Homework #5

9-2 (a). Assume the wavenumbe is $\vec{k} = \beta \hat{a}_z$.

From Maxwell's equations,

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \quad (1)$$

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E} \quad (2)$$

since

$$\vec{E} = (\hat{a}_x E_x(x,y) \cdot e^{-j\beta z} + \hat{a}_y E_y(x,y) e^{-j\beta z})$$

$$\vec{H} = (\hat{a}_x H_x(x,y) \cdot e^{-j\beta z} + \hat{a}_y H_y(x,y) e^{-j\beta z}).$$

plug into (1), (2)

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x(x,y)e^{-j\beta z} & E_y(x,y)e^{-j\beta z} & 0 \end{vmatrix}$$

$$= \hat{a}_x\, j \cdot \beta E_y \cdot e^{-j\beta z} + \hat{a}_y (-j\beta) E_x \cdot e^{-j\beta z}$$

$$+ \hat{a}_z \left(\frac{\partial}{\partial x} E_y - \frac{\partial}{\partial y} E_x\right) \cdot e^{-j\beta z}.$$

$$= -j\omega\mu(\hat{a}_x H_x e^{-j\beta z} + \hat{a}_y H_y e^{-j\beta z}).$$

so,

$$\beta E_y = -\omega\mu H_x \quad (3)$$

$$\beta E_x = \omega\mu H_y \quad (4)$$

$$\frac{\partial}{\partial x} E_y = \frac{\partial}{\partial y} E_x. \quad (5)$$

also, from (2),

$$\beta H_y = \omega\varepsilon E_x \quad (6)$$

$$\beta H_x = -\omega\varepsilon E_y \quad (7)$$

$$\frac{\partial}{\partial x} H_y = \frac{\partial}{\partial y} H_x. \quad (8)$$

from (3) and (7).

$$\beta H_x = -\frac{\beta}{w\mu} E_y = -w\varepsilon E_y.$$

$$\beta = \sqrt{w^2\mu\varepsilon}.$$

from (4). $$\frac{E_x}{H_y} = \frac{w\mu}{\beta} = \sqrt{\frac{\mu}{\varepsilon}} = \eta.$$

(b). consider

$$\frac{\partial^2 E_x}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial E_x}{\partial x}\right).$$

Since $E_x = \eta \cdot H_y$. $\frac{\partial E_x}{\partial x} = \eta \cdot \frac{\partial H_y}{\partial x}$.

and $\frac{\partial H_y}{\partial x} = \frac{\partial H_x}{\partial y}$.

so $$\frac{\partial^2 E_x}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial E_x}{\partial x}\right) = \frac{\partial}{\partial x}\left(\eta \cdot \frac{\partial H_y}{\partial x}\right) = \frac{\partial}{\partial x}\left(\eta \cdot \frac{\partial H_x}{\partial y}\right).$$

$$= \eta \cdot \frac{\partial}{\partial y}\left(\frac{\partial H_x}{\partial x}\right)$$

Since $\eta H_x = -E_y$. $\frac{\partial H_x}{\partial y} = -\frac{1}{\eta} \cdot \frac{\partial E_y}{\partial y}$.

so $$\frac{\partial^2 E_x}{\partial x^2} = \eta \frac{\partial}{\partial y}\left(\frac{\partial H_x}{\partial x}\right) = \eta \frac{\partial}{\partial y}\left(-\frac{1}{\eta}\frac{\partial E_y}{\partial y}\right) = -\frac{\partial^2 E_y}{\partial y^2}.$$

then $$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} = 0.$$

Similarly we can show that

$$\frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} = 0.$$

9-6. From Eq. 9-70,

$$\gamma = \alpha + j\beta = j\omega\sqrt{\mu\varepsilon'}\cdot\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{\frac{1}{2}}.$$

Squaring both sides and equating the real and imaginary parts respectively, we have

$$\alpha^2 - \beta^2 = -\omega^2\mu\varepsilon'$$

$$2\alpha\beta = \omega^2\mu\varepsilon'\left(\frac{\varepsilon''}{\varepsilon'}\right).$$

Solve these two equations,

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon'}{2}}\cdot\left[\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2} - 1\right]^{\frac{1}{2}}.$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon'}{2}}\cdot\left[\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2} + 1\right]^{\frac{1}{2}}.$$

9-10. (a).

$$Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = \sqrt{\frac{L}{C}}.$$

for two-wire transmission line, recall pp. 445,

$$C = \frac{\pi\epsilon}{\cosh^{-1}(D/2a)} \qquad L = \frac{\mu}{\pi}\cdot\cosh^{-1}\left(\frac{D}{2a}\right).$$

plug into Z_0,

$$Z_0 = \frac{1}{\pi}\sqrt{\frac{\mu}{\varepsilon}}\cdot\cosh^{-1}\left(\frac{D}{2a}\right) = \frac{120}{\sqrt{\varepsilon_r}}\cdot\ln\left(\frac{D}{2a}+\sqrt{\left(\frac{D}{2a}\right)^2-1}\right) = 30$$

$$\frac{D}{2a} = 21.27. \qquad D = 25.5\times10^{-3}\ (m).$$

(b). For the coaxial line, recall pp. 446, ④

$$L = \frac{\mu}{2\pi} \ln \frac{b}{a}. \qquad C = \frac{2\pi\epsilon}{\ln(b/a)}.$$

$$Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}} \cdot \ln\left(\frac{b}{a}\right) = \frac{60}{\sqrt{\epsilon_r}} \ln\left(\frac{b}{a}\right) = 75 \ (\Omega).$$

$$\frac{b}{a} = 6.52. \qquad b = 3.91 \times 10^{-3} \ (m).$$

9-11. Assume $Z_i = R_i + j \cdot X_i$. $Z_g = R_g + j \cdot X_g$.

Then,

$$V_i = \frac{Z_i}{Z_i + Z_g} \cdot V_g.$$

$$I_i = \frac{V_g}{Z_i + Z_g}.$$

$$(P_{av})_L = \frac{1}{2} \mathrm{Re}(V_i \, I_i^*)$$

$$= \frac{|V_g|^2 \cdot R_i}{(R_g + R_i)^2 + (X_g + X_i)^2}.$$

To find maximum $(P_{av})_L$,

$$\frac{\partial (P_{av})_L}{\partial R_i} = 0 \quad \rightarrow \quad R_i = R_g.$$

$$\frac{\partial (P_{av})_L}{\partial X_i} = 0 \quad \rightarrow \quad X_i = -X_g.$$

So $Z_i = Z_g^*$.

$$\max (P_{av})_L = \frac{|V_g|^2}{4R_i}$$

The maximum powertransfer efficiency is.

$$\frac{\frac{|V_g|^2}{4R_i}}{\frac{|V_g|^2}{2R_i}} = \frac{1}{2} = 50\%.$$

9-12.

(a) $V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$.

$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z}$.

At the input end,

$V(0) = V_0^+ + V_0^- = V_i$

$I(0) = I_0^+ + I_0^- = I_i$.

$$\frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = Z_0.$$

So $V_0^+ = \frac{1}{2}(V_i + I_i Z_0)$ $\quad V_0^- = \frac{1}{2}(V_i - I_i Z_0)$

$I_0^+ = \cancel{Z_0} \cdot V_0^+/Z_0$. $\quad I_0^- = -V_0^-/Z_0$.

then we can express $V(z)$ and $I(z)$ as.

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

$$= \frac{1}{2}(V_i + I_i Z_0) e^{-\gamma z} + \frac{1}{2}(V_i - I_i Z_0) e^{+\gamma z}.$$

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z}.$$

$$= \frac{1}{2Z_0}(V_i + I_i Z_0) e^{-\gamma z} + \frac{-1}{2Z_0}(V_i - I_i Z_0) \cdot e^{+\gamma z}.$$

(b). $V(z) = V_i \cdot \cosh(\gamma z) - I_i Z_0 \cosh(\gamma z)$.

$I(z) = I_i \cosh(\gamma z) - \frac{V_i}{Z_0} \cosh(\gamma z)$.

9-13. (a). From Eqs. 9-100 (a), (b).

$$\begin{cases} V_1 = V(l) = V_2 \cdot \cosh(\gamma l) + Z_0 I_2 \cdot \sinh(\gamma l) \\ I_1 = I(l) = \frac{V_2}{Z_0} \sinh(\gamma l) + I_2 \cosh(\gamma l) \end{cases}$$

and

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}.$$

so we can ~~see~~ compare the coefficients,

$$A = D = \cosh(\gamma l).$$

$$B = Z_0 \cdot \sinh(\gamma l).$$

$$C = \frac{1}{Z_0} \cdot \sinh(\gamma l).$$

$$AD - BC = \cosh^2(\gamma l) - \sinh^2(\gamma l)$$

$$= \frac{1}{4}(e^{2\gamma l} + e^{-2\gamma l} + 2) - \frac{1}{4}(e^{2\gamma l} + e^{-2\gamma l} - 2)$$

$$= 1.$$

(b). From Fig. (b).

$$V_1 = \frac{Z_1}{2} \cdot I_1 + \frac{1}{Y_2}(I_1 - I_2). \qquad (1)$$

$$= \left(\frac{Z_1}{2} + \frac{1}{Y_2}\right) I_1 - \frac{1}{Y_2} I_2.$$

$$I_1 = I_2 + \left(V_2 + \frac{Z_1}{2} I_2\right) Y_2. \qquad (2).$$

$$= Y_2 \cdot V_2 + \left(1 + \frac{Z_1 Y_2}{2}\right) I_2.$$

substituting ② in ①,

$$V_1 = \left(1 + \frac{Z_1 Y_2}{2}\right) V_2 + Z_1 \left(1 + \frac{Z_1 Y_2}{4}\right) I_2.$$

so $A = 1 + \frac{Z_1 Y_2}{2}$. $B = Z_1\left(1 + \frac{Z_1 Y_2}{4}\right)$.

$$Z_1 Y_2 = 2(A-1).$$

$$B = Z_1 \left(1 + \frac{2(A-1)}{4}\right)$$

so $$Z_1 = \frac{B}{1+\frac{2(A-1)}{4}} = \frac{Z_0 \cdot (e^{\gamma l} - e^{-\gamma l})\frac{1}{2}}{\frac{1}{4}(e^{\frac{1}{2}\gamma l} + e^{-\frac{1}{2}\gamma l})^2}$$

$$= \frac{2Z_0\,(e^{\frac{1}{2}\gamma l} - e^{-\frac{1}{2}\gamma l})}{(e^{\frac{1}{2}\gamma l} + e^{-\frac{1}{2}\gamma l})} = 2Z_0 \cdot \tan\left(\tfrac{1}{2}\gamma l\right).$$

$$Y_2 = \frac{2(A-1)}{Z_1} = \frac{1}{Z_0}\cdot \sinh(\gamma l).$$

9-17 (a). From Eq.(9-117),

$$Z_i = Z_0 \cdot \frac{1-e^{-2\gamma l}}{1+e^{-2\gamma l}}, \qquad l = \frac{\lambda}{4}. \quad \beta l = \frac{2\pi}{\lambda}\cdot l = \frac{\pi}{2}.$$

$\cancel{\gamma l = j\beta l = j\frac{\pi}{2}}$;

$$Z_i = Z_0 \cdot \frac{1-e^{-2\alpha\frac{\lambda}{4}}\cdot e^{-j\pi}}{1+e^{-2\alpha\frac{\lambda}{4}}\cdot e^{-j\pi}}$$

$$= Z_0 \cdot \frac{1+(1-\frac{1}{2}\alpha\lambda)}{1-(1-\frac{1}{2}\alpha\lambda)} \simeq \frac{4Z_0}{\alpha\lambda}.$$

(b). From Eq.(9-116)

$$Z_i = Z_0 \cdot \frac{1+e^{-2\gamma l}}{1-e^{-2\gamma l}}.$$

$$= Z_0 \cdot \frac{1+e^{-2\alpha\frac{\lambda}{4}}\cdot e^{-j\pi}}{1-e^{-2\alpha\frac{\lambda}{4}}\cdot e^{-j\pi}} = Z_0 \cdot \frac{1-(1-\frac{1}{2}\alpha\lambda)}{1+(1-\frac{1}{2}\alpha\lambda)}$$

$$= \frac{Z_0\cdot\alpha\lambda}{4}.$$

9-20 (a). Consider the wave length $\lambda = c/f = \frac{3\times10^8}{10^5} = 3000$ (

the length of the coaxial cable is 0.6 (m) << λ

so we can use Eqs. (9-111) and (9-113).

$$C = \frac{54\times10^{-12}}{0.6} = 9\times10^{-11}\ (F/m).$$

$$L = \frac{0.3\times10^{-6}}{0.6} = 5\times10^{-7}\ (H/m).$$

$$R_o = \sqrt{\frac{L}{C}} = 74.5(\Omega) = Z_o \quad \text{(lossless line)}.$$

$$\mu\varepsilon = LC \quad , \quad \varepsilon_r = \frac{LC}{\mu_o\varepsilon_o} = 4.05.$$

(b). Calculate the X_{io} and X_{is} at 10 (MHz).

$$\beta = \frac{w}{u_p} = w\cdot\sqrt{\mu\varepsilon} = w\cdot\sqrt{LC} = 0.42,$$

$$\beta\cdot l = 0.42\times0.6 = 0.252\ \text{(rad)}.$$

$$X_{io} = -R_o\cdot\cot(\beta l) = -289.3\ (\Omega).$$

$$X_{is} = R_o\tan(\beta l) = 19.2\ (\Omega)$$