UCSB

ECE 146A: Solutions to Problem Set 4

Problem 3.3. A message to be transmitted using AM is given by

$$m(t) = 3cos2\pi t + 4sin6\pi t$$

where the unit of time is milliseconds. It is to be sent using a carrier frequency of 600 KHz.

(a) What is the message bandwidth? Sketch its magnitude spectrum, clearly specifying the units used on the frequency axis.

(b) Find an expression for the normalized message mn(t).

(c) For a modulation index of 50%, write an explicit time domain expression for the AM signal.

(d) What is the power efficiency of the AM signal?

(e) Sketch the magnitude spectrum for the AM signal, again clearly specifying the units used on the frequency axis.

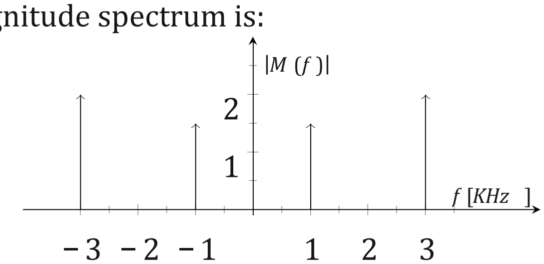
(f) The AM signal is to be detected using an envelope detector (as shown in Figure 3.8), with R = 50 ohms. What is a good range of choices for the capacitance C?

Solution 3.3.

(a) Fourier transform of message is given by

$$M(f) = 3(\frac{1}{2j})(\delta(f-1)+\delta(f+1)) + 4(-\frac{1}{2j})(\delta(f-3)-\delta(f+3))$$

Hence Magnitude spectrum is:



Clearly the message bandwidth is 3 KHz.

(b) Normalized message

$$m_n(t) = \frac{m(t)}{\texttt{Max}(m(t))} = \frac{m(t)}{6.63} = 0.45cos(2\pi t) + 0.60sin(2\pi t)$$

(c) Modulation index $a_{mod} = 0.5$, AM signal is given by:

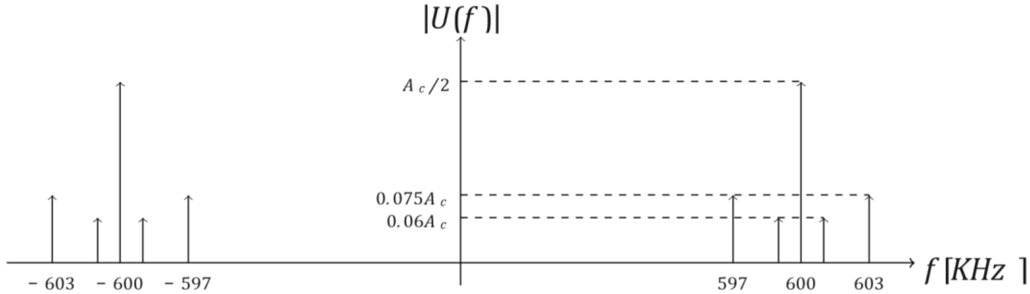
$$u(t) = A_c(1 + 0.5 * m_n(t))\cos(2\pi f_c t)$$

(d) RMS message power, $\overline{m_n^2(t)} = \frac{1}{2}(0.45^2 + 0.60^2) = 0.28$

Power Efficiency, η_{AM} is given by

$$\eta_A M = \frac{a_{mod}^2 \overline{m_n^2(t)}}{1 + a_{mod}^2 \overline{m_n^2(t)}} = \frac{0.5^2 * 0.28}{1 + (0.5^2 * 0.28)} = 6.6\%$$

(e) Magnitude spectrum of AM signal, $|U(f)|$ is



(f) RC Envelope detector designed such that $\frac{1}{f_c} < RC < \frac{1}{BW}$, hence

$$\Rightarrow \frac{1}{600KHz} < 50C < \frac{1}{6KHz}$$
$$\Rightarrow 33nF < C < 3\mu F$$

Problem 3.6 Consider a DSB signal corresponding to the message $m(t) = \text{sinc}(2t)$ and a carrier frequency f_c which is 100 times larger than the message bandwidth, where **the unit of time is milliseconds.**

(a) Sketch the magnitude spectrum of the DSB signal $10m(t)\cos 2\pi f_c t$, specifying the units on the frequency axis.

(b) Specify a time domain expression for the corresponding LSB signal.

(c) Now, suppose that the DSB signal is passed through a bandpass filter whose transfer function is given by

$$H_p(f) = (f - f_c + \frac{1}{2})I_{[f_c-\frac{1}{2}, f_c+\frac{1}{2}]} + I_{[f_c+\frac{1}{2}, f_c+\frac{3}{2}]}\ , \qquad f > 0$$

Sketch the magnitude spectrum of the corresponding VSB signal.

(d) Find a time domain expression for the VSB signal of the form

$$u_c(t)\cos 2\pi f_c t - u_s(t)\sin 2\pi f_c t$$

Carefully specifying u_c and u_s. the I and Q components.

Solution 3.6

(a) The DSB spectrum is given by $U_{DSB}(f) = 5M(f - 100) + 5M(f + 100)$ where $M(f)=0.5I_{[-1,1]}(f)$ and is shown in Figure 1

DSB spectrum

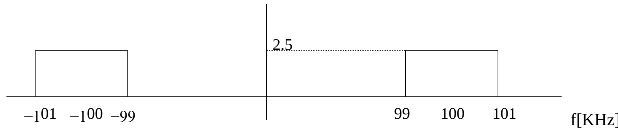


Figure 1: DSB spectrum for Problem 3.6(a).

(b) USB signal time domain expression:

$$
\begin{aligned}
u_{USB}(t) &= A[m(t)\cos(2\pi f_c t) - \hat{m}(t)\sin(2\pi f_c t)] \\
&= 10\left(sinc(2t)\cos(2\pi f_c t) + H[sinc(2t)]\sin(2\pi f_c t)\right) \\
&= 10\left(sinc(2t)\cos(2\pi f_c t) + \sin(\pi t) sinc(t)\sin(2\pi f_c t)\right) \\
&= 10\left(\frac{2sin(\pi t)cos(\pi t)}{2\pi t}\cos(2\pi f_c t) + sinc(t)\sin(\pi t)\sin(2\pi f_c t)\right) \\
&= 10\left(sinc(t)\cos(\pi t)\cos(2\pi f_c t) + sinc(t)\sin(\pi t)\sin(2\pi f_c t)\right) \\
&= sinc(t).10[\cos(\pi t)\cos(2\pi f_c t) + \sin(\pi t)\sin(2\pi f_c t)] \\
&= sinc(t).5\cos(2\pi f_c t - \pi t) \\
&= 5sinc(t)\cos(2\pi(100)t - \pi t) \\
&= 5sinc(t)\cos(199\pi t)
\end{aligned}
$$

U_{LSB} can be found in similar fashion as $A[m(t)\cos(2\pi f_c t) + \hat{m}(t)\sin(2\pi f_c t)]$

$$u_{LSB}(t) = 5sinc(t)\cos(201\pi t)$$

c) The resulting VSB signal has spectrum shown in Figure 2.

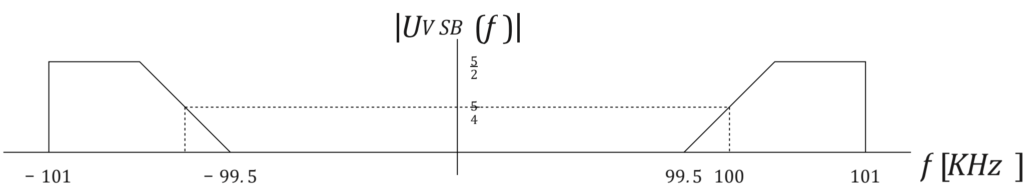
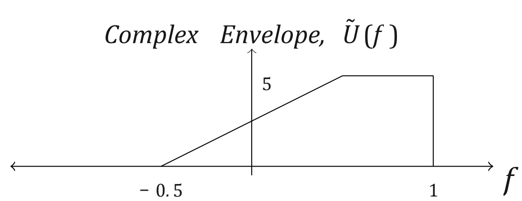


Figure 2: VSB spectrum for Problem 3.6(c)

d) The spectrum of corresponding complex envelope(Reference frequency=100KHz) is shown in Figure 3. Also shown are the I and Q components in the frequency domain, obtained in terms of the complex envelope as follows:

$$\tilde{U}_c(f) = \frac{1}{2}\left[\tilde{U}(f) + \tilde{U}^*(-f)\right]$$

$$\tilde{U}_s(f) = \frac{1}{2j}\left[\tilde{U}(f) - \tilde{U}^*(-f)\right]$$



$\tilde{U}_c(f)$ $\tilde{U}_s(f)$

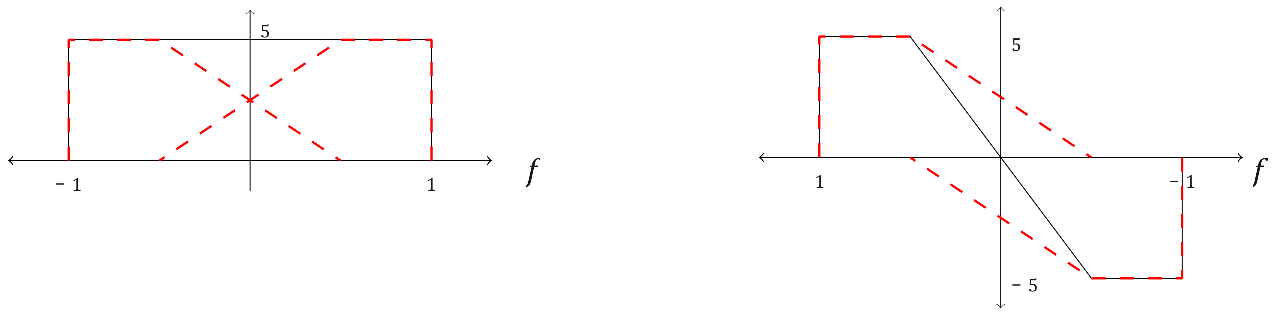
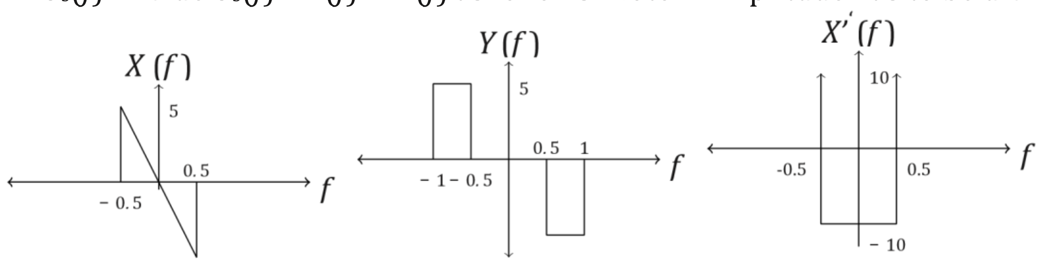


Figure 3. VSB complex envelope and I/Q components (Note- Amplitude has to further divided by 2)

Clearly, $U_c(f) = 5M(f)$. Hence, $u_c(t) = 5m(t)$. $u_s(t)$ can be calculated by taking inverse transform of $U_s(f)$. Divide $U_s(f) = X(f) + Y(f)$ as follows: Note – Amplitude has to be divided by a factor of 1/2



$$\tilde{U}_s(f) = X(f) + Y(f)$$
$$= \int_{-\infty}^{f} X'(k)dk + Y(f)$$

We know, $X'(f) \xrightarrow{\mathcal{F}^{-1}} 5\cos(\pi t) - 10sinc(t)$ and $Y(f) \xrightarrow{\mathcal{F}^{-1}} 5sinc(t)e^{-1.5\pi t} - 5sinc(t)e^{1.5\pi t}$

$$\tilde{U}_s(f) = \frac{X(-f)}{-j2\pi t} + Y(f)$$
$$u_s(t) = Im\left[\frac{5\cos(\pi t) - 10sinc(t)}{-j2\pi t} + \frac{10sinc(t)}{j}\left(\frac{1}{2j}(e^{1.5\pi t} - e^{-1.5\pi t})\right)\right]$$
$$= Im\left[j\left[\frac{5\cos(\pi t) - 10sinc(t)}{2\pi t} - 10sinc(t)\sin(1.5\pi t)\right]\right]$$
$$= \frac{5\cos(\pi t) - 10sinc(t)}{2\pi t} - 10sinc(t)\sin(1.5\pi t) \qquad /2$$

Problem 3.9 Consider a message signal $m(t)$ with spectrum $M(f) = I_{[-2,2]}(f)$.

(a) Sketch the spectrum of the DSB-SC signal $u_{DSB-SC} = 10m(t)\cos 300\pi t$. What is the power and bandwidth of u?

(b) The signal in (a) is passed through an envelope detector. Sketch the output, and comment on how it is related to the message.

(c) What is the smallest value of A such that the message can be recovered without distortion from the AM signal $u_{AM} = (A + m(t))\cos 300\pi t$ by envelope detection? (d) Give a time-domain expression of the form

$$u_p(t) = u_c(t)\cos 300\pi t - u_s(t)\sin 300\pi t$$

obtained by high-pass filtering the DSB signal in (a) so as to let through only frequencies above 150 Hz.

(e) Consider a VSB signal constructed by passing the signal in (a) through a passband filter with transfer function for positive frequencies specified by:

$$H_p(f) = \begin{cases} f - 149 & 149 \leq f \leq 151 \\ 2 & f \geq 151 \end{cases}$$

(you should be able to sketch $H_p(f)$ for both positive and negative frequencies.) Find a time domain expression for the VSB signal of the form

$$u_p(t) = u_c(t)\cos 300\pi t - u_s(t)\sin 300\pi t$$

Solution 3.9(a)

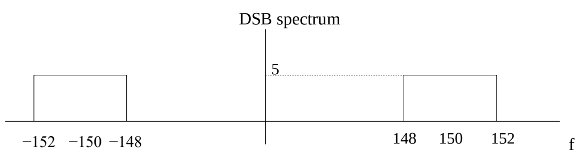


Figure 1: DSB spectrum for Problem 3.9(a).

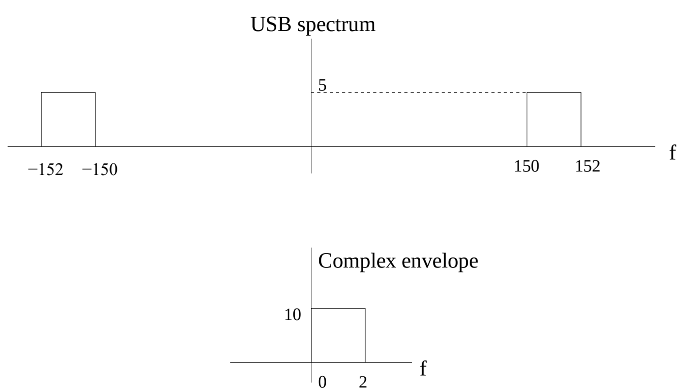


Figure 2: USB spectrum and complex envelope for Problem 3.9(d).

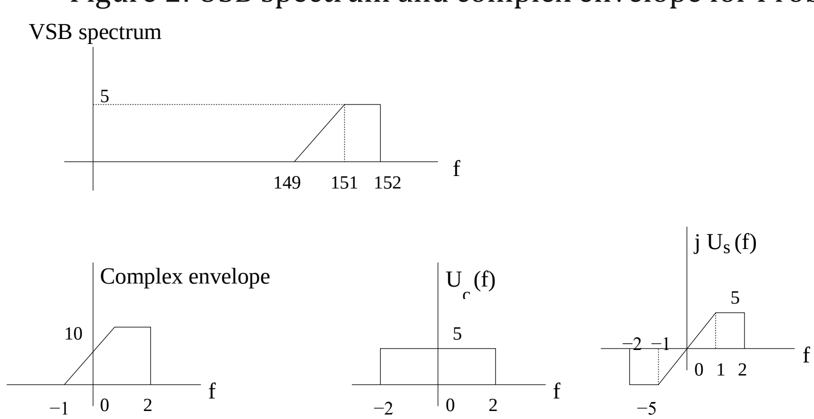


Figure 3: VSB spectrum, complex envelope, and I/Q components for Problem 3.9(e).

The DSB spectrum is given by $U_{DSB}(f) = 5M(f - 150) + 5M(f + 150)$, and is shown in Figure 1. Bandwidth is 4 Hz (our default assumption is that the unit of frequency is Hz, unless specified otherwise). The message is finite energy, and hence so is the DSB signal. Thus, the power is zero.

If we consider the periodic extension of the signal, its power could be computed by approximating the period. To compute energy, note that

$$u^2_{DSB}(t) = 100m^2(t)\cos^2 300\pi t = 50m^2(t) + 50m^2(t)\cos 600\pi t$$

The second term on the extreme right hand side is passband, and hence integrates to zero. Thus, the energy of the DSB signal is given by

$$E_{u_{DSB}} = 50E_m = 50\int |M(f)|^2 df = 50\int_{-2}^{2} 1^2 df = 200$$

using Parseval's identity.

(b) We have $m(t) = 4\text{sinc}4t$. The output of the envelope detector is $10|m(t)| = 40|\text{sinc}4t|$. Sketch omitted, since the shape is well-known. The envelope is the magnitude of the message, so sign information is lost.

(c) The AM signal is $u_{AM}(t) = (A+4\text{sinc}4t)\cos 300\pi t$. Thus, we must have $A > 4|\min_t \text{sinc}4t| = 4|\min_x \text{sinc}x|$. The minimum of the sinc function is near the first negative lobe, around $x = 1.5$. It is actually achieved at a value of x a little less than 1.5, but let's just approximate the minimum as the value at $x = 1.5$: $\min_x \text{sinc}x \approx \frac{\sin 1.5\pi}{1.5\pi} = -\frac{2}{3\pi} = -0.2122$. (The actual minimum can be numerically computed as -0.2172.) In particular, $A \geq 0.88$ will work.

(d) The USB signal has spectrum and complex envelope shown in Figure 2. Taking the inverse Fourier transform of the complex envelope (with respect to $f_c = 150$), we obtain

$$u(t) = 20\text{sinc}2t e^{j2\pi t}$$

Taking real and imaginary parts, we obtain:

$$u_c(t) = 20\text{sinc}2t\cos 2\pi t \qquad u_s(t) = 20\text{sinc}2t\sin 2\pi t$$

We see that

$$u_c(t) = 20\frac{\sin 2\pi t}{2\pi t}\cos 2\pi t = 10\frac{\sin 4\pi t}{2\pi t} = 20\text{sinc}4t = 5m(t)$$

We can also check that $u_s(t) = 5\check{m}(t)$

(e) Filtering the DSB spectrum in (a) using the VSB filter, we obtain the spectrum (only positive frequencies shown) and complex envelope as shown in Figure **??**. Also shown are the I and Q components in the frequency domain, obtained in terms of the complex envelope as follows:

$$U_c(f) = (U(f) + U^*(-f))/2, \qquad jU_s(f) = (U(f) - U^*(-f))/2$$

Clearly, $U_c(f) = 5M(f)$, so $u_c(t) = 5m(t) = 20\text{sinc}4t$. Finding the Q component in the time domain is an exercise in taking inverse Fourier transforms that we skip here.

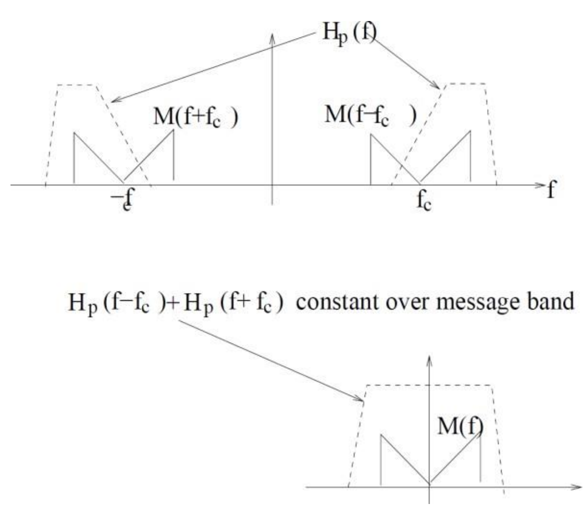


Figure 4 for Problem 3.10.

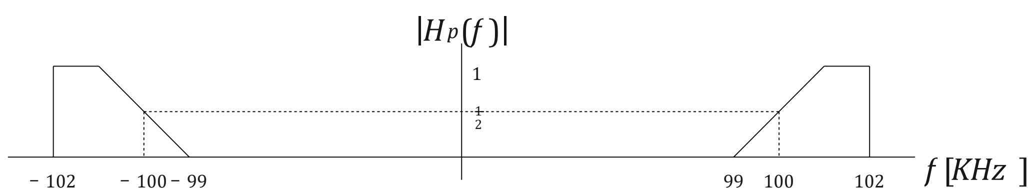
Problem 3.10 Consider Figure 3 depicting VSB spectra. Suppose that the passband VSB filter $H_p(f)$ is specified (for positive frequencies) as follows:

$$H_p(f) = \begin{cases} 1, & 101 \leq f < 102 \\ \frac{1}{2}(f-99), & 99 \leq f \leq 101 \\ 0, & else \end{cases}$$

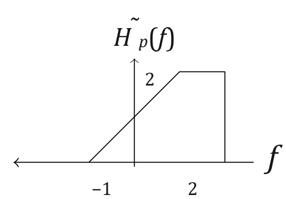
(a) Sketch the passband transfer function $H_p(f)$ for both positive and negative frequencies.
(b) Sketch the spectrum of the complex envelope $H(f)$, taking $f_c = 100$ as a reference.
(c) Sketch the spectra (show the real and imaginary parts separately) of the I and Q components of the impulse response of the passband filter.
(d) Consider a message signal of the form $m(t) = 4\text{sinc}4t - 2\cos 2\pi t$. Sketch the spectrum of the DSB signal that results when the message is modulated by a carrier at $f_c = 100$.
(e) Now, suppose that the DSB signal in (d) is passed through the VSB filter in (a)-(c). Sketch the spectra of the I and Q components of the resulting VSB signal, showing the real and imaginary parts separately.
(f) Find a time domain expression for the Q component.

Solution 3.10

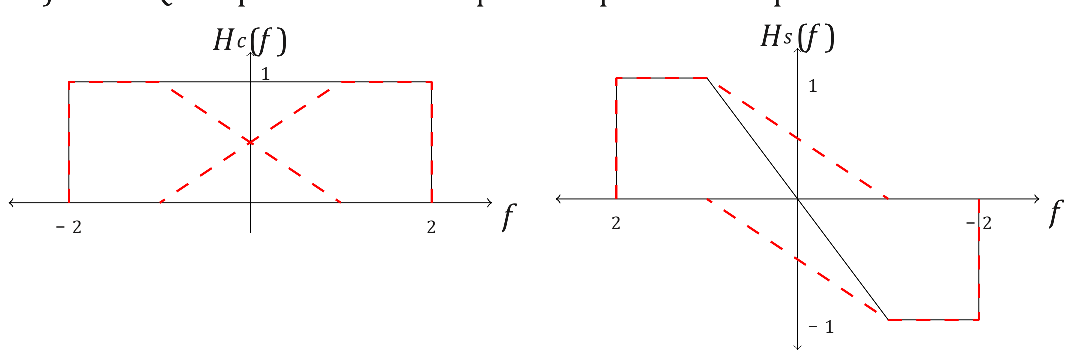
a) Passband transfer function $H_p(f)$



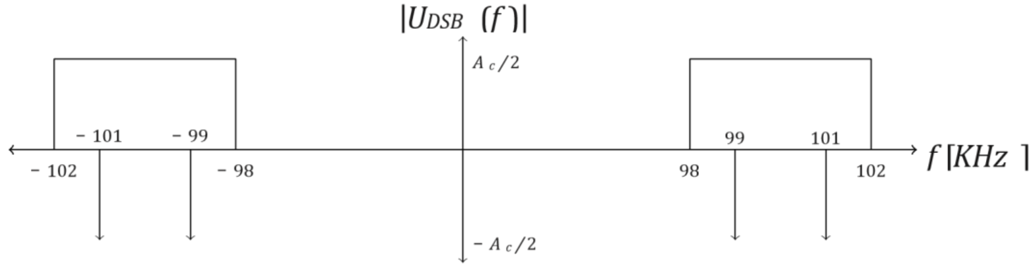
b) Spectrum of the complex envelope $\tilde{H}(f)$, $f_c = 100$



c) I and Q components of the impulse response of the passband filter are sketched in Figure



d) DSB Spectrum is sketched in the Figure below

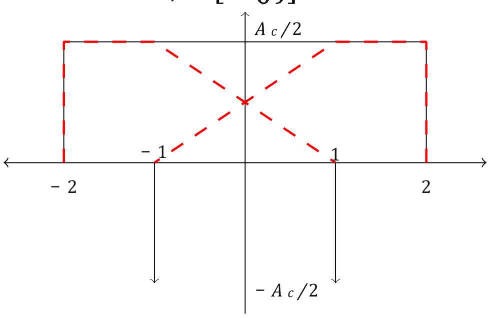


e) VSB signal's I/Q components is computed using the relations:

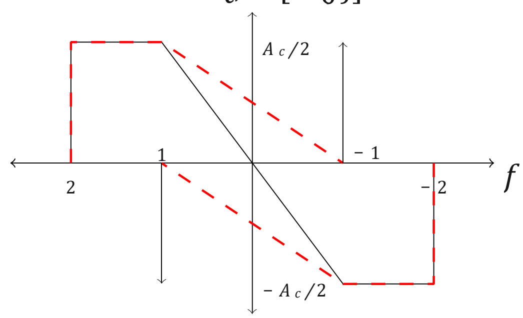
$$\tilde{U}_c(f) = \frac{1}{2}\left[\tilde{U}(f) + \tilde{U}^*(-f)\right]$$
$$\tilde{U}_s(f) = \frac{1}{2j}\left[\tilde{U}(f) - \tilde{U}^*(-f)\right]$$

The I component is real and Q component is imaginary so their imaginary and real parts will be 0 respectively. $Im[U_c(f)] = 0$ and $Re[U_s(f)] = 0$

$I, Re[U_c(f)]$

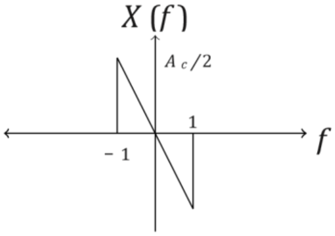


$Q, Im[U_s(f)]$

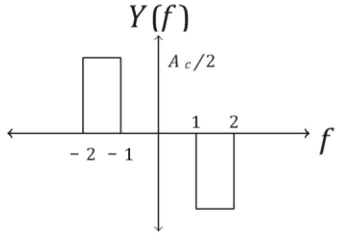


f) The time domain expression for Q component will be the inverse fourier transform of $U_s(f)$, divide it into three parts as $U_s(f) = X(f) + Y(f) + Z(f)$ where X(f),Y(f),Z(f) are

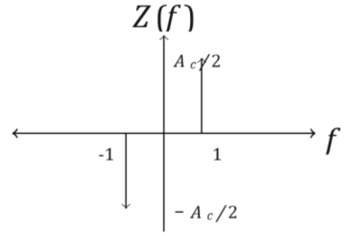
$X(f)$



$Y(f)$



$Z(f)$



$$\tilde{U}_s(f) = X(f) + Y(f) + Z(f)$$
$$= \int_{-\infty}^{f} X'(k)dk + Y(f) + Z(f)$$

We know, $X'(f) \xrightarrow{\mathcal{F}^{-1}} \frac{A_c}{4}\cos(2\pi t) - A_c sinc(2t)$, $Y(f) \xrightarrow{\mathcal{F}^{-1}} \frac{A_c}{2} sinc(t)e^{-3\pi t} - \frac{A_c}{2} sinc(t)e^{3\pi t}$ and $Z(f) = A_c j \sin(2\pi t)$

$$
\begin{aligned}
\tilde{U}_s(f) &= \frac{X(-f)}{-j2\pi t} + Y(f) + Z(f) \\
u_s(t) &= A_c Im\left[\frac{0.25\cos(2\pi t) - sinc(2t)}{-j2\pi t} + \frac{sinc(t)}{j}\left(\frac{1}{2j}(e^{3\pi t} - e^{-3\pi t})\right) + j\sin(2\pi t)\right] \\
&= A_c Im\left[j\left[\frac{0.25\cos(2\pi t) - sinc(2t)}{2\pi t} - sinc(t)\sin(3\pi t) + \sin(2\pi t)\right]\right] \\
&= A_c\left(\frac{0.25\cos(2\pi t) - sinc(2t)}{2\pi t} - sinc(t)\sin(3\pi t) + \sin(2\pi t)\right)
\end{aligned}
$$