ECE 146B: Solutions to Problem Set 4

5.26 $$x(t) = \sum_{n=-\infty}^{\infty} (-1)^n \, p(t-n)$$

$$X(t) = x(t-D). \qquad D \sim \text{uniform } [0, 1].$$

(a) $$R_{XX}(t_1, t_2) = E\{X(t_1)X(t_2)\}$$

$$= E\{x(t_1 - D)\, x(t_2 - D)\}$$

$$= E\left\{\sum_{n=-\infty}^{\infty} (-1)^n p(t_1 - n - D) \cdot \sum_{m=-\infty}^{\infty} (-1)^m p(t_2 - m - D)\right\}$$

$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} (-1)^{n+m} E\{p(t_1-n-D)\, p(t_2-m-D)\}.$$

$$E\{p(t_1-n-D)\, p(t_2-m-D)\} = \int_0^1 p(t_1-n-a)\, p(t_2-m-a)\, da$$

let $t_1 - n - a = u$ $\Rightarrow$ $-da = du$

$\Rightarrow t_2 - m - a = t_2 - m - (t_1 - n - u) = (t_2 - t_1) - (m - n) + u$

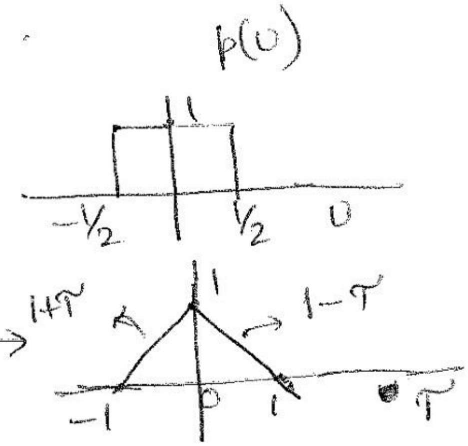
$$= -\int_{t_1-n}^{t_1-n-1} p(u)\, p((t_2-t_1) - (m-n) + u)\, du$$

$$= \int_{t_1-n-1}^{t_1-n} p(u)\, p(u + \underbrace{(t_2-t_1)}_{\tau} - (m-n))\, du$$

$$\therefore R_{XX}(t_1, t_2) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} (-1)^{m+n} \int_{t_1-n-1}^{t_1-n} p(u)\, p(u + \tau - (m-n))\, du$$

for $m = n$

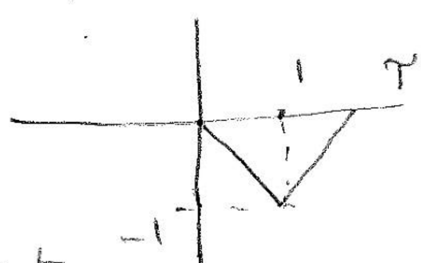
$$R_{xx}(t_1,t_2) = \sum_{m=-\infty}^{\infty} \int_{t_1-m-1}^{t_1-m} p(u)\, p(u+\tau)\, du$$



$$= \int_{-\infty}^{\infty} p(u)\, p(u+\tau)\, du \longrightarrow$$

for $(m-n)=1$

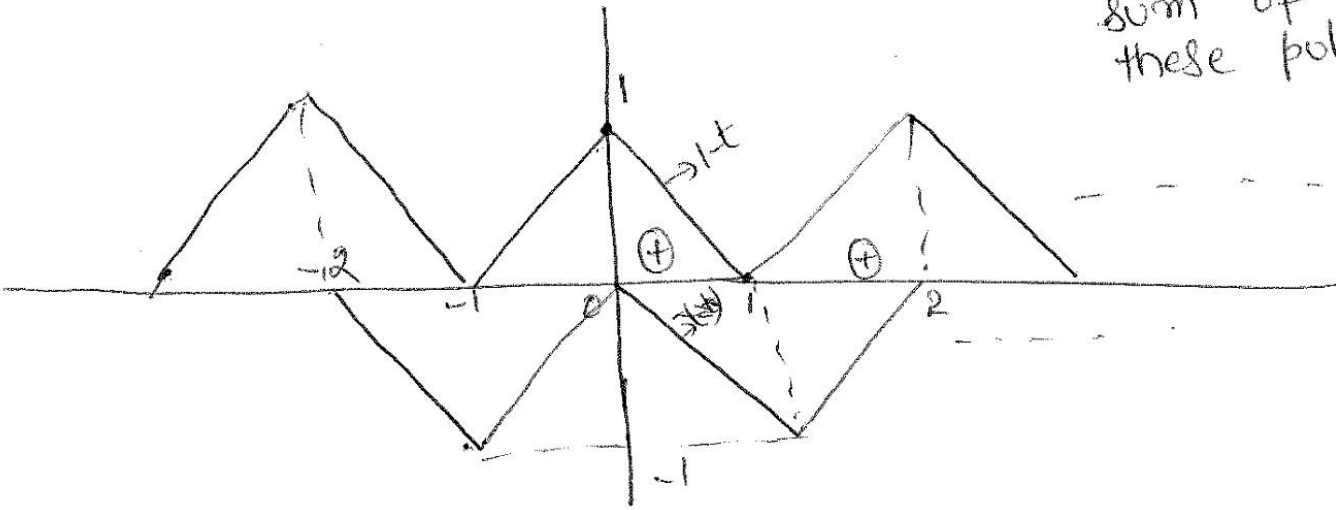
$$R_{xx}(t_1,t_2) = \sum_{m=-\infty}^{\infty} (-1)^1 \int_{t_1-(m-1)-1}^{t_1-(m-1)} p(u)\, p(u+\tau-1)\, du \longrightarrow$$



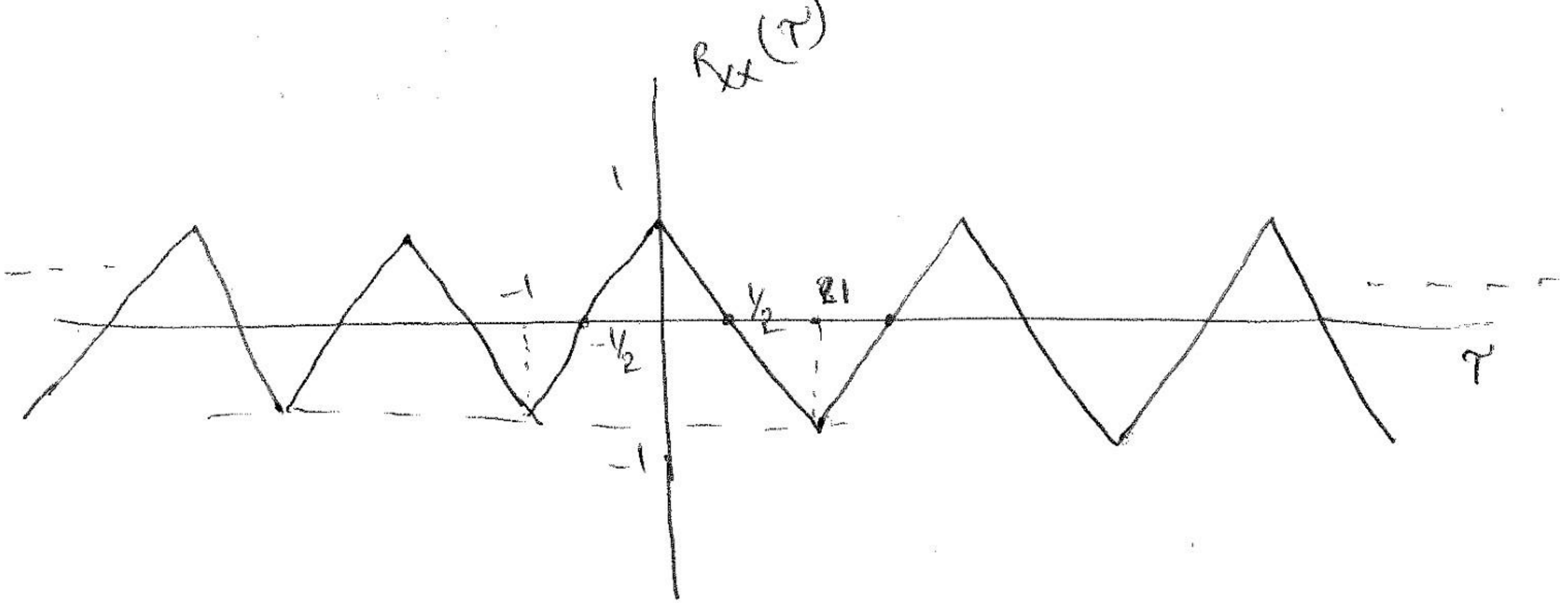
proceeding in a similar way we get

$R_{xx}(t_1,t_2) = R_{xx}(\tau)$

sum of all these pulses



This results in



(b).

$$E\{X(t)\} = E\{x(t-D)\}$$

$$= E\left\{ \sum_{n=-\infty}^{\infty} (-1)^n \, p(t-n-D) \right\}$$

$$= \sum_{n=-\infty}^{\infty} (-1)^n \, E\{p(t-n-D)\}$$

$$= \sum_{n=-\infty}^{\infty} (-1)^n \cdot \int_0^1 p(t-n-a) \cdot 1 \cdot da$$

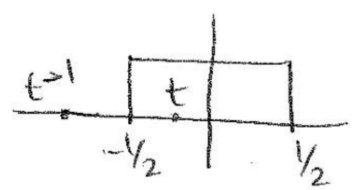
$$= \sum_{n=-\infty}^{\infty} (-1)^n \cdot \int_0^1 I_{[-1/2 \;\; 1/2]}(t-n-a) \; da.$$

let $t-n-a = u$

$$= \sum_{n=-\infty}^{\infty} (-1)^n \int_{t-n}^{t-n-1} I_{[-1/2 \;\; 1/2]}(u) \; (-du)$$

$$= \sum_{n=-\infty}^{\infty} (-1)^n \int_{t-(n+1)}^{t-n} I_{[-1/2 \;\; 1/2]}(u) \; du$$

for $n=0$



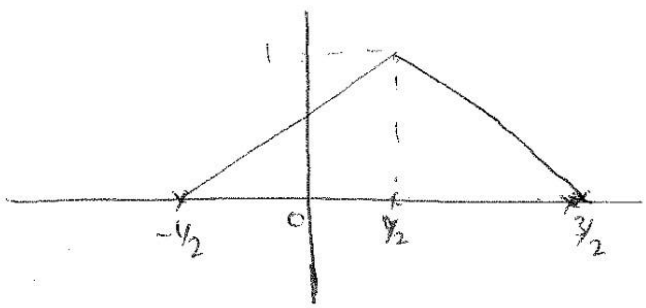
$$= \int_{t-1}^{t} I_{[-1/2 \;\; 1/2]}(u) \; du$$

$$= \int_{-1/2}^{t} du = t + 1/2 \quad \text{if } -1/2 < t < 1/2$$

$$= \int_{t-1}^{1/2} du \qquad \text{if} \qquad 1/2 < t < 3/2$$

$$= 1/2 - (t-1)$$

$$= 3/2 - t$$



f(6) $n = 1$

$$\int_{-1/2}^{t-1} dt \qquad -1/2 < t-1 < 1/2$$
$$1/2 < t < 3/2$$

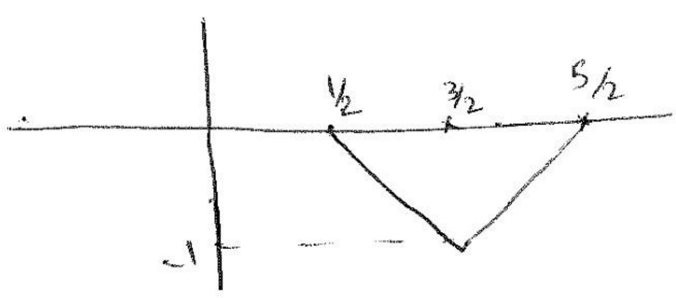
$$= t - 1 + 1/2$$

$$= t - 1/2.$$

$$= \int_{t-2}^{1/2} dt \qquad \text{if} \qquad -1/2 < t-2 < 1/2$$
$$3/2 < t < 5/2$$

$$= 1/2 - (t-2)$$

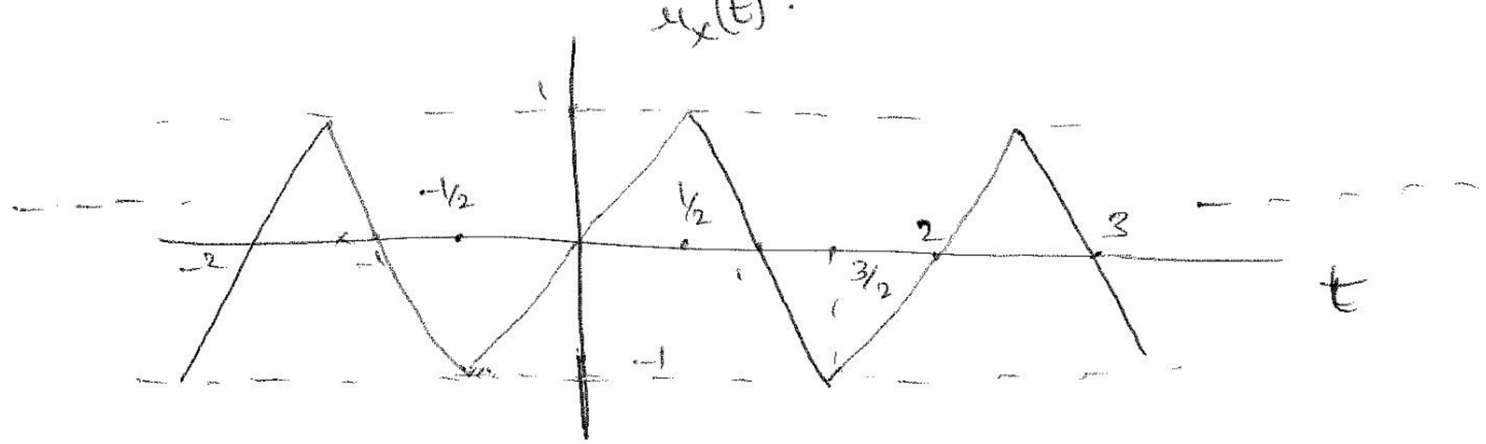
$$= -t + 3/2$$ (Simply shifts by 1)



(sign reversed because of $(-1)^n$).

III^{ly} for other n

by adding all these we get

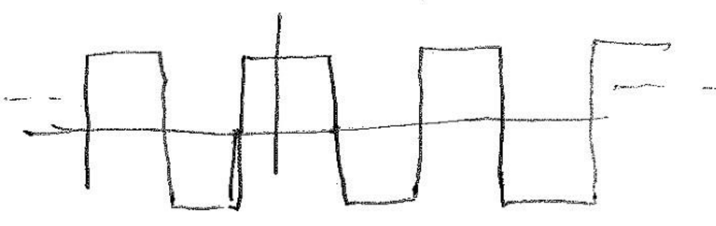


(b) $\because \mu_x(t)$ depends on t it is not WSS

(c) $\because$ X is not WSS it is not stationary

(d) consider time average of mean

$$\bar{\mu}_x(t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t-d)\, dt$$



over infinite intervals mean $= 0$

$$\bar{\mu}_x(t) = 0$$

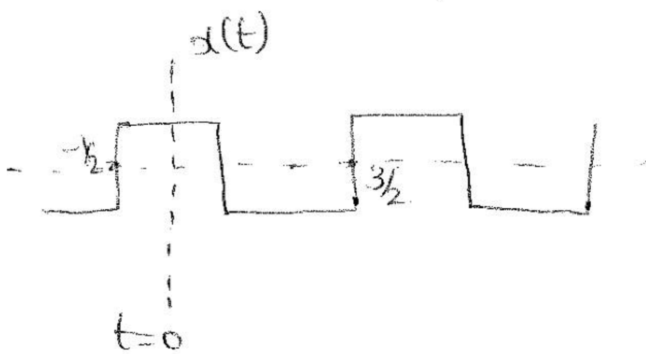
$\therefore$ X is not ergodic in mean.

from soln to 5.205 and 5.206(a)

we see that ensemble autocorrelation is same as time averaged autocorrelation.

So x is ergodic in autocorrelation.

Pdf approach to find mean



$x(t)$ at $t=t_1$ can take either $+1$ or -1.

$P\{x(t_1)=1\}$

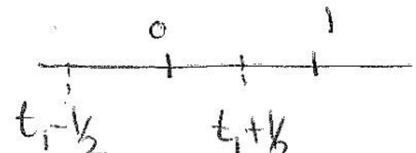
$= P\{x(t_1-D)=1\}$ consider $-\frac{1}{2} < t_1 \leq \frac{3}{2}$

$x(t)$ is periodic with period 2.

$= P\{\frac{-1}{2} \leq t_1-D \leq \frac{1}{2}\}$

$= P\{\frac{-1}{2} \leq D-t_1 \leq \frac{1}{2}\}$

$= P\{t_1-\frac{1}{2} \leq D \leq t_1+\frac{1}{2}\}$



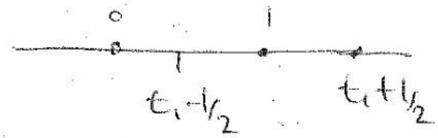
if $0 \leq t_1+\frac{1}{2} \leq 1$

$-\frac{1}{2} \leq t_1 \leq \frac{1}{2}$

then $P\{t_1-\frac{1}{2} \leq D \leq t_1+\frac{1}{2}\} = \left(t_1+\frac{1}{2}-0\right)\cdot 1$

$= t_1+\frac{1}{2}$

if $\quad 0 \leq t_1 - \frac{1}{2} \leq 1$



$\Rightarrow \frac{1}{2} \leq t_1 \leq \frac{3}{2}$

$$P\{t_1 - \tfrac{1}{2} \leq D \leq t_1 + \tfrac{1}{2}\} = \left[1 - \left(t_1 - \tfrac{1}{2}\right)\right] \cdot 1$$

$$= \tfrac{3}{2} - t_1$$

$$\therefore P\{x(t_1) = 1\} = \begin{cases} t_1 + \frac{1}{2} & -\frac{1}{2} \leq t_1 \leq \frac{1}{2} \\ \frac{3}{2} - t_1 & \frac{1}{2} \leq t_1 \leq \frac{3}{2} \end{cases}$$

$$E\{x(t_1)\} = 1 \cdot P\{x(t_1) = 1\} - 1 \; P\{x(t_1) = -1\}$$

$$= P\{x(t_1) = 1\} - \{1 - P\{x(t_1) = 1\}\}$$

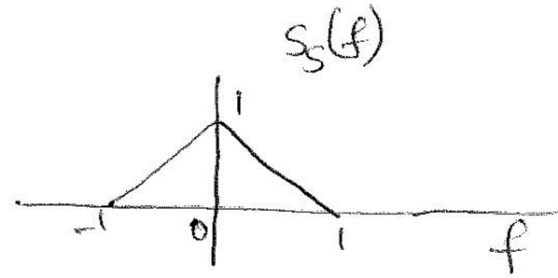
$$= 2P\{x(t_1) = 1\} - 1$$

$$\therefore \quad E\{x(t_1)\} = \begin{cases} 2t_1 - 1 & -\frac{1}{2} \leq t_1 \leq \frac{1}{2} \\ 3 - 2t_1 & \frac{1}{2} \leq t_1 \leq \frac{3}{2} \end{cases}$$

$\because$ $x(t)$ is periodic this results in same answer as previous.

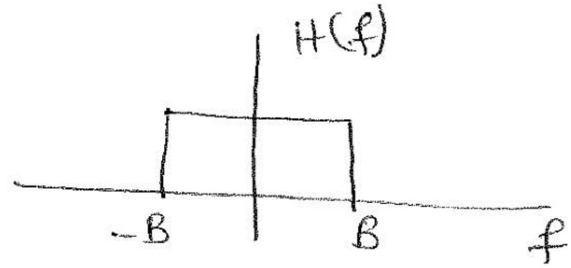
5.31

$$S_s(f) = (1-|f|)\, I_{[-1,1]}(f)$$



$$y(t) = s(t) + n(t)$$

$$S_n(f) \equiv 0.001.$$



(a) SNR at filter i/p

$$= \frac{\text{signal power}}{\text{noise power}}$$

$$\text{signal power} = \int_{-1}^{1} S_s(f)\, df = \frac{1}{2} \times 2 \times 1 = 1$$

$$\text{noise power} = 2 \times 0.001 = 0.002$$

$$\text{SNR} = \frac{1}{2 \times 10^{-3}} = 500.$$

(b) for $B = 1$

$$\text{SNR}_{o/p} = \text{SNR}_{i/p}$$

for $B = \frac{1}{2}$

$$\text{Signal power at the o/p} = 2\int_0^{1/2} (1-f)\, df$$

$$= \frac{3}{4}$$

noise power = 10^{-3}

$\therefore$ $SNR_{o/p} = \frac{3}{4} \times 1000$

$= 750$

$\therefore$ SNR is better for $B = 1/2$.

Problem 6.1
(a) Let $Z = (y * h)(t_0)$, where $h(t) = s(-t)$. Then $Z \sim N(m, v^2)$ if 1 sent, and $Z \sim N(0, v^2)$ if 0 sent, where

$$v^2 = \sigma^2 ||h||^2 = \sigma^2 4 \int_0^1 t^2 dt = \frac{4}{3}\sigma^2$$

$$m = (s * h)(t_0) = \int s(t)s(t - t_0)dt = \int_0^1 t(1-t)dt = \frac{1}{6}$$

Thus, for the ML decision rule

$$P_e = Q\left(\frac{|m|}{2v}\right) = Q\left(\frac{1}{8}\sqrt{\frac{E_b}{N_0}}\right)$$

using the fact that we must have

$$\frac{|m|}{2v} = a\sqrt{\frac{E_b}{N_0}}$$

(why?) where a is a constant determined by substituting $E_b = \frac{1}{2}||s||^2 = \frac{2}{3}$ and $N_0 = 2\sigma^2$.
(b) We can improve the error probability by sampling at $t_0 = 0$. We then have $m = ||s||^2 = \frac{4}{3}$, while v^2 is as before. This gives, reasoning as in (a),

$$P_e = Q\left(\frac{|m|}{2v}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

the usual formula for the performance of optimal reception of on-off keying in AWGN.
(c) For $h(t) = I_{[0,2]}$, we again have the same model for the decision statistic $Z = y * h(t_0)$, but with $v^2 = \sigma^2||h||^2 = 2\sigma^2$, and $m = (s * h)(t_0)$. The performance improves with $|m|$, which is maximized at $t_0 = 2$ ($m = 1$) or $t_0 = 4$ ($m = -1$). We therefore get that, for the ML decision rule,

$$P_e = Q\left(\frac{|m|}{2v}\right) = Q\left(\sqrt{\frac{3E_b}{8N_0}}\right)$$

(d) Note that we can approximate the matched filter $s(-t)$ using linear combinations of two shifted versions of $h(t) = I_{[0,2]}$, by approximating triangles by rectangles. That is, the matched filter shape is approximated as $\tilde{h}(t) = h(t+2) - h(t+4)$. Thus, we can use the decision statistic

$$Z = (y * \tilde{h})(0) = (y * h)(2) - (y * h)(4)$$

We now have $Z \sim N(m, v^2)$ if 1 sent, and $Z \sim N(0, v^2)$ if 0 sent, where

$$v^2 = \sigma^2||\tilde{h}||^2 = 4\sigma^2$$

$$m = (s * \tilde{h})(0) = 2$$

As before, we can enforce the scaling with $\frac{E_b}{N_0}$ to get

$$P_e = Q\left(\frac{|m|}{2v}\right) = Q\left(\sqrt{\frac{3E_b}{4N_0}}\right)$$

3 dB better than the performance in (c), and $10 \log_{10} \frac{4}{3} = 1.25$ dB worse than the optimal receiver in (b).

Problem 6.2 (a) We have

$$p(y|1) = \frac{e^{-(y-2)^2/18}}{\sqrt{18\pi}}$$

$$p(y|0) = \frac{e^{-(y+2)^2/8}}{\sqrt{8\pi}}$$

The optimal rule consists of comparing the log likelihood ratio to a threshold. The log likelihood ratio can be written as

$$\log L(y) = \log p(y|1) - \log p(y|0) = \left\{ -(y-2)^2/18 - \frac{1}{2}\log(18\pi) \right\} - \left\{ -(y+2)^2/8 - -\frac{1}{2}\log(8\pi) \right\}$$

which has the desired quadratic form. Simplifying, we get

$$\log L(y) = \frac{5y^2 + 52y + 20}{72} - \log 3/2$$

(b) For $\pi_0 = 1/4$, we compare $\log L(y)$ to the threshold $\log \frac{\pi_0}{\pi_1} = -\log 3$. Simplifying, we obtain the MPE rule

$$5y^2 + 52y \underset{H_0}{\overset{H_1}{\gtrless}} -72\log 2 - 20 = -69.9$$

(c) The conditional error probability is given by

$$P_{e|0} = P[5Y^2 + 52Y + 69.9 > 0|H_0]$$

The roots of the quadratic $5y^2 + 52y + 69.9$ are $\alpha_1 = -1.6$ and $\alpha_2 = -8.8$, hence we can write the conditional error probability as

$$\begin{aligned} P_{e|0} &= P[(Y - \alpha_1)(Y - \alpha_2) > 0|H_0] = P[Y > -1.6|H_0] + P[Y < -8.8|H_0] \\ &= Q\left(\tfrac{-1.6-(-2)}{\sqrt{4}}\right) + \Phi\left(\tfrac{-8.8-(-2)}{\sqrt{4}}\right) = Q(0.2) + \Phi(-3.4) \\ &= Q(0.2) + Q(3.4) \end{aligned}$$

Problem 6.3: The conditional densities and decision regions are sketched in Figure 1 (not to scale). The threshold γ satisfies $p(\gamma|1) = p(\gamma|0)$, or

$$\frac{1}{\sqrt{2\pi}} e^{-\gamma^2/2} = \frac{1}{4}$$

which yields $\gamma = \sqrt{3\log(2) - \log(\pi)} \approx 0.97$.

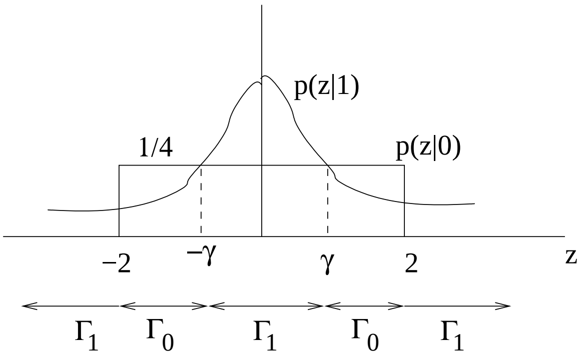


Figure 1: Conditional densities and decision regions for Problem 6.3.

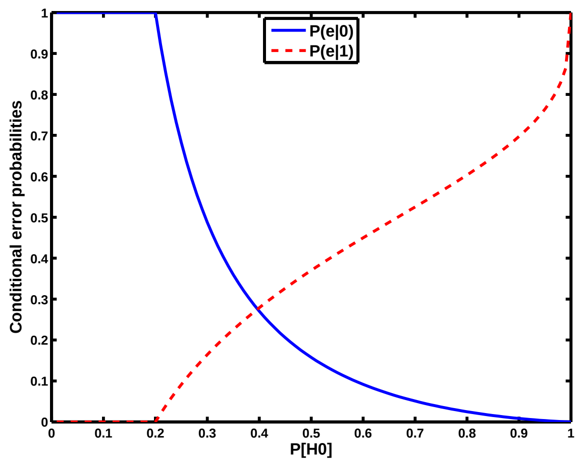


Figure 3: Conditional error probabilities for the MAP rule (as a function of π_0) for Problem 6.5.

Problem 6.5 The log likelihood ratio is given by

$$\log L(y) = \frac{\mu_1 e^{-\mu_1 y}}{\mu_0 e^{-\mu_0 y}} = (\mu_0 - \mu_1)y + \log \frac{\mu_1}{\mu_0}$$

where $\mu_1 = 1/4$, $\mu_0 = 1$. Plugging in, we obtain $\log L(y) = \frac{3}{4}y - \log 4$. The MAP/MPE rule

$$\log L(y) \underset{H_0}{\overset{H_1}{\gtrless}} \log \frac{\pi_0}{\pi_1} = \log \frac{\pi_0}{1 - \pi_0}$$

therefore simplifies to

$$y \underset{H_0}{\overset{H_1}{\gtrless}} \frac{4}{3} \log \frac{4\pi_0}{1 - \pi_0} = \gamma_{MAP} \quad \textbf{MAPrule}$$

Since $y \geq 0$, the MAP rule always says H_1 if $\gamma_{MAP} \leq 0$, which happens if the argument of the log is less than (or equal to) one: $\frac{4\pi_0}{1-\pi_0} \leq 1$, or $\pi_0 \leq \frac{1}{5}$.
(b) From (a), we see that for $\pi_0 \leq \frac{1}{5}$, the conditional error probabilities are given by $P_{e|0} = 1$ and $P_{e|1} = 0$, since the MAP rule always says H_1.
For $\pi_0 > \frac{1}{5}$, we have $\gamma_{MAP} > 0$, and the conditional error probabilities are given by

$$P_{e|0} = P[Y > \gamma_{MAP} | H_0] = e^{-\mu_0 \gamma_{MAP}} = e^{-\frac{4}{3} \log \frac{4\pi_0}{1-\pi_0}} = \left(\frac{1 - \pi_0}{4\pi_0} \right)^{4/3}$$

$$P_{e|1} = P[Y \leq \gamma_{MAP}|H_1] = 1 - e^{-\mu_1 \gamma_{MAP}} = 1 - e^{-\frac{1}{3}\log\frac{4\pi_0}{1-\pi_0}} = 1 - \left(\frac{1-\pi_0}{4\pi_0}\right)^{1/3}$$

These are plotted as a function of π_0 in Figure 3.