UCSB

ECE 146A: Solutions to Problem Set 3

Problem 2.18 The passband signal $u(t) = I_{[-1,1]}(t)\cos 100\pi t$ is passed through the passband filter $h(t) = I_{[0,3]}(t)\sin 100\pi t$. Find an explicit time domain expression for the filter output.

Solution 2.18 When the passband signal $u(t)$ is passed through the passband filter $h(t)$, we get a passband signal $y(t)$. Let us therefore find the complex envelope of y with respect to the reference $100\pi t$, by finding the complex envelopes of u and h with respect to the reference and then convolving them. The complex envelope of u(t) with respect to the $100\pi t$ carrier is $I_{[-1,1]}(t)$. The complex envelope of $h(t)$ with respect to the $100\pi t$ carrier reference is $I_{[0,3]}e^{-j\pi/2}\,(t)$. The complex envelope of the filter output is therefore given by

$$\tilde{y} = \frac{1}{2}\tilde{u} * \tilde{h} = \frac{1}{2}I_{[-1,1]}(t) * -jI_{[0,3]}(t) = \frac{-j}{2}p(t)$$

where $p(t) = I_{[-1,1]} * I_{[0,3]}$ is a trapezoidal pulse. The equivalent passband signal is [Sketch omitted]

$$y(t) = Re(\tilde{y}e^{j2\pi f_c t}) = Re(\frac{1}{2}p(t)e^{j100\pi t - \pi/2}) = \frac{1}{2}p(t)sin(100\pi t)$$

Problem 2.20 Consider the following two passband signals:

$$u_p(t) = \text{sinc}(2t)\cos 100\pi t$$

and

$$v_p(t) = \text{sinc}(t)\sin(101\pi t + \pi/4)$$

(a) Find the complex envelopes $u(t)$ and $v(t)$ for u_p and v_p, respectively, with respect to the frequency reference $f_c = 50$.

(b) What is the bandwidth of $u_p(t)$? What is the bandwidth of $v_p(t)$?

(c) Find the inner product hu_p,v_pi, using the result in (a).

(d) Find the convolution $y_p(t) = (u_p * v_p)(t)$, using the result in (a).

Solution 2.20 (a) For a frequency reference of $f_c = 50$, or a phase reference of $100\pi t$, the complex envelope of $u_p(t)$ can be read off as $u(t) = \text{sinc}(2t)$. The complex envelope of $v_p(t)$ with respect to $101\pi t + \frac{\pi}{4}$ can be read off as $\tilde{v}(t) = -j\text{sinc}t$. Switching the reference to $100\pi t$ means retarding it by $\pi t + \frac{\pi}{4}$, hence we must advance the complex envelope by that amount to get

$$v(t) = -j\text{sinc}te^{j(\pi t + \frac{\pi}{4})} = \text{sinc}te^{j(\pi t - \frac{\pi}{4})}$$

as the complex envelope with respect to $100\pi t$.

(b) The bandwidth of the passband signal equals the two-sided bandwidth of the complex envelope. We have $U(f) = \frac{1}{2}I_{[-1,1]}(f\ \ )$, which has two-sided bandwidth 2, and $V(f) = e^{-j\frac{\pi}{4}}I_{[0,1]}(f$, which has two-sided bandwidth 1. Thus, u_p has bandwidth 2 and v_p has bandwidth 1.

(c) The passband inner product is related to the complex baseband inner product as follows:

$$\langle u_p, v_p \rangle = \frac{1}{2}\mathrm{Re}\langle u, v \rangle$$

Since the complex envelopes are bandlimited, it is convenient to use Parseval's identity to compute the inner product in the frequency domain:

$$\langle u, v \rangle = \langle U, V \rangle = \int U(f)V^*(f)df = \int \frac{1}{2}I_{[-1,1]}(f)e^{j\frac{\pi}{4}}I_{[0,1]}(f)df = \frac{e^{j\frac{\pi}{4}}}{2}$$

We therefore obtain

$$\langle u_p, v_p \rangle = \frac{1}{2}\mathrm{Re}\left(\frac{e^{j\frac{\pi}{4}}}{2}\right) = \frac{1}{4}\cos\frac{\pi}{4} = \frac{1}{4\sqrt{2}}$$

d) The convolution $y_p(t)$ can be obtained using the Fourier transform of complex envelopes X(f) and Y(f):

$$Y(f) = \tfrac{1}{2}U(f)V(f) = \tfrac{1}{4}I_{[-1,1]}(f)I_{[0,1]}(f)e^{-j\pi/4} = \tfrac{1}{4}I_{[0,1]}(f)e^{-j\pi/4}$$
$$y(t) = \mathcal{F}^{-1}\left[\tfrac{1}{4}I_{[-\frac{1}{2},\frac{1}{2}]}(f-\tfrac{1}{2})e^{-j\pi/4}\right] = \tfrac{1}{4}sinc(t)e^{j(\pi t-\pi/4)}$$
$$y_p(t) = Re(y(t)e^{j100\pi t}) = Re(\tfrac{1}{4}sinc(t)e^{j(100\pi t+\pi t-\frac{\pi}{4})}) = \tfrac{1}{4}sinc(t)\cos(101\pi t - \tfrac{\pi}{4})$$

Problem 3.1 Figure 3.45 shows a signal obtained after amplitude modulation by a sinusoidal message. The carrier frequency is difficult to determine from the figure, and is not needed for answering the questions below.

(a) Find the modulation index.
(b) Find the signal power.
(c) Find the bandwidth of the AM signal.

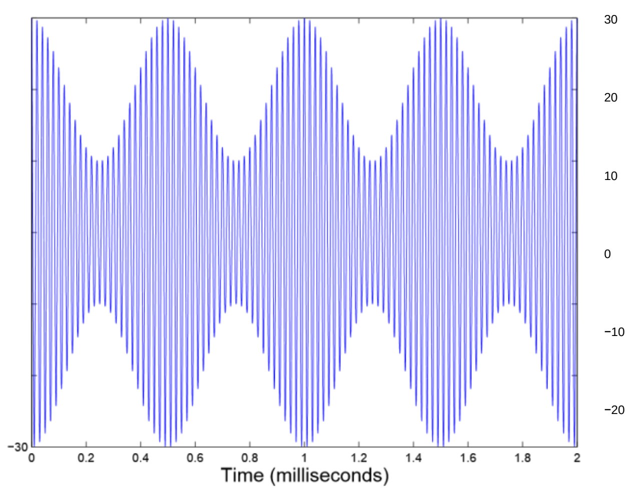

Figure 3.45: Amplitude modulated signal for Problem 3.1.

Solution 3.1(a) The AM signal is given by

$$u(t) = (A_c + m(t))\cos 2\pi f_c t$$

where $m(t) = A\cos 2\pi f_m t$. The largest negative excursion of this message has magnitude A, so that the modulation index is $a_{mod} = A/A_c$. The largest value of $u(t)$ is $A_c + A = 30$, and the smallest value is $A_c - A = 10$, so that $A_c = 20$, $A = 10$, and the modulation index is 0.5.

(b) The power is given by

$$\overline{u2} = \overline{(Ac + m(t))^2 \cos^2(2\pi fct)} = \overline{({A_c}^2 + m^2(t) + 2A_c m(t))(1 + \cos 4\pi f_c t)/2}$$
$$= {A_c}^2/2 + \overline{m^2}/2 + A_c\overline{m} + \frac{1}{2}\overline{({A_c}^2 + m^2(t) + 2A_c m(t))\cos 4\pi f_c t}$$

The term involving $\cos 4\pi f_c t$ is passband around $2f_c$, and hence has zero DC value. We also have $\overline{m^2} = A^2/2$ and $\overline{m} = 0$ for our sinusoidal message, so that the power is given by

$$Power, P_u = \overline{u}^2 = \frac{A_c^2 + A^2/2}{2} = \frac{400 + 50}{2} = 225$$

(c) We see from the figure that the envelope has period 0.5 ms, i.e., our sinusoidal message has frequency 2 KHz. Thus, the bandwidth of the AM signal is 4 KHz.

Problem 3.2 Consider a message signal $m(t) = 2\cos\ 2\pi t + \frac{\pi}{4})$.

(a) Sketch the spectrum $U(f)$ of the DSB-SC signal $u_p(t) = 8m(t)\cos 400\pi t$. What is the power of u?

(b) Carefully sketch the output of an ideal envelope detector with input u_p. On the same plot, sketch the message signal $m(t)$.

(c) Let $v_p(t)$ denote the waveform obtained by high-pass filtering the signal $u(t)$ so as to let through only frequencies above 200 Hz. Find $v_c(t)$ and $v_s(t)$ such that we can write

$$v_p(t) = v_c(t)\cos 400\pi t - v_s(t)\sin 400\pi t$$

and sketch the envelope of v.

Solution 3.2(a) We have

$$m(t) = 2\cos(2\pi t + \frac{\pi}{4}) = e^{j(2\pi t + \frac{\pi}{4})} + e^{-j(2\pi t + \frac{\pi}{4})} \leftrightarrow M(f) = e^{j\frac{\pi}{4}}\delta(f-1) + e^{-j\frac{\pi}{4}}\delta(f+1)$$

For the DSB signal, we have $u(t) = 8m(t)\cos 400\pi t = 4m(t)e^{j400\pi t} + 4m(t)e^{-j400\pi t} \leftrightarrow U(f) = 4M(f - 200) + 4M(f + 200)$ which is sketched in Figure 5.

Power of u, $P_u = \frac{A_c^2}{2} a_{mod}^2 \overline{m_n^2(t)} = \frac{(16)^2}{2} 1_{mod}^2(0.5) = 64$

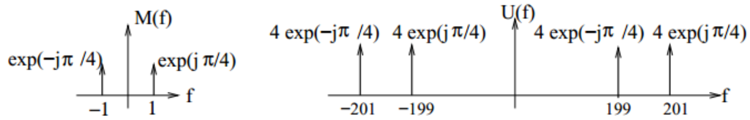

Figure 5: Spectra of message and DSB signal (Problem 3.2).

(b) The output of the envelope detector is $8|m(t)|$, which is a rectified version of the message (illustrating that envelope detection does not work for extracting the message from a DSB signal). Plot omitted.

(c) The spectrum of the passband signal and its complex envelope with respect to 200 Hz are sketched in Figure 6. Thus, the complex envelope is given by

$$V(f) = 8e^{j\frac{\pi}{4}}\delta(f-1) \leftrightarrow v(t) = 8e^{j(2\pi t+\frac{\pi}{4})}$$

Taking the real and imaginary parts of $v(t)$, we obtain the I and Q components of the USB signal

$$v_c(t) = 8cos(2\pi t + \frac{\pi}{4}), v_s(t) = 8sin(2\pi t + \frac{\pi}{4})$$

(We decided to do this from scratch, but we could have used the expression for a USB signal in terms of the message and its Hilbert transform given in the text.)

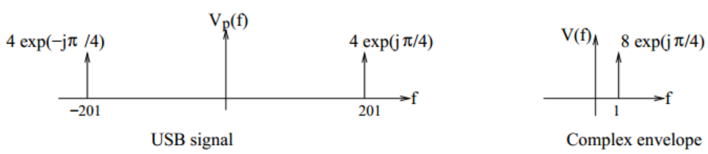

Figure 6: Spectra of the USB signal and its complex envelope (Problem 3.2).