

Homework 8

1. From Maxwell's Equations,

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \quad (1)$$

$$\nabla \times \vec{H} = j\omega\epsilon\vec{E} \quad (2)$$

$$\nabla = \nabla_T + \nabla_z = \nabla_T + \hat{a}_z \frac{\partial}{\partial z}$$

$$\vec{E} = \vec{E}_T + \hat{a}_z E_z, \quad \vec{H} = \vec{H}_T + \hat{a}_z H_z$$

Plug in (1), (2)

$$(\nabla_T + \hat{a}_z \frac{\partial}{\partial z}) \times (\vec{E}_T + \hat{a}_z E_z) = -j\omega\mu (\vec{H}_T + \hat{a}_z H_z)$$

$$\nabla_T \times \vec{E}_T + r \vec{E}_T \times \hat{a}_z + \nabla_T \times (\hat{a}_z E_z) = -j\omega\mu (\vec{H}_T + \hat{a}_z H_z)$$

Since the wave has an arbitrary cross section,

$$\nabla_T \times \vec{E}_T = 0$$

So that

$$\nabla_T E_z \times \hat{a}_z + r \vec{E}_T \times \hat{a}_z = -j\omega\mu (\vec{H}_T + \hat{a}_z H_z)$$

Consider xy-plane,

$$\nabla_T E_z \times \hat{a}_z + r \vec{E}_T \times \hat{a}_z = -j\omega\mu \vec{H}_T \quad (3)$$

Also, from (2)

$$\nabla_T H_z \times \hat{a}_z + r \vec{H}_T \times \hat{a}_z = j\omega\epsilon \vec{E}_T \quad (4)$$

From (3), (4)

$$-j\omega\mu (\nabla_T H_z \times \hat{a}_z) + r (-j\omega\mu \vec{H}_T) \times \hat{a}_z = \omega^2\mu\epsilon \vec{E}_T$$

$$-j\omega\mu (\nabla_T H_z \times \hat{a}_z) + r (\nabla_T E_z \times \hat{a}_z + r \vec{E}_T \times \hat{a}_z) \times \hat{a}_z = \omega^2\mu\epsilon \vec{E}_T$$

$$j\omega\mu \hat{a}_z \times \nabla_T H_z - r \nabla_T E_z - r^2 \vec{E}_T = \omega^2\mu\epsilon \vec{E}_T$$

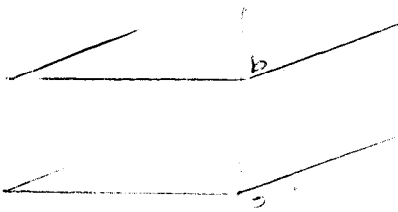
$$(\omega^2\mu\epsilon + r^2) \vec{E}_T = h^2 \vec{E}_T = j\omega\mu \hat{a}_z \times \nabla_T H_z - r \nabla_T E_z$$

$$\vec{E}_T = -\frac{1}{h^2} (r \nabla_T E_z - j\omega\mu \hat{a}_z \times \nabla_T H_z)$$

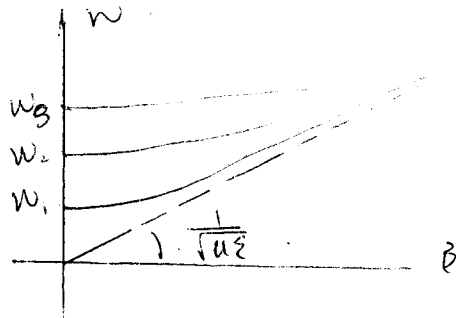
$$\text{Similarly, } \vec{H}_T = -\frac{1}{h^2} (r \nabla_T E_z + j\omega\epsilon \hat{a}_z \times \nabla_T E_z)$$

3.

$$\omega^2 \mu \epsilon - \beta^2 = \left(\frac{n\pi}{b} \right)^2$$



$$\omega_1 = \frac{\pi}{b\sqrt{\mu\epsilon}}, \quad \omega_2 = \frac{2\pi}{b\sqrt{\mu\epsilon}}, \quad \omega_3 = \frac{3\pi}{b\sqrt{\mu\epsilon}}$$



(a). b affects the cutoff frequency
 μ, ϵ affect the cutoff frequency
 and the slope of the ω - β curves

As $\beta \rightarrow +\infty$, $\omega \rightarrow +\infty$, all curves
 turn out to be $\omega^2 \mu \epsilon = \beta^2$

$$\omega = \frac{1}{\sqrt{\mu\epsilon}} \beta$$

(b). From Maxwell's equations, we can also get

$$\omega^2 \mu \epsilon - \beta^2 = \left(\frac{n\pi}{b} \right)^2$$

for TE modes.

So they share the same curves.

5. For TE_n mode.

$$H_z(y) = B_n \cos(n\pi y/b)$$

$$H_y(y) = \frac{r}{h} B_n \sin(n\pi y/b)$$

$$E_x(y) = \frac{j\omega\mu}{h} B_n \sin(n\pi y/b)$$

$$y=0, \quad \vec{J}_{s0} = \hat{a}_y \times \vec{H}(0) = \hat{a}_y \times B_n$$

$$y=b, \quad \vec{J}_{sb} = (-\hat{a}_y) \times \vec{H}(b) = \hat{a}_y (-1)^{n+1} B_n$$

$$\begin{cases} \vec{J}_{s0} & n \text{ is odd} \\ -\vec{J}_{s0} & n \text{ is even} \end{cases}$$

7

$$\vec{P}_{av} = \frac{1}{2} \operatorname{Re} (\vec{E} \times \vec{H}^*) = \frac{1}{2} \operatorname{Re} [\hat{a}_z E_x \cdot H_y^* - \hat{a}_y E_x \cdot H_z^*]$$

along z -direction

$$\vec{P}_{av} \cdot \hat{a}_z = \frac{1}{2} \operatorname{Re} [E_x \cdot H_y^*] = \frac{\omega \mu \beta}{2h^2} B_n^2 \cdot \sin^2\left(\frac{n\pi y}{b}\right)$$

$$(P_z)_{av} = \int_0^b \vec{P}_{av} \cdot \hat{a}_z dy = \frac{\omega \mu \beta}{4h^2} B_n^2 \quad \text{per unit width}$$

$$(W_e)_{av} = \frac{\epsilon}{4} \operatorname{Re} [\vec{E} \cdot \vec{E}^*] = \frac{\epsilon \omega^2 u^2}{4h^2} B_n^2 \cdot \sin^2\left(\frac{n\pi y}{b}\right)$$

$$(W_e)_{av} = \int_0^b (w_e)_{av} dy = \frac{\epsilon \omega^2 u^2 b}{8h^2} B_n^2 = (W_m)_{av}$$

$$u_{en} = \frac{(P_z)_{av}}{(W_e)_{av} + (W_m)_{av}} = \frac{\omega \mu \beta}{\epsilon \omega^2 u^2} = \frac{\omega \beta}{k^2} = u \cdot \sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

8. (a). TEM mode:

$$\beta = \omega \sqrt{\mu \epsilon} = 314.2 \quad (\text{rad/m})$$

$$\alpha_d = \frac{\sigma}{2} \sqrt{\frac{\mu}{\epsilon}} = 1.257 \times 10^{-8} \quad (\text{Np/m})$$

$$\alpha_c = \frac{1}{b} \sqrt{\frac{\mu f \epsilon}{\sigma_c}} = 2.076 \times 10^{-3} \quad (\text{Np/m})$$

$$u_p = u_g = u = \frac{1}{\sqrt{\mu \epsilon}} = 2 \times 10^8 \quad (\text{m/s})$$

$$\lambda_g = \lambda = \frac{u}{f} = 2 \times 10^{-2} \quad (\text{m})$$

(b). TM₁ mode. $f_c = \frac{1}{2b \sqrt{\mu \epsilon}} = 2 \times 10^9 \text{ Hz}$

$$\beta = \omega \sqrt{\mu \epsilon} \cdot \sqrt{1 - \left(\frac{f_c}{f}\right)^2} = 307.8 \quad (\text{rad/m})$$

$$\alpha_d = \frac{\sigma}{2 \sqrt{1 - \left(\frac{f_c}{f}\right)^2}} = 1.283 \times 10^{-8} \quad (\text{Np/m})$$

$$\alpha_c = \frac{2 R_s}{b \sqrt{1 - \left(\frac{f_c}{f}\right)^2}} = 4.238 \times 10^{-3} \quad (\text{Np/m})$$

$$u_p = u \cdot \frac{1}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}} = 2.041 \times 10^8 \text{ (m/s)}$$

$$u_g = u \cdot \sqrt{1 - \left(\frac{f_c}{f}\right)^2} = 1.960 \times 10^8 \text{ (m/s)}$$

$$\lambda_g = \lambda \cdot \frac{1}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}} = 2.041 \times 10^{-2} \text{ (m)}$$

TM₂ mode,

$$(c) \quad f_c = \frac{1}{b\sqrt{\mu\epsilon}} = 4 \times 10^9 \text{ (Hz)}$$

$$\beta = n\sqrt{\mu\epsilon} \cdot \sqrt{1 - \left(\frac{f_c}{f}\right)^2} = 287.9 \text{ (rad/m)}$$

$$\alpha_d = \frac{\sigma\eta}{2\sqrt{1 - \left(\frac{f_c}{f}\right)^2}} = 1.371 \times 10^{-8} \text{ (Np/m)}$$

$$\alpha_c = \frac{2}{b\sqrt{1 - \left(\frac{f_c}{f}\right)^2}} \cdot \frac{\pi f \epsilon}{\sigma_c} = 4.530 \times 10^{-3} \text{ (Np/m)}$$

$$u_p = u \cdot \frac{1}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}} = 2.182 \times 10^8 \text{ (m/s)}$$

$$u_g = u \cdot \sqrt{1 - \left(\frac{f_c}{f}\right)^2} = 1.833 \times 10^8 \text{ (m/s)}$$

$$\lambda_g = \frac{\lambda}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}} = 2.182 \times 10^{-2} \text{ (m)}$$

$$\#9. (a) \text{ TE}_1 \quad f_c = (f_c)_{\text{TM}_1}$$

$$\alpha_c = \frac{2}{b\sqrt{1 - \left(\frac{f_c}{f}\right)^2}} \cdot \frac{\pi f \epsilon}{\sigma_c} \cdot \left(\frac{f_c}{f}\right)^2 = 1.695 \times 10^{-4} \text{ (Np/m)}$$

All others are the same as F_{TM} parameters of TM₁.

(b). TE₂ mode. $(f_c) = (f_c)_{TM_2} = 4 \times 10^9 \text{ Hz}$

$$\alpha_c = \frac{2}{b \sqrt{1 - \left(\frac{f_c}{f}\right)^2}} \cdot \sqrt{\frac{\bar{n} f \epsilon}{\gamma_c}} \cdot \left(\frac{f_c}{f}\right)^2 = 7.249 \times 10^{-4} \text{ (Np/m)}$$

All others are the same as parameters of TM₂ mode.