

2. homogeneous Helmholtz's equation is

$$\nabla^2 \vec{E} + k^2 \vec{E} = 0$$

$$k^2 = \omega^2 \mu \epsilon \quad \text{from (8-22)}$$

$$\nabla^2 \vec{E} = \nabla^2 \vec{E}(x, y, z) = \nabla^2 (\vec{E}_0 \cdot e^{-jk_x x - jk_y y - jk_z z})$$

$$= \hat{a}_x \cdot \nabla^2 E_{0x} e^{-jk_x x - jk_y y - jk_z z}$$

$$+ \hat{a}_y \cdot \nabla^2 E_{0y} e^{-jk_x x - jk_y y - jk_z z}$$

$$+ \hat{a}_z \cdot \nabla^2 E_{0z} e^{-jk_x x - jk_y y - jk_z z}$$

$$= \hat{a}_x E_{0x} (k_x^2 - k_y^2 - k_z^2) \cdot e^{-jk_x x - jk_y y - jk_z z}$$

$$+ \hat{a}_y E_{0y} (-k_x^2 - k_y^2 - k_z^2) \cdot e^{-jk_x x - jk_y y - jk_z z}$$

$$+ \hat{a}_z E_{0z} (-k_x^2 - k_y^2 - k_z^2) \cdot e^{-jk_x x - jk_y y - jk_z z}$$

$$= -(k_x^2 + k_y^2 + k_z^2) \cdot e^{-jk_x x - jk_y y - jk_z z} (\hat{a}_x E_{0x} + \hat{a}_y E_{0y} + \hat{a}_z E_{0z})$$

$$= -(k_x^2 + k_y^2 + k_z^2) \cdot \vec{E}_0 \cdot e^{-jk_x x - jk_y y - jk_z z}$$

$$= -k^2 \vec{E} \quad (\text{from (8-23)})$$

so

$$\nabla^2 \vec{E} + k^2 \vec{E} = -k^2 \vec{E} + k^2 \vec{E} = 0$$

Notation: $\vec{E}$ means vector E

$$\vec{E}_0 = \hat{a}_x E_{0x} + \hat{a}_y E_{0y} + \hat{a}_z E_{0z}$$

$\hat{a}_x, \hat{a}_y, \hat{a}_z$ are unit vectors

(2)

3.

Assume that the vehicle moves with a velocity u in $(+z)$ direction, which is the direction of wave propagation.

(a). for the vehicle

if, $\vec{E}_{in} = \hat{a}_x E_0 e^{j(\omega t - kz)}$

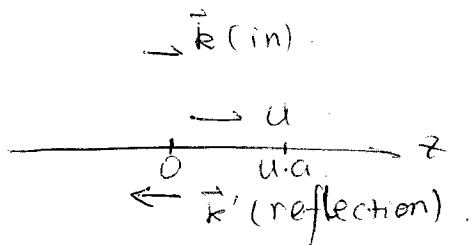
then, $\vec{E}_{ref} = -\hat{a}_x E_0 e^{j(\omega' t + k' z)}$

} perfectly reflecting $\Rightarrow$ the amplitude (E_0) keeps unchanged.

Note, $\omega \neq \omega'$ $k \neq k'$

Consider the reflecting surface, we have:

$$\vec{E}_{in} + \vec{E}_{ref} = 0 \quad \text{at the reflecting surface}$$



(1) $t = 0$

$$\vec{E}_{in} = \hat{a}_x E_0 e^{j(-kz)}$$

$$\vec{E}_{ref} = -\hat{a}_x E_0 e^{j(+k'z)}$$

$$\vec{E}_{in}(z) + \vec{E}_{ref}(z) = 0 \quad \text{at the reflecting surface}$$

$$\text{so, } z = 0$$

(2) $t = a$ the reflecting surface is located at

$$z = u \cdot t = u \cdot a$$

$$\vec{E}_{in}(ua) + \vec{E}_{ref}(ua) = 0$$

$$\text{so that, } \omega a - k u a = \omega' a + k' u a$$

$$\underline{\omega - k \cdot u = \omega' + k' u}$$

since $k = \frac{w}{c}$, $k' = \frac{w'}{c}$.

(3)

$$w - ku = w' + k'u$$

$$w - w' = \frac{u}{c} (w + w')$$

$$(1 - \frac{u}{c}) w = (1 + \frac{u}{c}) w'$$

$$\frac{w'}{w} = \frac{2\pi f'}{2\pi f} = \frac{f'}{f} = \frac{1 - \frac{u}{c}}{1 + \frac{u}{c}} = \frac{(1 - \frac{u}{c})^2}{(1 - \frac{u^2}{c^2})}$$

$$= \frac{1 - \frac{2u}{c} + \frac{u^2}{c^2}}{1 - \frac{u^2}{c^2}}$$

because $u \ll c$, $\frac{u}{c} \ll 1$,

$$\frac{f'}{f} = \frac{1 - \frac{2u}{c}}{1} = 1 - \frac{2u}{c}$$

$$\Delta f = f' - f = f(1 - \frac{2u}{c}) - f = f \cdot (-\frac{2u}{c})$$

(b). $\Delta f = 2.33 \text{ kHz}$

$f = 10.5 \text{ GHz}$

$$u = -\frac{\Delta f \times c}{2f} = -\frac{2.33 \times 10^3 \times 3 \times 10^8}{2 \times 10.5 \times 10^9}$$

$$= -33.28 \text{ m/s} = -119.8 \text{ km/hr} = 74.6 \text{ (miles/hr)}$$

4.

(4)

$$\vec{E} = \vec{E}_0 e^{-j\vec{k} \cdot \vec{R}}$$

$$\vec{H} = \vec{H}_0 e^{-j\vec{k} \cdot \vec{R}}$$

$\vec{E}_0, \vec{H}_0$ are constant vectors.

consider $\nabla \times \vec{E}$.

$$\nabla \times \vec{E} = \nabla \times (\vec{E}_0 e^{-j\vec{k} \cdot \vec{R}})$$

$$\text{assume } \vec{E}_0 = \hat{a}_x E_x + \hat{a}_y E_y + \hat{a}_z E_z$$

E_x, E_y, E_z are constant scalars.

$$\nabla \times (\vec{E}_0 e^{-j\vec{k} \cdot \vec{R}}) = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x e^{-j\vec{k} \cdot \vec{R}} & E_y e^{-j\vec{k} \cdot \vec{R}} & E_z e^{-j\vec{k} \cdot \vec{R}} \end{vmatrix}$$

$$= \hat{a}_x \left[E_z \cdot (-jk_y) \cdot e^{-j\vec{k} \cdot \vec{R}} - E_y \cdot (-jk_z) \cdot e^{-j\vec{k} \cdot \vec{R}} \right]$$

$$+ \hat{a}_y \left[E_x \cdot (-jk_z) \cdot e^{-j\vec{k} \cdot \vec{R}} - E_z \cdot (-jk_x) \cdot e^{-j\vec{k} \cdot \vec{R}} \right]$$

$$+ \hat{a}_z \left[E_y \cdot (-jk_x) \cdot e^{-j\vec{k} \cdot \vec{R}} - E_x \cdot (-jk_y) \cdot e^{-j\vec{k} \cdot \vec{R}} \right]$$

$$= (-j \cdot e^{-j\vec{k} \cdot \vec{R}}) \left[\hat{a}_x (E_z k_y - E_y k_z) + \hat{a}_y (E_x k_z - E_z k_x) \right. \\ \left. + \hat{a}_z (E_y k_x - E_x k_y) \right]$$

$$= (-j \cdot e^{-j\vec{k} \cdot \vec{R}}) (\vec{k} \times \vec{E}_0)$$

$$= -j \cdot \vec{k} \times (\vec{E}_0 e^{-j\vec{k} \cdot \vec{R}})$$

$$= \underline{-j \cdot \vec{k} \times \vec{E}}$$

In the same way,

$$\nabla \times \vec{H} = -j \vec{k} \times \vec{H}$$

$$\begin{aligned}
 \nabla \cdot \vec{E} &= \nabla \cdot (\vec{E}_0 e^{-j\vec{k} \cdot \vec{r}}) \quad (5) \\
 &= E_x (-jk_x) e^{-j\vec{k} \cdot \vec{r}} + E_y (-jk_y) e^{-j\vec{k} \cdot \vec{r}} + E_z (-jk_z) e^{-j\vec{k} \cdot \vec{r}} \\
 &= -j \cdot \vec{k} \cdot \vec{E}_0 e^{-j\vec{k} \cdot \vec{r}} \\
 &= -j (\vec{k} \cdot \vec{E}).
 \end{aligned}$$

Also ,

$$\nabla \cdot \vec{H} = -j \vec{k} \cdot \vec{H}$$

Now consider Maxwell's equations.

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t} \vec{B} = -j\omega \vec{B} = -j\omega \mu \vec{H}$$

$$\nabla \times \vec{E} = -j \vec{k} \times \vec{E}$$

$$\text{So that } -j \vec{k} \times \vec{E} = -j\omega \mu \vec{H}$$

$$\underline{\vec{k} \times \vec{E} = \omega \mu \vec{H}}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial}{\partial t} \vec{D} = \vec{J} + (+j\omega) \epsilon \vec{E}$$

$$= 0 + j\omega \epsilon \vec{E} \quad (\text{source-free})$$

$$\nabla \times \vec{H} = -j \vec{k} \times \vec{H}$$

$$\text{So that } -j \vec{k} \times \vec{H} = +j\omega \epsilon \vec{E}$$

$$\underline{\vec{k} \times \vec{H} = -\omega \epsilon \vec{E}}$$

$$\nabla \cdot \vec{E} = -j \vec{k} \cdot \vec{E} = 0 \quad \Rightarrow \quad \vec{k} \cdot \vec{E} = 0$$

$$\nabla \cdot \vec{H} = -j \vec{k} \cdot \vec{H} = 0 \quad \Rightarrow \quad \vec{k} \cdot \vec{H} = 0$$

(6)

8-7. Consider $E(0, t) = \hat{a}_x E_{10} \sin(\omega t) + \hat{a}_y E_{20} \sin(\omega t + \phi)$
 $= \hat{a}_x E_x + \hat{a}_y E_y$

$$E_x = E_{10} \sin(\omega t)$$

$$E_y = E_{20} \sin(\omega t + \phi) = E_{20} [\sin(\omega t) \cos \phi + \sin \phi \cos(\omega t)]$$

then,

$$\frac{E_x}{E_{10}} = \sin(\omega t) \quad (1)$$

$$\frac{E_y}{E_{20}} = \sin(\omega t) \cos \phi + \sin \phi \cos(\omega t) \quad (2)$$

$$= \frac{E_x}{E_{10}} \cos \phi + \sin \phi \cos(\omega t)$$

so, from (2),

$$\frac{E_y}{E_{20}} - \frac{E_x}{E_{10}} \cos \phi = \sin \phi \cos(\omega t)$$

$$\left(\frac{E_y}{E_{20}} - \frac{E_x}{E_{10}} \cos \phi \right)^2 = \sin^2 \phi \cos^2(\omega t)$$

$$= \sin^2 \phi (1 - \sin^2(\omega t))$$

$$= \sin^2 \phi \left(1 - \left(\frac{E_x}{E_{10}} \right)^2 \right)$$

expand the left-side,

$$\left(\frac{E_y}{E_{20}} \right)^2 + \left(\frac{E_x}{E_{10}} \right)^2 \cos^2 \phi - 2 \frac{E_x E_y}{E_{10} E_{20}} \cos \phi = \sin^2 \phi - \left(\frac{E_x}{E_{10}} \right)^2 \sin^2 \phi$$

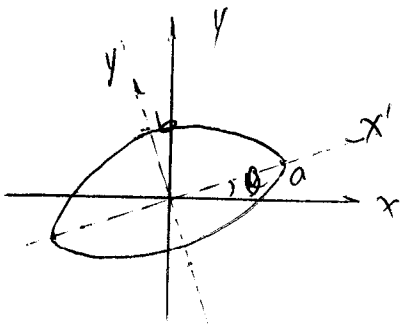
we have,

$$\left(\frac{E_y}{E_{20}} \right)^2 + \left(\frac{E_x}{E_{10}} \right)^2 - 2 \frac{E_x E_y}{E_{10} E_{20}} \cos \phi = \sin^2 \phi$$

$$\left(\frac{E_x}{E_{10} \sin \phi} \right)^2 + \left(\frac{E_y}{E_{20} \sin \phi} \right)^2 - 2 \frac{E_x E_y}{E_{10} E_{20}} \frac{\cos \phi}{\sin^2 \phi} = 1 \quad (*)$$

This is already an ellipse.

(7)



To find the parameters of the polarizat ellipse , we need to rotate the coordinate axes $x-y$ counterclockwise by an angle θ , to $x'-y'$.

Assume the ellipse in terms the new coordinates is

$$\left(\frac{E_{x'}}{a}\right)^2 + \left(\frac{E_{y'}}{b}\right)^2 = 1 \quad (3)$$

$$\begin{cases} E_{x'} = E_x \cos \theta + E_y \sin \theta \\ E_{y'} = -E_x \sin \theta + E_y \cos \theta \end{cases} \quad (4)$$

put (4) in (3)

$$E_x^2 \left(\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right) + E_y^2 \left(\frac{\sin^2 \theta}{a^2} + \frac{\cos^2 \theta}{b^2} \right) + 2 \frac{E_x E_y \sin \theta \cos \theta}{a^2} - 2 \frac{E_x E_y \sin \theta \cos \theta}{b^2} = 1$$

so that compare with the former equation (*)

$$\begin{cases} \frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} = \frac{1}{E_{10}^2 \sin^2 \varphi} \\ \frac{\sin^2 \theta}{a^2} + \frac{\cos^2 \theta}{b^2} = \frac{1}{E_{20}^2 \sin^2 \varphi} \\ \sin \theta \cos \theta \cdot \left(\frac{1}{a^2} - \frac{1}{b^2} \right) = \frac{1}{E_{10} E_{20} \sin^2 \varphi} \end{cases}$$

The solutions are :

$$\theta = \frac{1}{2} \cdot \tan^{-1} \left(\frac{2 E_{10} E_{20} \cos \varphi}{E_{10}^2 - E_{20}^2} \right)$$

$$a = \left[\frac{2}{\frac{1}{E_{10}^2} (1 + \sec(2\theta)) + \frac{1}{E_{20}^2} (1 - \sec(2\theta))} \right]^{\frac{1}{2}} \sin \varphi$$

$$b = \left[\frac{2}{\frac{1}{E_{10}^2} (1 - \sec(2\theta)) + \frac{1}{E_{20}^2} (1 + \sec(2\theta))} \right]^{\frac{1}{2}} \sin \varphi$$

(8)

8-8. (a). Consider the phasor at $z=0$.

An elliptically polarized wave can be written

$$\vec{E} = \hat{a}_x E_1 + \hat{a}_y E_2 \cdot e^{j\alpha}$$

Assume the Right-hand circularly polarized wave

$$\vec{E}_{rc} = E_{rc} (\hat{a}_x - \hat{a}_y \cdot j)$$

+ the Left-hand circularly polarized wave:

$$\vec{E}_{lc} = E_{lc} (\hat{a}_x + \hat{a}_y \cdot j)$$

$$\text{if } \vec{E}_{rc} = \frac{1}{2} (E_1 + j E_2 \cdot e^{j\alpha})$$

$$\vec{E}_{lc} = \frac{1}{2} (E_1 - j E_2 \cdot e^{j\alpha})$$

$$\text{then } \vec{E} = \vec{E}_{rc} + \vec{E}_{lc}$$

(b).

$$\vec{E}_{rc} = E (\hat{a}_x - \hat{a}_y \cdot j)$$

$$= E (\hat{a}_x \frac{1}{2} - \hat{a}_y \cdot j \cdot 2) + E (\hat{a}_x \cdot \frac{1}{2} + \hat{a}_y \cdot j)$$

These two terms are oppositely directed elliptically polarized waves.

The solution is not unique.

8-9. for a conducting media,

$$\frac{\sigma}{\omega} \gg \epsilon$$

consider a general expression,

$$\Gamma = \alpha + j\beta = j k_c = j \omega \sqrt{\mu \epsilon_c}$$

9

from Maxwell's equations,

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} = \sigma \cdot \vec{E} + j\omega \epsilon \vec{E}$$
$$= j\omega \cdot \left(\epsilon - j \cdot \frac{\sigma}{\omega} \right) \vec{E} = j \cdot \omega \cdot \epsilon_c \cdot \vec{E}.$$

$$\epsilon_c = \epsilon - j \cdot \frac{\sigma}{\omega}.$$

$$(\alpha + j\beta)^2 = -\omega^2 \mu \epsilon_c.$$

here, $\alpha, \beta, \omega, \mu, \epsilon, \sigma$ are all real number,

$$\alpha^2 - \beta^2 + j \cdot 2\alpha\beta = -\omega^2 \mu \cdot \left(\epsilon - j \cdot \frac{\sigma}{\omega} \right).$$
$$= -\omega^2 \mu \epsilon + j \cdot \omega \mu \sigma$$

so,

$$\alpha^2 - \beta^2 = -\omega^2 \mu \epsilon.$$

$$2\alpha\beta = \omega \mu \sigma.$$

~~then~~ and

$$(\alpha^2 + \beta^2) = |\alpha + j\beta|^2 = |- \omega^2 \mu \epsilon_c|^2.$$

$$= | \omega^2 \mu \epsilon - j \cdot \omega \mu \sigma |$$

$$= \sqrt{\omega^4 \mu^2 \epsilon^2 + \omega^2 \mu^2 \sigma^2} = \omega^2 \mu \epsilon \cdot \sqrt{1 + \left(\frac{\sigma}{\omega \epsilon} \right)^2} \quad (2)$$

from (1), (2).

$$\alpha^2 = \frac{\omega^2 \mu \epsilon}{2} \cdot \left(\sqrt{1 + \left(\frac{\sigma}{\omega \epsilon} \right)^2} - 1 \right).$$

$$\beta^2 = \frac{\omega^2 \mu \epsilon}{2} \cdot \left(\sqrt{1 + \left(\frac{\sigma}{\omega \epsilon} \right)^2} + 1 \right).$$

we can show that

$$\alpha = \omega \sqrt{\frac{\mu \epsilon}{2}} \cdot \left(\sqrt{1 + \left(\frac{\sigma}{\omega \epsilon} \right)^2} - 1 \right)^{\frac{1}{2}}.$$

$$\beta = \omega \sqrt{\frac{\mu \epsilon}{2}} \cdot \left(\sqrt{1 + \left(\frac{\sigma}{\omega \epsilon} \right)^2} + 1 \right)^{\frac{1}{2}}.$$

(10)

8-12.

(a). $\sigma = 4 \text{ (S/m)}$.

so seawater is a conducting media.

$$\frac{\sigma}{\omega\epsilon} = \frac{4}{10^{10}\pi \times 80 \times \frac{1}{36\pi} \times 10^{-9}} = 0.18$$

so the seawater is neither a perfect conductor, nor an insulator,

(Choose the equations shown in Problem 8-9.

the attenuation constant $\alpha = \omega \cdot \sqrt{\frac{\mu\epsilon}{2}} \cdot \left(\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1 \right)^{\frac{1}{2}} = 84 \text{ (Np/m)}$

the phase constant $\beta = \omega \cdot \sqrt{\frac{\mu\epsilon}{2}} \cdot \left(\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1 \right)^{\frac{1}{2}} = 300\pi \text{ (rad/m)}$.

the intrinsic impedance $\eta_c = \sqrt{\frac{\mu}{\epsilon_c}} = \sqrt{\frac{\mu}{\epsilon_0}} \cdot \frac{1}{\sqrt{\epsilon_r \cdot \left[1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{\frac{1}{2}}}} \cdot e^{\frac{j}{2} \cdot \arctan\left(\frac{\sigma}{\omega\epsilon}\right)}$
 $= 41.8 e^{j \cdot 0.0283} \text{ (}\Omega\text{)}$

the ~~group~~ phase velocity

$$v_p = \frac{\omega}{\beta} = \frac{10^{10}\pi}{300\pi} = 3.3 \times 10^7 \text{ (m/s)}$$

the wavelength

$$\lambda = \frac{2\pi}{\beta} = 0.67 \text{ (cm)}$$

the skin depth

$$\delta = \frac{1}{\alpha} = \frac{1}{84} = 1.19 \times 10^{-2} \text{ (m)}$$

(11)

(b).

$$0.1 \cdot e^{-\alpha y} = 0.01$$

$$y = \frac{1}{\alpha} \cdot \ln 10 = 2.74 \text{ (cm)}.$$

(c).

from Maxwell's equation,

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial}{\partial t} \vec{D} = j\omega \cdot \epsilon_c \vec{E}.$$

$$-j \vec{k} \times \vec{H} = j\omega \epsilon_c \vec{E}.$$

$$\vec{E} = -\frac{k}{\omega \epsilon_c} \hat{a}_y \times \hat{a}_x \left(0.1 \cdot e^{j(-ky - \frac{\pi}{3} - \frac{\pi}{2})} \right).$$

phasor.

$$\text{Here, } k = \omega \sqrt{\mu \epsilon_c}.$$

$$\begin{aligned} \vec{E} &= \hat{a}_z \cdot \sqrt{\frac{\mu}{\epsilon_c}} \cdot 0.1 \cdot e^{j(-ky - \frac{\pi}{3} - \frac{\pi}{2})} \\ &= \hat{a}_z \cdot \eta_c \cdot 0.1 \cdot e^{-j(k \cdot y + \frac{\pi}{3} + \frac{\pi}{2})} \\ &= \hat{a}_z \cdot \eta_c \cdot 0.1 \cdot e^{-\gamma \cdot \alpha} \cdot e^{-j(\beta y + \frac{\pi}{3} + \frac{\pi}{2})}. \end{aligned}$$

$$\text{We have: } \eta_c = 41.8 e^{j0.0283\pi}$$

$$\alpha = 84$$

$$\beta = 300\pi$$

$$\vec{E}(y, t) = \Re(\vec{E} \cdot e^{j\omega t})$$

so

$$\vec{E}(y=0.5, t) = \hat{a}_z \cdot 2.41 \times 10^{-18} \times \sin\left(10^{10}\pi t - \frac{\pi}{3} + 0.0283\pi\right)$$

8-15 .

(12)

$$(a). \quad u_p = \frac{w}{\beta} \quad w = \beta \cdot u_p .$$

$$\text{so} \quad u_g = \frac{dw}{d\beta} = \frac{d}{d\beta} (\beta \cdot u_p) = u_p + \beta \cdot \frac{du_p}{d\beta} .$$

Note : u_p is a function of w .

$$(b). \quad \lambda = \frac{2\pi}{\beta} \quad \frac{d\lambda}{d\beta} = -\frac{2\pi}{\beta^2} = -\frac{\lambda}{\beta} .$$

Now ,

$$\begin{aligned} u_g = \frac{dw}{d\beta} &= u_p + \beta \cdot \frac{du_p}{d\beta} \\ &= u_p + \beta \cdot \left(\frac{du_p}{d\lambda} \cdot \frac{d\lambda}{d\beta} \right) \\ &= u_p + \beta \cdot \left(-\frac{\lambda}{\beta} \cdot \frac{du_p}{d\lambda} \right) \\ &= u_p - \lambda \cdot \frac{du_p}{d\lambda} . \end{aligned}$$

8-17 . Assume

$$\vec{E} = \hat{a}_x E_0 \cos(\omega t - kz + \phi) + \hat{a}_y E_0 \sin(\omega t - kz + \phi) .$$

$$\vec{H} = \hat{a}_y \frac{E_0}{\eta} \cos(\omega t - kz + \phi) - \hat{a}_x \frac{E_0}{\eta} \sin(\omega t - kz + \phi) .$$

then

$$\begin{aligned} \vec{P} = \vec{E} \times \vec{H} &= \hat{a}_z \frac{E_0^2}{\eta} \cdot [\cos^2(\omega t - kz + \phi) + \sin^2(\omega t - kz + \phi)] \\ &= \hat{a}_z \cdot \frac{E_0^2}{\eta} . \end{aligned}$$

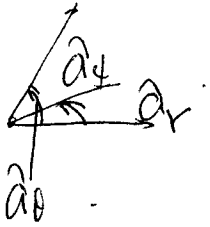
So $\vec{P}$ is a constant, independent of time and distance .

(13)

8-18.

$$\vec{E} = \hat{a}_\theta E_\theta + \hat{a}_\varphi E_\varphi$$

$$\vec{H} = \frac{1}{\eta} \hat{a}_r \times \vec{E} = \frac{1}{\eta} (-\hat{a}_\varphi E_\theta + \hat{a}_\theta E_\varphi)$$



$$\vec{P}_{av} = \frac{1}{2} \cdot \text{Re}(\vec{E} \times \vec{H}^*)$$

$$= \frac{1}{2\eta} \cdot \text{Re}[\hat{a}_r E_\theta \cdot E_\theta^* + \hat{a}_r E_\varphi \cdot E_\varphi^*]$$

$$= \frac{1}{2\eta} \cdot \hat{a}_r (|E_\theta|^2 + |E_\varphi|^2)$$