t) + 0.362\sin(4.84\times10^{11}t)\right]\right]\ mV$$

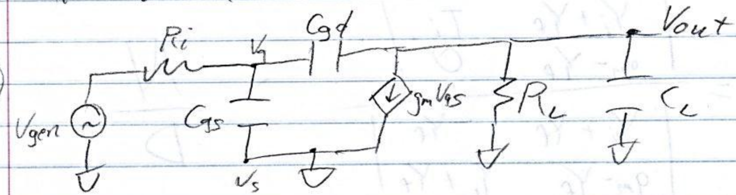
Note $\omega = 4.84\times10^{11}$ rad/s

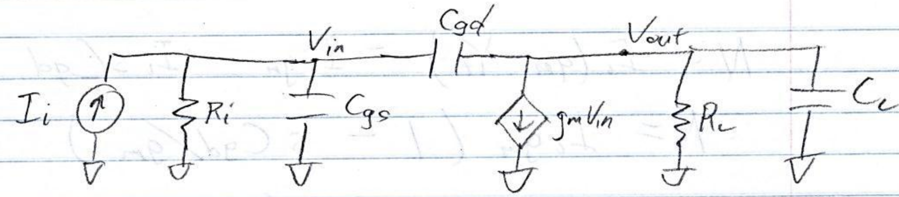
$\approx 77\times10^{9}\,s^{-1} = 77\,GHz \approx f_n$

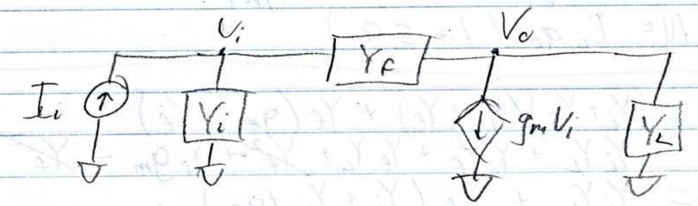


Problem 3

a.)







$Y_i = 1/R_i + sC_{gs}$

b.)

$Y_f = sC_{gd}$

$Y_L = 1/R_L + sC_L$

$-I_i + V_i Y_i + (V_i - V_o) Y_f = 0$

① $(Y_i + Y_f) V_i - Y_f V_o = I_i$

$g_m V_i + V_o Y_L + (V_o - V_i) Y_f = 0$

② $(g_m - Y_f) V_i + (Y_L + Y_f) V_o = 0$

Combine ① and ②

$$\begin{bmatrix} Y_i + Y_f & -Y_f \\ g_m - Y_f & Y_L + Y_f \end{bmatrix} \begin{bmatrix} V_i \\ V_o \end{bmatrix} = \begin{bmatrix} I_i \\ 0 \end{bmatrix}$$

Solve for V_o by Cramer's rule

$$V_o = \frac{\begin{vmatrix} Y_i + Y_f & I_i \\ g_m - Y_f & 0 \end{vmatrix}}{\begin{vmatrix} Y_i + Y_f & -Y_f \\ g_m - Y_f & Y_L + Y_f \end{vmatrix}} = \frac{N}{D}$$

$$N = I_i(g_m - Y_f) = I_i g_m - I_i s C_{gd}$$

$$N = I_i g_m (1 - s C_{gd}/g_m)$$

$$a_1 = -C_{gd}/g_m$$

$$N = I_i g_m (1 + s a_1)$$

$$D = (Y_i + Y_f)(Y_L + Y_f) + Y_f(g_m - Y_f)$$

$$= Y_i Y_L + Y_i Y_f + Y_f Y_L + \cancel{Y_f^2} + Y_f g_m - \cancel{Y_f^2}$$

$$= Y_i Y_L + Y_f(Y_i + Y_L + g_m)$$

$$Y_i = \frac{1}{R_i} + sC_{gs}, \quad Y_f = sC_{gd}, \quad Y_L = \frac{1}{R_L} + sC_L$$

$$D = \frac{1}{R_i R_L} + s\left(\frac{C_L}{R_i} + \frac{C_{gs}}{R_L}\right) + s^2(C_{gs}C_L)$$

$$+ sC_{gd}\left(\frac{1}{R_i} + sC_{gs} + \frac{1}{R_L} + sC_L + g_m\right)$$

$$D = \frac{1}{R_i R_L} + s\left(\frac{C_L}{R_i} + \frac{C_{gs}}{R_L} + \frac{C_{gd}}{R_i} + \frac{C_{gd}}{R_L} + C_{gd} g_m\right)$$

$$+ s^2(C_{gs}C_L + C_{gd}C_{gs} + C_{gd}C_L)$$

Pull out $[R_i R_L]^{-1}$

$$D = \frac{1}{R_i R_L}\Big[1 + s\big(R_L(C_L + C_{gd}) + R_i(C_{gs} + C_{gd}) + R_i R_L(C_{gd} g_m)\big) + s^2(R_i R_L)(C_{gs}C_L + C_{gd}C_{gs} + C_{gd}C_L)\Big]$$

$$b_1 = R_L(C_L + C_{gd}) + R_i(C_{gs} + C_{gd}) + R_L R_i C_{gd} g_m$$

$$b_2 = R_i R_L \left(C_{gs} C_L + C_{gd} C_{gs} + C_{gd} \cdot C_L\right)$$

$$D = \frac{1}{R_i R_L}\left(1 + s b_1 + s^2 b_2\right)$$

$$V_o = \frac{N}{D} = I_i g_m R_i R_L \left(\frac{1 + s a_1}{1 + s b_1 + s^2 b_2}\right)$$

$$V_{gen} = I_i R_i$$

$$\frac{V_o}{V_{gen}} = g_m R_L \left(\frac{1 + s a_1}{1 + s b_1 + s^2 b_2}\right)$$

c, d.) $a_1 = -C_{gd}/g_m = 10\,fF/50\,mS$
$= -2 \times 10^{-13}\ s$

zero at $s = -\frac{1}{a_1} = 5 \times 10^{12}$ rad/s
$\approx +800$ GHz

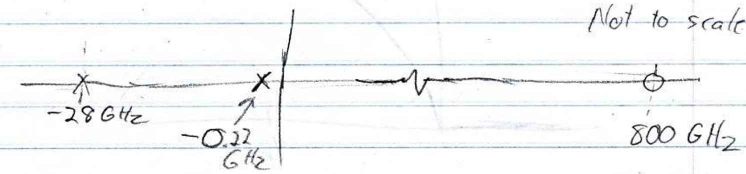
Plug in $C_{gs} = 200\,fF$, $C_{gd} = 10\,fF$, $g_m = 50\,mS$,
$R_L = 1000\,\Omega$, $R_i = 1000\,\Omega$, $C_L = 10\,fF$

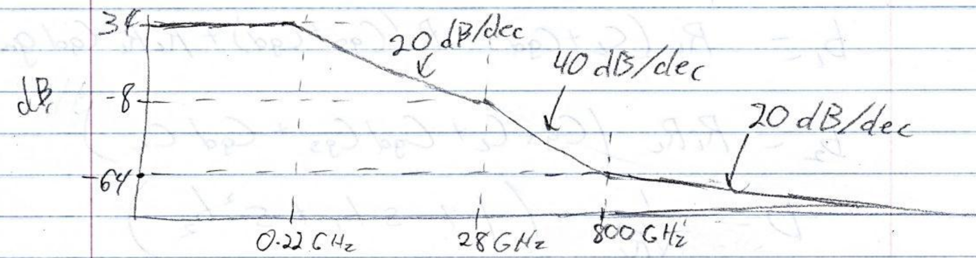
$b_1 = 7.3 \times 10^{-10}\ s$
$b_2 = 4.1 \times 10^{-21}\ s^2$

$$s = \frac{-b_1}{2b_2} \pm \frac{\sqrt{b_1^2 - 4b_2}}{2b_2}$$

ω $\quad s = -1.4 \times 10^{9}$, -1.77×10^{11} rad/s
f $\quad s = -0.22$ GHz, -28 GHz

Not to scale





$$20\log_{10}(g_m R_L) = 34\,dB$$

$$\log_{10}(28/.22)\times 20 = 42\,dB$$

$$34-42=-8\,dB$$

$$\log_{10}(800/28)\times 40 = 58\,dB$$

$$-8-58=-64\,dB$$

$b_1 = \tau_1 = 730\,ps \qquad \tau_2 = \sqrt{b_2} = 64\,ps$

Approximate using only τ_1

$$\frac{V_o}{V_{gen}} \approx \frac{g_m R_L}{1+s(730ps)}$$

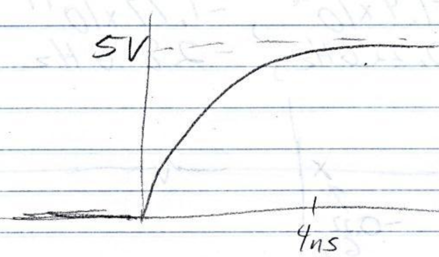
$$V_{out} \approx \frac{100mV}{s} \times \frac{g_m R_L}{1+s(730ps)}$$

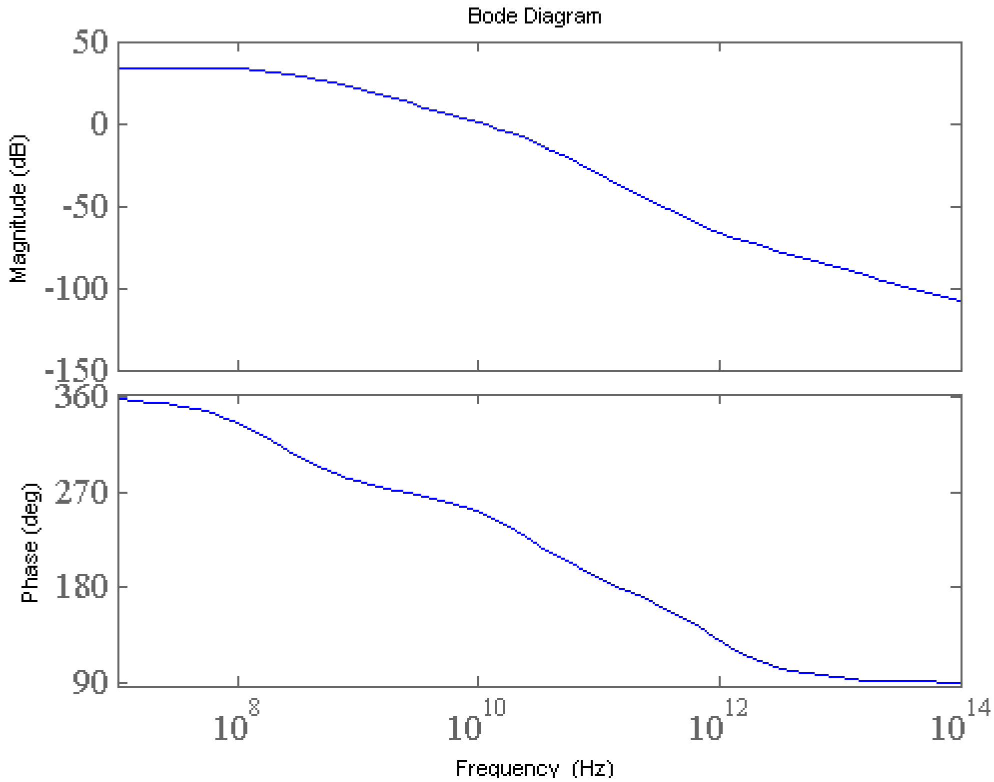
$$V_{out} \approx \frac{100 g_m R_L}{s} - \frac{100 g_m R_L (730ps)}{1+s(730ps)}$$

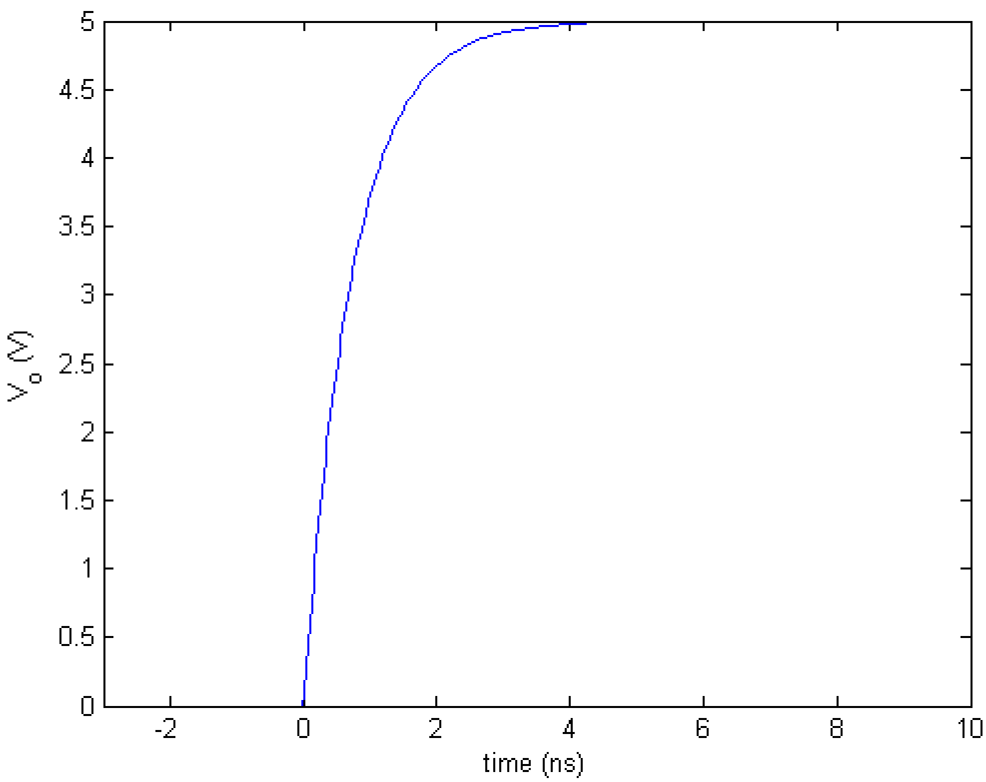
$$V_{out} \approx 100 g_m R_L \left[\frac{1}{s} - \frac{730ps}{1+s(730ps)}\right]$$

$$V_{out} = 5000\left[u(t) - u(t)\exp(-t/730ps)\right]$$

$$V_{out} = 5\left(1-\exp\left(\frac{-t}{730ps}\right)\right)u(t)\ V$$




Bode Diagram
50
0
-50
-100
-150
Magnitude (dB)
360
270
180
90
Phase (deg)
10^{8}
10^{10}
10^{12}
10^{14}
Frequency (Hz)


5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
V_o (V)
-2
0
2
4
6
8
10
time (ns)