

Homework #4

(1)

9-2. (a) Suppose the wave-number is $\vec{k} = \beta \hat{a}_z$

From Maxwell's equations,

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \quad (1)$$

$$\nabla \times \vec{H} = j\omega\epsilon\vec{E} \quad (2)$$

$$\text{since } \vec{E} = \hat{a}_x E_x(x,y) e^{-j\beta z} + \hat{a}_y E_y(x,y) e^{-j\beta z}$$

$$\vec{H} = \hat{a}_x H_x(x,y) e^{-j\beta z} + \hat{a}_y H_y(x,y) e^{-j\beta z}$$

$$\begin{aligned} \nabla \times \vec{E} &= \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x(x,y)e^{-j\beta z} & E_y(x,y)e^{-j\beta z} & 0 \end{vmatrix} \\ &= -j\omega\mu(\hat{a}_x H_x e^{-j\beta z} + \hat{a}_y H_y e^{-j\beta z}) \end{aligned}$$

$$\text{thus, } \beta E_y = -\omega\mu H_x \quad (3)$$

$$\beta E_x = \omega\mu H_y \quad (4)$$

$$\frac{\partial}{\partial x} E_y = \frac{\partial}{\partial y} E_x \quad (5)$$

$$\text{Also, for (2), } \beta H_y = \omega\epsilon E_x \quad (6)$$

$$\beta H_x = -\omega\epsilon E_y \quad (7)$$

$$\frac{\partial}{\partial x} H_y = \frac{\partial}{\partial y} H_x \quad (8)$$

from (3) and (7).

(2)

$$\beta H_x = -\frac{\beta}{\omega \mu} E_y = -\omega \varepsilon E_y.$$

$$\beta = \sqrt{\omega^2 \mu \varepsilon}.$$

from (4) . $\frac{E_x}{H_y} = \frac{\omega \mu}{\beta} = \sqrt{\frac{\mu}{\varepsilon}} = \eta.$

(b) . consider

$$\frac{\partial^2 E_x}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial E_x}{\partial x} \right).$$

Since $E_x = \eta \cdot H_y$. $\frac{\partial E_x}{\partial x} = \eta \cdot \frac{\partial H_y}{\partial x}$.

and $\frac{\partial H_y}{\partial x} = -\frac{\partial H_x}{\partial y}$.

So
$$\begin{aligned} \frac{\partial^2 E_x}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial E_x}{\partial x} \right) = \frac{\partial}{\partial x} \left(\eta \cdot \frac{\partial H_y}{\partial x} \right) = \frac{\partial}{\partial x} \left(\eta \cdot \frac{\partial H_x}{\partial y} \right) \\ &= \eta \cdot \frac{\partial}{\partial y} \left(\frac{\partial H_x}{\partial x} \right) . \end{aligned}$$

Since $\eta H_x = -E_y$. $\frac{\partial H_x}{\partial y} = -\frac{1}{\eta} \cdot \frac{\partial E_y}{\partial y}$.

So
$$\frac{\partial^2 E_x}{\partial x^2} = \eta \frac{\partial}{\partial y} \left(\frac{\partial H_x}{\partial x} \right) = \eta \frac{\partial}{\partial y} \left(-\frac{1}{\eta} \frac{\partial E_y}{\partial y} \right) = -\frac{\partial^2 E_y}{\partial y^2} .$$

then
$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} = 0 .$$

Similarly we can show that

$$\frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} = 0 .$$

9-6

(3)

From Eq. 9-70,

$$\Gamma = \alpha + j\beta = j\omega\sqrt{\mu\epsilon'} \cdot \left(1 - j\frac{\epsilon''}{\epsilon'}\right)^{\frac{1}{2}}.$$

Squaring both sides and equating the real and imaginary parts respectively, we have

$$\alpha^2 - \beta^2 = -\omega^2\mu\epsilon'.$$

$$2\alpha\beta = \omega^2\mu\epsilon'\left(-\frac{\epsilon''}{\epsilon'}\right).$$

Solve these two equations,

$$\alpha = \omega\sqrt{\frac{\mu\epsilon'}{2}} \cdot \left[\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'}\right)^2} - 1\right]^{\frac{1}{2}}.$$

$$\beta = \omega\sqrt{\frac{\mu\epsilon'}{2}} \cdot \left[\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'}\right)^2} + 1\right]^{\frac{1}{2}}.$$

9-10. (a).

$$\gamma_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{L}{C}}.$$

for two-wire transmission line, recall pp. 445,

$$C = \frac{\pi\epsilon}{\cosh^{-1}(D/2a)} \quad L = \frac{\mu}{\pi} \cdot \cosh^{-1}\left(\frac{D}{2a}\right).$$

plug into γ_0 ,

$$\gamma_0 = \frac{1}{\pi} \sqrt{\frac{\mu}{\epsilon}} \cdot \cosh^{-1}\left(\frac{D}{2a}\right) = \frac{120}{\sqrt{\epsilon_r}} \cdot \ln\left(\frac{D}{2a} + \sqrt{\left(\frac{D}{2a}\right)^2 - 1}\right) = 30$$

$$\frac{D}{2a} = 21.27, \quad D = 25.5 \times 10^{-3} \text{ (m)}.$$

(b). For the coaxial line, recall pp. 446,

④

$$L = \frac{\mu}{2\pi} \ln \frac{b}{a} \quad C = \frac{2\pi\epsilon}{\ln(\frac{b}{a})}$$

$$Z_0 = \frac{1}{2\pi} \sqrt{\frac{\mu}{\epsilon}} \cdot \ln\left(\frac{b}{a}\right) = \frac{10}{\sqrt{\epsilon_r}} \ln\left(\frac{b}{a}\right) = 75 \quad (\Omega)$$

$$\frac{b}{a} = 6.52 \quad b = 3.91 \times 10^{-3} \text{ (m)}$$

9-11. Assume $Z_i = R_i + j \cdot X_i$ $Z_g = R_g + j \cdot X_g$.

$$\text{Then, } V_i = \frac{Z_i}{Z_i + Z_g} \cdot V_g$$

$$I_i = \frac{V_g}{Z_i + Z_g}$$

$$\begin{aligned} (P_{av})_L &= \frac{1}{2} \operatorname{Re}(V_i I_i^*) \\ &= \frac{|V_g|^2 \cdot R_i}{(R_g + R_i)^2 + (X_g + X_i)^2} \end{aligned}$$

To find maximum $(P_{av})_L$,

$$\frac{\partial (P_{av})_L}{\partial R_i} = 0 \quad \rightarrow R_i = R_g$$

$$\frac{\partial (P_{av})_L}{\partial X_i} = 0 \quad \rightarrow X_i = -X_g$$

$$\text{So } Z_i = Z_g^*$$

$$\max (P_{av})_L = \frac{|V_g|^2}{4R_i}$$

The maximum power transfer efficiency is

$$\frac{\frac{|V_g|^2}{4R_i}}{\frac{|V_g|^2}{2R_i}} = \frac{1}{2} = 50\%$$

9-12

(5)

$$(a) \quad V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$$

$$\text{At } z=0, \quad V(0) = V_i = V_0^+ + V_0^-$$

$$I(0) = I_i = I_0^+ + I_0^- = \frac{1}{Z_0} (V_0^+ - V_0^-)$$

$$\text{thus, } V_0^+ = \frac{1}{2} (V_i + Z_0 I_i)$$

$$V_0^- = \frac{1}{2} (V_i - Z_0 I_i)$$

$$V(z) = \frac{1}{2} (V_i + I_i Z_0) e^{-\gamma z} + \frac{1}{2} (V_i - I_i Z_0) e^{\gamma z}$$

$$I(z) = \frac{1}{2Z_0} (V_i + I_i Z_0) e^{-\gamma z} + \frac{1}{2Z_0} (I_i Z_0 - V_i) e^{\gamma z}$$

$$(b) \quad V(z) = \frac{1}{2} V_i (e^{\gamma z} + e^{-\gamma z}) + \frac{1}{2} I_i Z_0 (e^{-\gamma z} - e^{\gamma z})$$

$$= V_i \cosh(\gamma z) - I_i Z_0 \sinh(\gamma z)$$

$$I(z) = \frac{1}{2Z_0} V_i (e^{-\gamma z} - e^{\gamma z}) + \frac{1}{2} I_i (e^{\gamma z} - e^{-\gamma z})$$

$$= I_i \sinh(\gamma z) - \frac{V_i}{Z_0} \cosh(\gamma z)$$

9-13. (a). From Eqs. 9-100 (a), (b).

(6)

$$\begin{cases} V_1 = V(L) = V_2 \cosh(rL) + Z_0 I_2 \sinh(rL) \\ I_1 = I(L) = \frac{V_2}{Z_0} \sinh(rL) + I_2 \cosh(rL) \end{cases}$$

and

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

so we can compare the coefficients,

$$A = D = \cosh(rL)$$

$$B = Z_0 \sinh(rL)$$

$$C = \frac{1}{Z_0} \sinh(rL)$$

$$\begin{aligned} AD - BC &= \cosh^2(rL) - \sinh^2(rL) \\ &= \frac{1}{4} (e^{2rL} + e^{-2rL} + 2) - \frac{1}{4} (e^{2rL} + e^{-2rL} - 2) \\ &= 1 \end{aligned}$$

(b). From Fig. (b),

$$\begin{aligned} V_1 &= \frac{Z_1}{2} I_1 + \frac{1}{Y_2} (I_1 - I_2) \\ &= \left(\frac{Z_1}{2} + \frac{1}{Y_2} \right) I_1 - \frac{1}{Y_2} I_2 \end{aligned} \quad (1)$$

$$\begin{aligned} I_1 &= I_2 + (V_2 + \frac{Z_1}{2} I_2) Y_2 \\ &= Y_2 V_2 + \left(1 + \frac{Z_1 Y_2}{2} \right) I_2 \end{aligned} \quad (2)$$

substituting (2) in (1),

$$V_1 = \left(1 + \frac{Z_1 Y_2}{2} \right) V_2 + Z_1 \left(1 + \frac{Z_1 Y_2}{4} \right) I_2$$

$$\text{so } A = 1 + \frac{Z_1 Y_2}{2} \quad B = Z_1 \left(1 + \frac{Z_1 Y_2}{4} \right)$$

$$Z_1 Y_2 = 2(A-1)$$

(7)

$$B = Z_1 \left(1 + \frac{2(A-1)}{4} \right)$$

$$\begin{aligned} \text{so } Z_1 &= \frac{B}{1 + \frac{2(A-1)}{4}} = \frac{Z_0 \cdot (e^{rL} - e^{-rL})^{\frac{1}{2}}}{\frac{1}{4} (e^{\frac{1}{2}rL} + e^{-\frac{1}{2}rL})^2} \\ &= \frac{2Z_0 (e^{\frac{1}{2}rL} - e^{-\frac{1}{2}rL})}{(e^{\frac{1}{2}rL} + e^{-\frac{1}{2}rL})} = 2Z_0 \cdot \tanh\left(\frac{1}{2}rL\right) \end{aligned}$$

$$Y_2 = \frac{2(A-1)}{Z_1} = \frac{1}{Z_0} \cdot \sinh(rL)$$

9-17. (a) From Eq. (9-117),

$$Z_i = Z_0 \cdot \frac{1 - e^{-2\alpha L}}{1 + e^{-2\alpha L}}$$

$$l = \frac{\lambda}{4}, \quad \beta l = \frac{2\pi}{\lambda} \cdot l = \frac{\pi}{2}$$

$$rL = j\beta l = j\frac{\pi}{2}$$

$$Z_i = Z_0 \cdot \frac{1 - e^{-2\alpha \frac{\lambda}{4}} \cdot e^{-j\pi}}{1 + e^{-2\alpha \frac{\lambda}{4}} \cdot e^{-j\pi}}$$

$$= Z_0 \cdot \frac{1 + (1 - \frac{1}{2}\alpha\lambda)}{1 - (1 - \frac{1}{2}\alpha\lambda)} = \frac{4Z_0}{2\lambda}$$

(b) From Eq. (9-116),

$$Z_i = Z_0 \cdot \frac{1 + e^{-2\alpha L}}{1 - e^{-2\alpha L}}$$

$$= Z_0 \cdot \frac{1 + e^{-2\alpha \frac{\lambda}{4}} \cdot e^{-j\pi}}{1 - e^{-2\alpha \frac{\lambda}{4}} \cdot e^{-j\pi}} = Z_0 \cdot \frac{1 - (1 - \frac{1}{2}\alpha\lambda)}{1 + (1 - \frac{1}{2}\alpha\lambda)}$$

$$= \frac{Z_0 \cdot \alpha\lambda}{4}$$

9-20 (a). Consider the wavelength $\lambda = c/f = \frac{3 \times 10^8}{10^5} = 3000$ (8)
 the length of the coaxial cable is $0.6 \text{ (m)} \ll \lambda$
 so we can use Eqs. (9-111) and (9-113).

$$C = \frac{54 \times 10^{-12}}{0.6} = 9 \times 10^{-11} \text{ (F/m)}.$$

$$L = \frac{0.3 \times 10^{-6}}{0.6} = 5 \times 10^{-7} \text{ (H/m)}.$$

$$R_0 = \sqrt{\frac{L}{C}} = 74.5 (\Omega) = Z_0 \text{ (lossless line)}.$$

$$\mu\epsilon = LC, \quad \epsilon_r = \frac{LC}{\mu_0\epsilon_0} = 4.05.$$

(b). Calculate the X_{io} and X_{is} at 10 (MHz).

$$\beta = \frac{\omega}{u_p} = \omega \sqrt{\mu\epsilon} = \omega \sqrt{LC} = 0.42,$$

$$\beta \cdot l = 0.42 \times 0.6 = 0.252 \text{ (rad)}.$$

$$X_{io} = -R_0 \cdot \cot(\beta l) = -289.3 (\Omega).$$

$$X_{is} = R_0 \tan(\beta l) = 19.2 (\Omega)$$