UCSB

ECE 146A: Solutions to Problem Set 2

Problem 2.8 Consider the tent signal $s(t) = (1 - |t|)I_{[-1,1]}(t)$.

(a) Find and sketch the Fourier transform $S(f)$.

(b) Compute the 99% energy containment bandwidth in KHz, assuming that the unit of time is milliseconds.

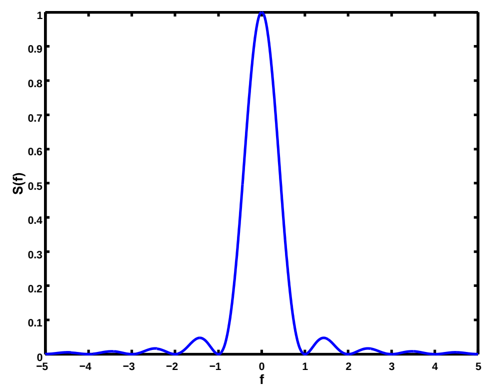


Figure 1: Spectrum of the tent signal in Problem 2.8(a) is $\text{sinc}^2(f)$.

Solution 2.8 (a) The tent signal can be written as a convolution of two boxes $s(t) = I_{[-1/2,1/2]}(t) * I_{[-1/2,1/2]}(t)$. Since $I_{[-1/2,1/2]}(t) \leftrightarrow \text{sinc}(f)$, we have $S(f) = \text{sinc}^2(f)$, plotted in Figure 1.

(b) The 99% energy containment bandwidth W satisfies

$$\int_{-W}^{W} |S(f)|^2 df = 2\int_0^W \text{sinc}^4 f\, df = 0.99E_s$$

where the energy is given by

$$E_s = \int_{-\infty}^{\infty} |S(f)|^2 df = \int_{-\infty}^{\infty} |s(t)|^2 dt = 2\int_0^1 (1-t)^2 dt = 2/3$$

Using the symmetry of $|S(f)|^2$ and plugging in its expression, we see that we need to numerically solve the following equation for W:

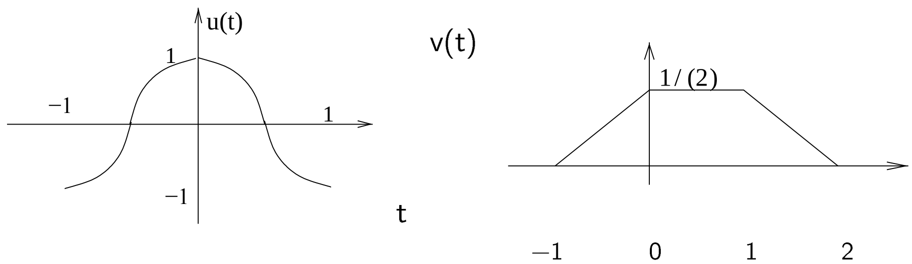
$$\int_0^W \text{sinc}^4 f\, df = 0.33$$

We get $W = 0.65$. Since the unit of time is ms, the unit of frequency is KHz, so the 99% bandwidth is 0.65 KHz.

Problem 2.15 Consider the signal $s(t) = I_{[-1,1]}(t)\cos 400\pi t$.

(a) Find and sketch the baseband signal $u(t)$ that results when $s(t)$ is downconverted as shown in the upper branch of Figure 2.39.

(b) The signal $s(t)$ is passed through the bandpass filter with impulse response $h(t) = I_{[0,1]}(t)\sin(400\pi t + \frac{\pi}{4})$. Find and sketch the baseband signal $v(t)$ that results when the filter output $y(t) = (s*h)(t)$ is downconverted as shown in the lower branch of Figure 2.39.



(a)Downconverter output in 2.15(a) (b) Downconverter output in 2.15(b)

Figure 5: Sketches for Problem 2.15.

Solution 2.15 (a) This problem does not have subscripts denoting passband signals, but clearly, $s(t)$ is passband, and $u(t)$ is the I component with respect to the $401\pi t$ reference. The complex envelope with respect to the $400\pi t$ carrier reference is clearly $\tilde{s}_1(t) = I_{[-1,1]}(t)$. When we advance the carrier reference by πt to $401\pi t$, we must correspondingly retard the complex envelope by πt to get $\tilde{s}_2(t) = I_{[-1,1]}(t)e^{-j\pi t}$. We now have

$$u(t) = \tilde{s}_{2c}(t) = \mathrm{Re}\ \ I_{[-1,1]}(t)e^{-j\pi t}) = I_{[-1,1]}(t)\cos\pi t$$

which is sketched in Figure (a).

(b) When the passband signal $s(t)$ is passed through the passband filter $h(t)$, we get a passband signal $y(t)$. Let us therefore find the complex envelope of y with respect to the reference $400\pi t$, by finding the complex envelopes of s and h with respect to the reference and then convolving them. The complex envelope of $s(t)$ with respect to the $400\pi t$ carrier reference is$s_1(t) = I_{[-1,1]}(t$). The complex envelope of $h(t)$ with respect to the $400\pi t$ carrier reference is $I_{[0,1]}e^{-j\pi/4}$ (t). The complex envelope of the filter output is therefore given by

$$\tilde{y} = \frac{1}{2}\tilde{s_1} * \tilde{h} = \frac{1}{2}I_{[-1,1]}(t) * I_{[0,1]}e^{-j\pi/4} = \frac{1}{2}p(t)e^{-j\pi/4}$$

where $p(t) = I_{[-1,1]} * I_{[0,1]}$ is a trapezoidal pulse. The equivalent passband signal is

$$y_p(t) = Re(\tilde{y}e^{j2\pi f_c t}) = \frac{1}{2}p(t)\cos(400\pi t - \pi/4)$$

The output from the mixer $v_1(t)$ is given by

$$v_1(t) = y_p(t).2\sin(400\pi t + \frac{\pi}{4}) = p(t)\cos(400\pi t - \frac{\pi}{4})\sin(400\pi t + \frac{\pi}{4})$$

$$v_1(t) = \frac{p(t)}{2}(\sin(400\pi t) + \sin(\pi/2))$$

$$v(t) = LPF(v_1(t)) = \frac{p(t)}{2}$$

The LPF filters off the $\sin(400\pi t)$ term giving the downconverter output v(t) which is sketched in Figure (b).

Problem 2.16 Consider the signals $u_1(t) = I_{[0,1]}(t)\cos 100\pi t$ and $u_2(t) = I_{[0,1]}(t)\sin 100\pi t$.
(a) Find the numerical value of the inner product $\int_{-\infty}^{\infty} u_1(t)u_2(t)dt$.
(b) Find an explicit time domain expression for the convolution $y(t) = (u_1 * u_2)(t)$.
(c) Sketch the magnitude spectrum $|Y(f)|$ for the convolution in (b).

Solution 2.16 (a) The inner product is zero by I-Q orthogonality. Alternatively, let $\tilde{u}_i$ denote the complex envelope for the passband signals u_i, $i = 1,2$ with respect to the $100\pi t$ reference. Then $\tilde{u}_1(t) = I_{[0,1]}(t)$ and $\tilde{u}_2(t) = -jI_{[0,1]}(t)$. The passband inner product is related to the complex baseband inner product as follows: $\langle u_1, u_2\rangle = \frac{1}{2}\mathrm{Re}\left(\langle \tilde{u}_1, \tilde{u}_2\rangle\right) = 0$, since

$$\langle \tilde{u}_1, \tilde{u}_2\rangle = \int_{-\infty}^{\infty} \tilde{u}_1(t)\tilde{u}_2^*(t)dt = j\int_0^1 dt = j$$

(b) The complex envelope $y = \frac{1}{2}\tilde{u}_1 * \tilde{u}_2 = -j\frac{1}{2}s(t) = y_c + jy_s$, where $s(t) = I_{[0,1]} * I_{[0,1]}$ is a triangular pulse over [0,2]. Thus, the passband signal $y(t) = \frac{1}{2}s(t)\sin 100\pi t$ ($y_c = 0$, $y_s = -\frac{1}{2}s(t)$).
(c) Since $Y(f) = \frac{1}{2}\tilde{Y}(f - f_c) + \frac{1}{2}\tilde{Y}^*(-f - f_c)$, the magnitude spectrum is given by

$$|Y(f)| = \frac{1}{2}|\tilde{Y}(f - f_c)| + \frac{1}{2}|\tilde{Y}(-f - f_c)|$$

Now plug in $\tilde{Y}(f) = \frac{1}{2}|S(f)| = \frac{1}{2}\mathrm{sinc}^2(f)$ (sketch omitted).

Problem 2.17 Consider a real-valued passband signal $v_p(t)$ whose Fourier transform for positive frequencies is given by

$$\mathrm{Re}(V_p(f)) = \begin{cases} 2, & 30 \le f \le 32 \\ 0, & 0 \le f < 30 \\ 0, & 32 < f < \infty \end{cases}$$

$$\mathrm{Im}(V_p(f)) = \begin{cases} 1 - |f - 32|, & 31 \le f \le 33 \\ 0, & 0 \le f < 31 \\ 0, & 33 < f < \infty \end{cases}$$

(a) Sketch the real and imaginary parts of $V_p(f)$ for both positive and negative frequencies. (b) Specify, in both the time domain and the frequency domain, the waveform that you get when you pass $v_p(t)\cos(60\pi t)$ through a low pass filter.

Solution 2.17(a) Since $v_p(t)$ is real-valued, $V_p(f)$ is conjugate symmetric, hence $\mathrm{Re}(V_p(f))$ is symmetric and $\mathrm{Im}(V_p(f))$ is antisymmetric. We use this, together with the given information, to plot the spectrum in Figure 6.

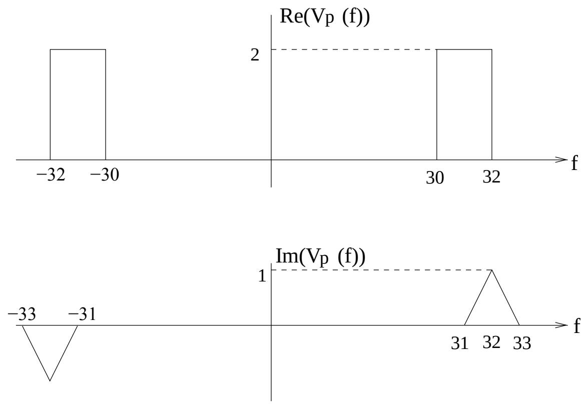


Figure 6: Spectrum of passband waveform in Problem 2.17 (frequency axis not to scale).

(b) Denote the output by $a(t) \leftrightarrow A(f)$. Let us do this in two ways.

Method 1: The first does not invoke the complex envelope, and uses the result that $v_p \cos(60\pi t)$ $\leftrightarrow$ $\frac{1}{2}\left(V_p(f-30)+V_p(f+30)\right)$, with the LPF rejecting the spectrum component around $2f_c = 60$. The resulting spectrum $A(f)$ is shown in Figure 7. We have $\mathrm{Re}(A(f)) \leftrightarrow 4\mathrm{sinc}(4t)$, and $j\mathrm{Im}(A(f)) \leftrightarrow j\frac{1}{2}\mathrm{sinc}^2 t\,(e^{j4\pi t}-e^{-j4\pi t}) = -\mathrm{sinc}^2 t \sin 4\pi t$. Adding up these two contributions, we have $\mathrm{Re}(A(f)) + j\mathrm{Im}(A(f)) \leftrightarrow a(t) = 4\mathrm{sinc}(4t) - \mathrm{sinc}^2 t \sin 4\pi t$.

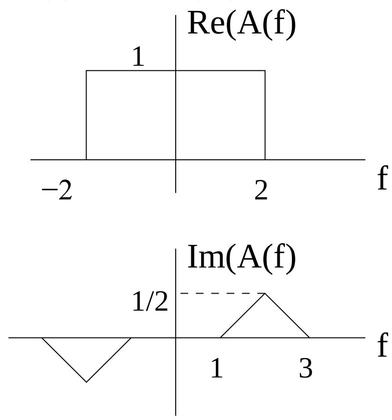


Figure 7: Spectrum of downconverter output in Problem 2.17(b).

Method 2: We recognize that $a(t) = \frac{1}{2}v_c(t) = \frac{1}{2}\mathrm{Re}(v(t))$ is the output of a downconversion operation, where $v(t)$ is the complex envelope with respect to $f_c = 30$, and $v_c(t)$ the I component. Weknow that we can get $V(f)$ from $V_p(f)$ by throwing away the spectrum for negative frequencies, and moving the spectrum for positive frequencies to the left by $f_c = 30$ and doubling it. The result is shown in Figure 8. We see that $\mathrm{Re}(V(f)) \leftrightarrow 8\mathrm{sinc}2te^{j2\pi t}$ and $j\mathrm{Im}(V(f)) \leftrightarrow j2\mathrm{sinc}^2 te^{j4\pi t}$ so that $V(f) = \mathrm{Re}(V(f)) + j\mathrm{Im}(V(f)) \leftrightarrow v(t) = 8\mathrm{sinc}2te^{j2\pi t} + j2\mathrm{sinc}^2 te^{j4\pi t}$. We can now take the real part to find the I component as follows:

$$v_c(t) = \mathrm{Re}(v(t)) = 8\mathrm{sinc}2t\cos 2\pi t - 2\mathrm{sinc}^2 t \sin 4\pi t$$

Check that $a(t) = \frac{1}{2}v_c(t)$ matches the answer from Method 1. We can also find the I component in the frequency domain directly from $V(f)$: $V_c(f) = \frac{V(f)+V^*(-f)}{2}$, which means that $\mathrm{Re}(V_c(f)) =$

$\frac{\mathrm{Re}(V(f))+\mathrm{Re}(V(-f))}{2}$ and Im($V_c(f)) = \frac{\mathrm{Im}V(f))+\mathrm{Im}(V(-f))}{2}$. Applying this to the spectrum in Figure 8 , check that the answer you get for $A(f) = \frac{1}{2}V_c(f)$ matches Figure 7.

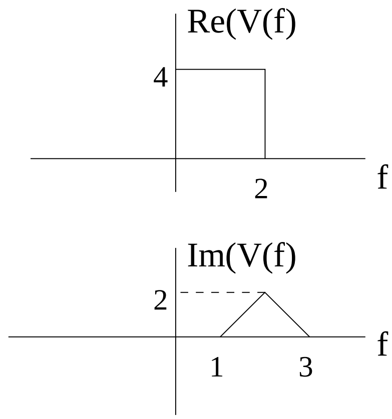


Figure 8: Complex envelope with respect to $f_c = 30$ for Method 2 in Problem 2.17(b).