UCSB

ECE 146A: Solutions to Problem Set 1

Problem 1 A system with input x(t) has output given by

$$y(t) = \int_{-\infty}^{t} e^{u-t}\, x(u)du$$

(a)Show that the system is LTI and find its impulse response.

(b)Find the transfer function H(f) and plot |H(f)|.

Solution 1 (a) We do this in two ways. The first is to directly show that $y = x * h$ for some h, which shows that the system is LTI with impulse response h. We can rewrite

$$y(t) = \int_{-\infty}^{\infty} e^{-(t-u)} I_{\{u \le t\}} x(u)du = \int_{-\infty}^{\infty} e^{-(t-u)} I_{\{t-u \ge 0\}} x(u)du$$

We see that this is in the convolution form

$$y(t) = \int_{-\infty}^{\infty} h(t-u)x(u)du$$

where $h(t) = e^{-t} I_{t \ge 0}$.

The second approach is to show that the system is LTI and then feed in an impulse to find the impulse response. Linearity of y in x is clear, hence let us check for time invariance. Let $x_1(t) = x(t - t_0)$ be a delayed version of the input. The corresponding output is given by

$$y_1(t) = \int_{-\infty}^{t} e^{u-t} x_1(u)du = \int_{-\infty}^{t} e^{u-t} x(u - t_0)du$$

Making the change of variables $v = u - t_0$, we obtain that

$$y_1(t) = \int_{-\infty}^{t-t_0} e^{v+t_0-t} x(v)dv = \int_{-\infty}^{t-t_0} e^{v-(t-t_0)} x(v)dv$$

Comparing with the original expression for $y(t)$, it is clear that we have simply replaced t by $t - t_0$. That is, $y_1(t) = y(t - t_0)$ and the system is time invariant. We now find the impulse response by setting the input to an impulse:

$$h(t) = \int_{-\infty}^{t} e^{u-t} \delta(u)du$$

The impulse at time zero falls into the integration interval only if $t \ge 0$, in which case we select the value of the integrant at $u = 0$. We therefore obtain $h(t) = e^{-t} I_{\{t \ge 0\}}$ as before.

(b) It is easy to see that the Fourier transform of the impulse response is $H(f) = \frac{1}{j2\pi f+1}$, with magnitude $|H(f)| = \frac{1}{\sqrt{4\pi^2 f^2+1}}$. The plot is omitted, but it is clear that this is a sloppy lowpass filter.

(c) We compute the energy in the frequency domain using Parseval's identity. The input $x(t)=2\text{sinc}(2t) \leftrightarrow X(f) = I_{[-1,1]}(f)$, and the output $Y(f) = H(f)X(f) = \frac{1}{j2\pi f+1} I_{[-1,1]}(f)$. The energy is given by

$$E_y = \int_{-\infty}^{\infty} |Y(f)|^2 df = \int_{-1}^{1} \frac{1}{4\pi^2 f^2 + 1} df$$

Making the standard substitution $2\pi f = \tan\theta$, so that $2\pi df = \sec^2\theta \, d\theta$ and $4\pi^2 f^2 + 1 = \tan^2\theta + 1 = \sec^2\theta$, we obtain

$$E_y = \int_{arctan(-2\pi)}^{arctan(2\pi)} \frac{1}{2\pi} d\theta = 0.45$$

Problem 2 Find and sketch y = x1 ∗ x2 for the following:
(a) $e^{-t} I_{[0,\infty)}(t), x2(t) = x1(-t)$.
(b) $x_1(t) = I_{[0,2]}(t) - 3I_{[1,4]}(t), x2(t) = I_{[0,1]}(t)$.
Hint: In (b), you can use the LTI property and the known result on the convolution of two boxes.

Solution 2 (a) The signal $x_2(t) = x_1(-t)$ is the matched filter for x_1, and the convolution output at time t is simply the correlation of x_1 with itself, delayed by time t:

$$y(t) = (x_1 * x_2)(t) = \int x_1(u) x_2(t-u) du = \int x_1(u) x_1(u-t) du$$

It is easy to check that $y(t) = y(-t)$, so we only need to evaluate the convolution for $t \geq 0$, as follows (see Figure 1):

$$y(t) = \int_t^{\infty} e^{-u} e^{-(u-t)} du = e^t \int_t^{\infty} e^{-2u} du = e^t \frac{-e^{-2u}}{2} |_t^{\infty} = \frac{1}{2} e^{-t}$$

Since $y(t)$ is symmetric, we can replace t by $|t$ to obtain

$$y(t) = \frac{1}{2} e^{-|t|}$$

sketched in Figure 1.

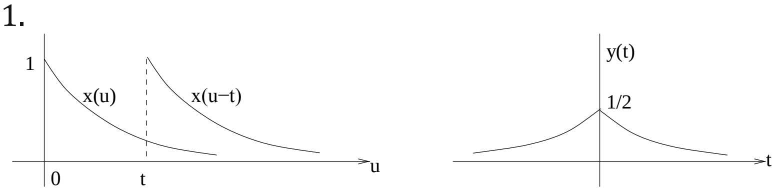

Figure 1: Convolution of a signal and its matched filter in 2(a).

(b) A graphical solution is provided in Figure 2.

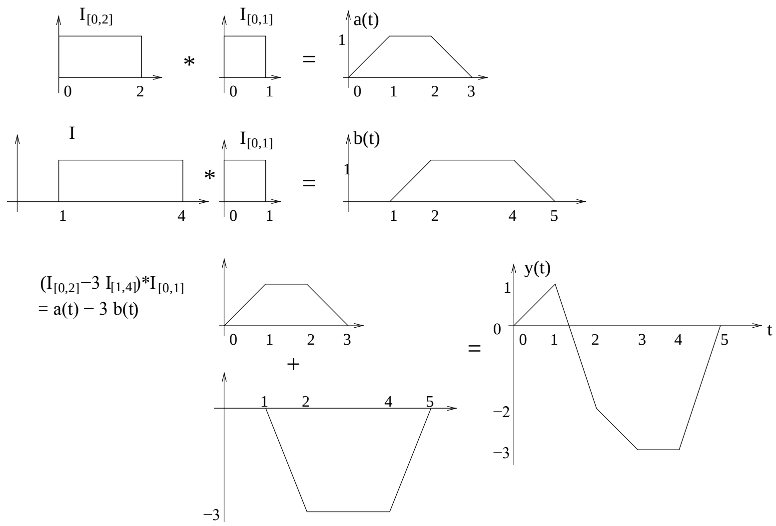

Figure 2: Graphical solution of convolution in Problem 2(b).

Problem 3 (a) For u(t) = sinc(t) sinc(2t), where t is in microseconds, find and plot the magnitude spectrum |U(f)|, carefully labeling the units of frequency on the x axis.

(b) Now, consider s(t) = u(t) cos 200πt. Plot the magnitude spectrum |S(f)|, again labeling the units of frequency and carefully showing the frequency intervals over which the spectrum is nonzero.

Solution 3 (a) We have $u(t) = u_1(t)u_2(t)$, where $u_1(t) = \mathrm{sinc}(t) \leftrightarrow U_1(f) = I_{[-\frac{1}{2},\frac{1}{2}]}(f)$ and $u_2(t) = \mathrm{sinc}(2t) \leftrightarrow U_2(f) = \frac{1}{2} I_{[-1,1]}(f)$, so that $U(f) = (U_1 * U_2)(f)$ is a trapezoidal pulse shown in Figure 3. Since the unit of time is microseconds, the unit of frequency is MHz. (b) The signal $s(t) = u(t)\cos 200\pi t \leftrightarrow (U(f - 100) + U(f + 100))/2$ is sketched in Figure 3.

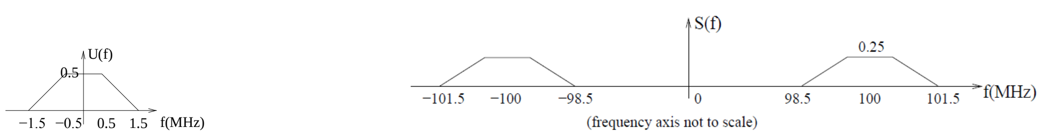

Figure 3: Frequency domain plots for Problem 3.

Problem 4 For a signal s(t), the *matched filter* is defined as a filter with impulse response h(t) = smf(t) = s∗(−t) (we allow signals to be complex valued, since we want to handle complex baseband signals as well as physical real-valued signals).

(a)Sketch the matched filter impulse response for s(t) = $I_{[1,3]}(t) - 2I_{[2,5]}(t)$.

(b)Find and sketch the convolution y(t) = (s ∗ smf)(t). This is the output when the signal is passed through its matched filter. Where does the peak of the output occur?

(c)(True or False) Y (f) ≥ 0 for all f.

Solution 4 (a) The signal $s(t)$ and its matched filter $h(t) = s_{mf}(t) = s(-t)$ are sketched in Figure 4.

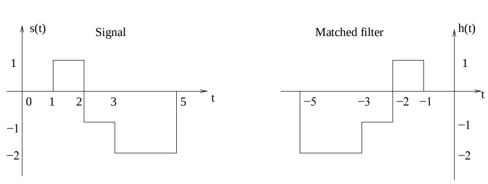

Figure 4: Sketch of signal and its matched filter for Problem 4(a).

(b) Let us break down the convolution into simpler parts: $s(t) = a(t)+b(t)$, where $a(t) = I_{[1,3]}(t)$ and $b(t) = -2I_{[2,5]}(t)$, so that $h(t) = a(-t) + b(-t)$ and

$$y(t) = (s * h)(t) = a(t) * a(-t) + b(t) * b(-t) + a(t) * b(-t) + a(-t) * b(t)$$

All of the terms above are convolutions between boxes. The first two terms give triangles centered at the origin, while the third and fourth terms are trapezoids which are reflections of each other. Figure 5 shows the 4 components of $y(t)$ and then adds them up to obtain the final signal. The figure also shows the building blocks (convolution between boxes starting at zero) used to obtain these 4 components (by translation and scaling of these building blocks). In adding up the 4 components of $y(t)$, we can use the simple observation that it is piecewise linear (since its components are piecewise linear), and simply compute $y(t)$ for $t \geq 0$ at the end points of the segments ($t = 0,1,2,3$) and then join them by lines. The signal for $t \leq 0$ can now be obtained by reflection, noting that the convolution of a signal with its matched filter is symmetric (for complex-valued signals, the matched filter is defined as $s_{mf}(t) = s^*(-t)$, and the convolution is conjugate symmetric).

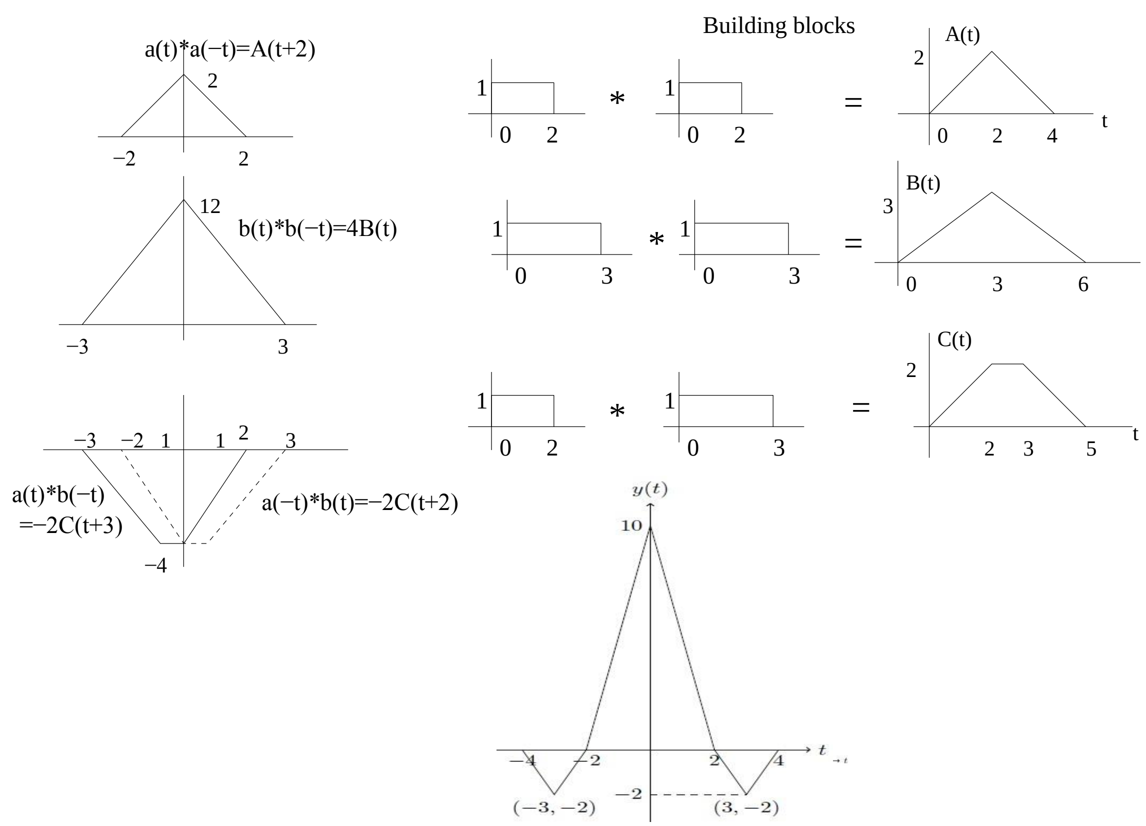

Figure 5: Computing the convolution of the signal with its matched filter in Problem 4.

(c) Since $s_{mf}(t) = s^*(-t)$, we have $S_{mf}(f) = S^*(f)$. Thus,

$$y(t) = (s * s_{mf})(t) \leftrightarrow Y(f) = S(f)S_{mf}(f) = S(f)S^*(f) = |S(f)|^2$$

Clearly, $Y(f) \geq 0$ for all f.