ECE146A :

Solutions to Problem Set 6

Problem 5.3 We have $Y \sim Exp(1)$ if 0 sent, $Y \sim Exp(\frac{1}{10})$ if 1 sent, and $P[0\ sent] = 1 - P[1\ sent] = 0.6$. Recall that the complementary CDF of an exponential random variable is given by $P[Exp(\mu) > z] = e^{-\mu z}$, $z \geq 0$.
(a) $P[Y > 5|0\ sent] = e^{-5}$.
(b) $P[Y > 5|1\ sent] = e^{-\frac{5}{10}} = e^{-\frac{1}{2}}$.
(c) Using the law of total probability,

$$P[Y > 5] = P[Y > 5|0\ sent]P[0\ sent] + P[Y > 5|1\ sent]P[1\ sent] = e^{-5} \times 0.6 + e^{-\frac{1}{2}} \times 0.4 = 0.2467$$

(d) Using Bayes's rule,

$$P[0\ sent|Y > 5] = \frac{P[Y > 5|0\ sent]P[0\ sent]}{P[Y > 5]} = \frac{e^{-5} \times 0.6}{0.2467} = 0.0164$$

(e) Using Bayes' rule,

$$P[0\ sent|Y = 5] = \frac{p_{Y|0}(5|0)P[0\ sent]}{p_Y(5)}$$

where, for $y \geq 0$, we have the conditional densities $p_{Y|0}(y|0) = e^{-y}$ and $p_{Y|1}(y|1) = \frac{1}{10}e^{-\frac{y}{10}}$, and the unconditional density

$$p_Y(y) = p_{Y|0}(y|0)P[0\ sent] + p_{Y|1}(y|1)P[1\ sent] = 0.6e^{-y} + 0.4\frac{1}{10}e^{-\frac{y}{10}}$$

Plugging these in, we obtain

$$P[0\ sent|Y = 5] = \frac{e^{-5} \times 0.6}{e^{-5} \times 0.6 + \frac{1}{10}e^{-\frac{5}{10}} \times 0.4} = 0.1428$$

Problem 5.6 (a) Using Bayes' rule and the law of total probability, the posterior probabilities are given by

$$P[X = 0|Y = y] = \frac{P[Y = y|X = 0]P[X = 0]}{P[Y = y]} = \frac{P[Y = y|X = 0]P[X = 0]}{P[Y = y|X = 0]P[X = 0] + P[Y = y|X = 1]P[X = 1]}$$

For 0, 1 equally likely, this reduces to

$$P[X = 0|Y = y] = \frac{P[Y = y|X = 0]}{P[Y = y|X = 0] + P[Y = y|X = 1]} \quad \textbf{0, 1 equiprobable}$$

which gives

$$P[X = 0|Y = y] = \begin{cases} \frac{1-p-q-r}{1-q-r} = 0.917 & y = +3 \\ \frac{r}{r+q} = 0.75 & y = +1 \\ \frac{q}{r+q} = 0.25 & y = -1 \\ \frac{p}{1-q-r} = 0.083 & y = -3 \end{cases}$$

(b) The LLRs are given by $L(y) = \log \frac{P[X=0|Y=y]}{P[X=1|Y=y]} = \log \frac{P[Y=y]}{1-P[X=0|Y=y]}$, and can therefore be calculated by plugging in answers from (a). However, it is worth noting that they can also be written as

$$\begin{aligned} L(y) &= \log \left(\tfrac{P[Y=y|X=0]P[X=0]}{P[Y=y]}\right) / \left(\tfrac{P[Y=y|X=1]P[X=1]}{P[Y=y]}\right) \\ &= \log \tfrac{P[Y=y|X=0]}{P[Y=y|X=1]} + \log \tfrac{P[X=0]}{P[X=1]} \end{aligned}$$

Thus, the LLR is a sum of two terms, one corresponding to the transition probabilities $\{P[Y = y|X = i]\}$ and one to the prior probabilities $\{P[X = i]\}$, where $i = 0, 1$. For equiprobable priors, the second term is zero, hence the LLRs are given by

$$L(y) = \log \frac{P[Y = y|X = 0]}{P[Y = y|X = 1]} = \begin{cases} \log \frac{1-p-q-r}{p} = 2.4 & y = +3 \\ \frac{r}{q} = 1.1 & y = +1 \\ \frac{q}{r} = -1.1 & y = -1 \\ \frac{p}{1-p-q-r} = -2.4 & y = -3 \end{cases}$$

(c) Since the channel uses are conditionally independent, the required conditional probabilities are given by

$$\begin{aligned} P[\mathbf{Y} = \mathbf{y}|X = i] &= P[Y_1 = y_1, Y_2 = y_2, Y_3 = y_3|X = i] \\ &= P[Y_1 = y_1|X = i]P[Y_2 = y_2|X = i]P[Y_3 = y_3|X = i], \quad i = 0, 1 \end{aligned}$$

For $\mathbf{y} = (+1, +3, -1)^T$, we get

$$P[\mathbf{Y} = \mathbf{y}|X = 0] = r(1 - p - q - r)q = 0.0165, \quad P[\mathbf{Y} = \mathbf{y}|X = 1] = qpr = 0.0015$$

(d) We replicate the argument in (a) to emphasize that the key ideas apply to vector observations gathered over multiple channel uses as well. Using Bayes' rule, we have

$$P[X = 0|\mathbf{Y} = \mathbf{y}] = \frac{P[\mathbf{Y} = \mathbf{y}|X = 0]P[X = 0]}{P[\mathbf{Y} = \mathbf{y}]} = \frac{P[\mathbf{Y} = \mathbf{y}|X = 0]P[X = 0]}{P[\mathbf{Y} = \mathbf{y}|X = 0]P[X = 0] + P[\mathbf{Y} = \mathbf{y}|X = 1]P[X = 1]}$$

For 0, 1, equiprobable, we have

$$P[X = 0|\mathbf{Y} = \mathbf{y}] = \frac{P[\mathbf{Y} = \mathbf{y}|X = 0]}{P[\mathbf{Y} = \mathbf{y}|X = 0] + P[\mathbf{Y} = \mathbf{y}|X = 1]} = 0.917 \quad \mathbf{0,\ 1\ equiprobable}$$

where we have used the results of (d). As before, the LLR can be written as

$$L(\mathbf{y}) = \log \frac{P[X = 0|\mathbf{Y} = \mathbf{y}]}{P[X = 1|\mathbf{Y} = \mathbf{y}]} = \log \frac{P[\mathbf{Y} = \mathbf{y}|X = 0]}{P[\mathbf{Y} = \mathbf{y}|X = 1]} + \log \frac{P[X = 0]}{P[X = 1]}$$

Thus, for equal priors, we have

$$L(\mathbf{y}) = \log \frac{\mathbf{Y} = \mathbf{y}|X = 0]}{P[\mathbf{Y} = \mathbf{y}|X = 1]} = \log(\frac{r(1 - p - q - r)q}{qpr} = \log \frac{1 - p - q - r}{p} = 2.4$$

Remark: Actually, for independent channel uses, we can write the LLR as

$$L(\mathbf{y}) = \sum_{k=1}^{3} \log \frac{P[Y = y_k|X = 0]}{P[Y = y_k|X = 1]} + \log \frac{P[X = 0]}{P[X = 1]}$$

so that the contributions from the different channel uses and the priors simply add up. This illustrates why the LLR is an attractive means of combining information from prior probabilities and observations.
(e) Since the posterior probability of 0 is much higher than that of 1, we would decide on 0 based on the channel output +1,+3,-1.

Problem 5.13: While all the derivations in this problem are there in the text and other problems, we do them from scratch here in order to reinforce the concepts.

(a) Since $U_1$and U_2 are independent, so are $Z = -2\ln U_1$ and $\Theta = 2\pi U_2$. Clearly, Θ is uniform over $[0,2\pi]$. Since U_1 takes values in $[0,1]$, the random variable Z takes values in $[0,\infty)$. TheCDF of Z is given by

$$P[Z\leq z] = P[-2\ln U_1 \leq z] = P[U_1 \geq e^{-z/2}] = 1 - e - ^{z/2}, z \geq 0$$

We recognize that $Z \sim \mathrm{Exp}\left(\frac{1}{2}\right)$, or an exponential random variable with mean $\mathrm{E}[Z] = 2$.

Remark: This provides a good opportunity to emphasize that one does not always have to write down explicit expressions for the joint CDF or density in order to specify the joint distribution. Any specification that would allow one to write down such expressions if needed is sufficient. In the preceding, we provided such a specification by stating that Z and Θ are independent, with $Z \sim \mathrm{Exp}\left(\frac{1}{2}\right)$) and $\Theta \sim \mathit{Unif}[0,2\pi]$.

(b)Let us do this from scratch instead of using results from prior examples and problems.

$$p(z,\theta) = p(z)p(\theta) = \frac{1}{2}e^{-z/2}\frac{1}{2\pi}, z \geq 0, 0 \leq \theta \leq 2\pi \tag{1}$$

and

$$p(x_1,x_2) = \left.\frac{p(z,\theta)}{|\det(J(x_1,x_2;z,\theta))|}\right|_{z=x_1^2+x_2^2, \theta=\tan^{-1}\frac{x_2}{x_1}} \tag{2}$$

where the Jacobian can be computed as

$$J(x_1,x_2;z,\theta) = \begin{pmatrix} \frac{1}{2\sqrt{z}}\cos\theta & \frac{1}{2\sqrt{z}}\sin\theta \\ -\sqrt{z}\sin\theta & \sqrt{z}\cos\theta \end{pmatrix}$$

so that $\det(J(x_1,x_2;z,\theta)) = \frac{1}{2}\cos^2\theta + \frac{1}{2}\sin^2\theta = \frac{1}{2}$. Plugging this and (1) into (2), we obtain that the joint density of X_1X_2 is given by

$$p(x_1,x_2) = \frac{1}{2\pi}e^{-(x_1^2+x_2^2)/2}, -\infty < x_1, x_2 < \infty$$

We recognize that this is a product of two $N(0,1)$ densities, so that X_1, X_2 are i.i.d. $N(0,1)$ random variables.

(c) A code fragment using the preceding to generate $N(0,1)$ random variables is provided below, and the histogram generated is shown in Figure 1.

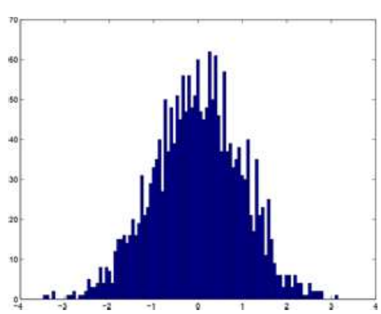

Figure 1. Histogram based on 2000 $N(0,1)$ random variables generated using the method in Problem 5.13.

```
%generating Gaussian random variables from uniform random variables
N=1000; %half the number of Gaussians needed
%generate uniform random variables
U1 = rand(N,1);
U2 = rand(N,1);
Z = -2*log(U1); %exponentials, mean 2 theta=2*pi*U2; % uniform over [0,2
pi]
%transform to standard Gaussian
X1=sqrt(Z).*cos(theta);
X2=sqrt(Z).*sin(theta);
X = [X1;X2]; %2N independent N(0,1) random variables hist(X,100);
%histogram with hundred bins
```

(d) We estimate $\mathrm{E}[X^2]$ as the empirical mean by adding the following code fragment:

```
estimated_power = sum(X.^2)/(2*N)
```

The answer should be close to the theoretical answer $\mathrm{E}[X^2] = \mathrm{var}(X) + (\mathrm{E}[X])^2 = 1 + 0^2 = 1$.

(e) The desired probability $P[X^3 + X > 3]$ can be estimated by adding the following code fragment.

```
E = (X.^3 + X > 3); %indicates whether the desired event occurs
probability = sum(E)/(2*N) %counts fraction of times desired event
occurs
```

We get an answer of about 0.11.

5.21

$x_1 = R\cos\phi \qquad x_2 = R\sin\phi$

(a) $\underline{X} = [R, \phi] \qquad \underline{Y} = [x_1, x_2]$

$$J(\underline{Y}; \underline{X}) = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & -r\cos\phi \end{bmatrix}$$

$$p_{R,\phi}(r, \phi) = \left. \frac{p_{x_1,x_2}(x_1, x_2)}{|\det(J(Y; X))|} \right|_{\substack{x_1 = r\cos\phi \\ x_2 = r\sin\phi}}$$

$$p_{x_1,x_2} = \frac{1}{\sqrt{2\pi}\, v} e^{-x_1^2/2v^2} \frac{1}{\sqrt{2\pi}\, v} e^{-x_2^2/2v^2}$$

$$\Rightarrow p_{R,\phi}(r, \phi) = \frac{1}{2\pi v^2} \cdot e^{-r^2/2v^2} \cdot \frac{1}{(r^{-1})}$$

$$= \frac{r}{2\pi v^2} e^{-r^2/2v^2} \cdot I_{[0,\infty)}(r)$$

(b) $$p_{R,\phi}(r,\phi) = \frac{1}{2\pi} \cdot \frac{r}{v^2} e^{-r^2/2v^2}$$

$$= p_R(r)\, p_\phi(\phi)$$

$\therefore$ R and ϕ are independent.

$$p_R(r) = \frac{r}{v^2} e^{-r^2/2v^2} \qquad p_\phi(\phi) = \frac{1}{2\pi}.$$

(c) $$y = R^2$$

$\Rightarrow R = \pm\sqrt{y}$ since R is +ve

$R = \sqrt{y}$.

$$\Rightarrow p_y(y) = \frac{p_R(\sqrt{y})}{|2\sqrt{y}|} \qquad p_R(\sqrt{y}) = \frac{\sqrt{y}}{v^2} e^{-y/2v^2}$$

$$p_y(y) = \frac{1}{2v^2} e^{-y/2v^2}, \qquad y = R^2.$$

(d) let $\lambda = \frac{1}{2v^2}$

$$\Rightarrow p_y(y) = \lambda e^{-\lambda y} \qquad \mu = E\{y\} = \frac{1}{\lambda}.$$

we need $-10\log(y) + 10\log(\mu) > 20$

$$\Rightarrow \log\left(\frac{\mu}{y}\right) > 2$$

$$\Rightarrow \mu/y > 100$$

$\Rightarrow \quad y < \frac{u}{100}$

$$= \int_0^{\frac{u}{100}} \lambda e^{-\lambda y} \, dy$$

$$= e^{-\lambda y} \Big|_{\frac{u}{100}}^{0}$$

$$= 1 - e^{-\lambda \cdot \frac{u}{100}} = 1 - e^{-1/100}$$

it doesn't depend on v^2.

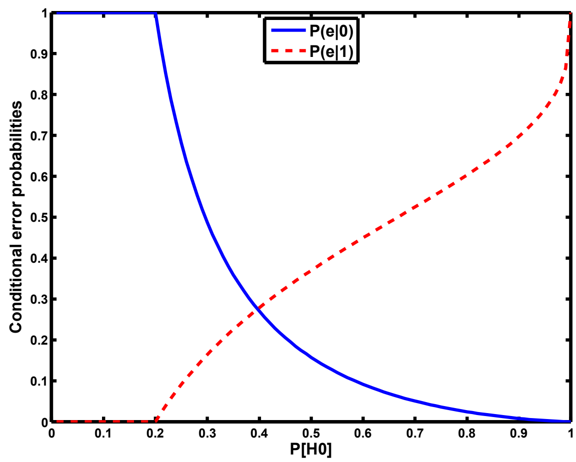

Figure 3: Conditional error probabilities for the MAP rule (as a function of π_0) for Problem 6.5.

Problem 6.5 The log likelihood ratio is given by

$$\log L(y) = \frac{\mu_1 e^{-\mu_1 y}}{\mu_0 e^{-\mu_0 y}} = (\mu_0 - \mu_1)y + \log \frac{\mu_1}{\mu_0}$$

where $\mu_1 = 1/4$, $\mu_0 = 1$. Plugging in, we obtain $\log L(y) = \frac{3}{4}y - \log 4$. The MAP/MPE rule

$$\log L(y) \underset{H_0}{\overset{H_1}{\gtrless}} \log \frac{\pi_0}{\pi_1} = \log \frac{\pi_0}{1-\pi_0}$$

therefore simplifies to

$$y \underset{H_0}{\overset{H_1}{\gtrless}} \frac{4}{3} \log \frac{4\pi_0}{1-\pi_0} = \gamma_{MAP} \quad \textbf{MAPrule}$$

Since $y \geq 0$, the MAP rule always says H_1 if $\gamma_{MAP} \leq 0$, which happens if the argument of the log is less than (or equal to) one: $\frac{4\pi_0}{1-\pi_0} \leq 1$, or $\pi_0 \leq \frac{1}{5}$.
(b) From (a), we see that for $\pi_0 \leq \frac{1}{5}$, the conditional error probabilities are given by $P_{e|0} = 1$ and $P_{e|1} = 0$, since the MAP rule always says H_1.
For $\pi_0 > \frac{1}{5}$, we have $\gamma_{MAP} > 0$, and the conditional error probabilities are given by

$$P_{e|0} = P[Y > \gamma_{MAP}|H_0] = e^{-\mu_0 \gamma_{MAP}} = e^{-\frac{4}{3}\log\frac{4\pi_0}{1-\pi_0}} = \left(\frac{1-\pi_0}{4\pi_0}\right)^{4/3}$$

$$P_{e|1} = P[Y \leq \gamma_{MAP}|H_1] = 1 - e^{-\mu_1 \gamma_{MAP}} = 1 - e^{-\frac{1}{3}\log\frac{4\pi_0}{1-\pi_0}} = 1 - \left(\frac{1-\pi_0}{4\pi_0}\right)^{1/3}$$

These are plotted as a function of π_0 in Figure 3.