ECE 146B : Solutions to Problem Set 6

Problem 6.17

(a) Signal space representations with respect to the given orthonormal basis are:
Signal Set A: $\mathbf{s}_1 = (1,0,0,0)^T$, $\mathbf{s}_2 = (0,1,0,0)^T$, $\mathbf{s}_3 = (0,0,0,1)^T$ and $\mathbf{s}_4 = (0,0,0,1)^T$
Signal Set B: $\mathbf{s}_1 = (1,0,0,1)^T$, $\mathbf{s}_2 = (0,1,1,0)^T$, $\mathbf{s}_3 = (1,0,1,0)^T$ and $\mathbf{s}_4 = (0,1,0,1)^T$
(b) For Signal Set A, the pairwise distance between any two points satisfies $d^2 = d^2_{min} = 2$, while the energy per symbol is $E_s = 1$. Thus, $E_b = E_s/(\log_2 4) = 1/2$, and $d^2_{min}/E_b = 4$. The union bound on symbol error probability is therefore given by

$$P_e(\text{signal set A}) \le 3Q\left(\sqrt{d^2_{min}/E_b}\sqrt{E_b/2N_0}\right) = 3Q\left(\sqrt{2E_b/N_0}\right)$$

For signal set B, each signal has one neighbor at distance given by $d_1^2 = 4$ and two at distance given by $d_2^2 = d^2_{min} = 2$. The energy per symbol is $E_s = 2$, so that $E_b = 1$. The union bound is given by

$$\begin{aligned} P_e(\text{signal set B}) &\le 2Q\left(\sqrt{d_2^2/E_b}\sqrt{E_b/2N_0}\right) + Q\left(\sqrt{d_1^2/E_b}\sqrt{E_b/2N_0}\right) \\ &= 2Q\left(\sqrt{E_b/N_0}\right) + Q\left(\sqrt{2E_b/N_0}\right) \end{aligned}$$

(c) For exact analysis of error probability for Signal Set B, suppose that the received signal in signal space is given by $\mathbf{Y} = (Y_1, Y_2, Y_3, Y_4)$. Condition on the signal $\mathbf{s}_0 = (1,0,0,1)^T$ being sent. Then

$$Y_1 = 1 + N_1,\ Y_2 = N_2,\ Y_3 = N_3,\ Y_4 = 1 + N_4$$

where $N_1, ..., N_4$ are i.i.d. $N(0, \sigma^2)$ random variables. A correct decision is made if $\langle \mathbf{Y}, \mathbf{s}_0 \rangle > \langle \mathbf{Y}, \mathbf{s}_k \rangle$, $k = 1,2,3$. These inequalities can be written out as

$$\begin{aligned} \langle \mathbf{Y}, \mathbf{s}_1 \rangle = Y_1 + Y_4 > \langle \mathbf{Y}, \mathbf{s}_2 \rangle = Y_2 + Y_3 \\ \langle \mathbf{Y}, \mathbf{s}_1 \rangle = Y_1 + Y_4 > \langle \mathbf{Y}, \mathbf{s}_3 \rangle = Y_1 + Y_3 \\ \langle \mathbf{Y}, \mathbf{s}_1 \rangle = Y_1 + Y_4 > \langle \mathbf{Y}, \mathbf{s}_4 \rangle = Y_2 + Y_4 \end{aligned}$$

The second and third inequalities give $Y_1 > Y_2$ and $Y_4 > Y_3$, and imply the first inequality. Thus, the conditional probability of correct reception is given by

$$P_{c|1} = P[Y_1 > Y_2 \text{ } and \text{ } Y_4 > Y_3 | s_0 \text{ sent}] = P[Y_1 > Y_2 | s_0 \text{ sent}] P[Y_4 > Y_3 | s_0 \text{ sent}]$$

since Y_k are conditionally independent given the transmitted signal. Noting that $Y_1 - Y_2$ and $Y_4 - Y_3$ are independent $N(1, 2\sigma^2)$, we have

$$P_{e|1} = 1 - Pc|1 = 1 - \left(1 - Q\left(\frac{1}{\sqrt{2\sigma^2}}\right)\right)^2 = 2Q\left(\frac{1}{\sqrt{2\sigma^2}}\right) - Q^2\left(\frac{1}{\sqrt{2\sigma^2}}\right)$$

Setting $\frac{1}{\sqrt{2\sigma^2}} = a\sqrt{E_b/N_0}$, with $E_b = 1$ and $N_0 = 2\sigma^2$, we have $a = 1$. Further, by symmetry, we have $P_e = P_{e|1}$. We therefore obtain that

$$P_e = 2Q\left(\sqrt{E_b/N_0}\right) - Q^2\left(\sqrt{E_b/N_0}\right), \quad \textbf{exact error probability for signal set B}$$

Problem 6.19

(a) For 8-PSK, the symbol energy $E_s = R^2$. For the QAM constellations,

$$\begin{aligned} E_s(QAM1) &= \text{I} - \text{channel avg energy } + \text{ Q} - \text{channel avg energy} \\ &= \tfrac{4}{8}[(d_1/2)^2 + (3d_1/2)^2] + (d_1/2)^2 = \tfrac{3}{2}d_1^2 \end{aligned}$$

$$E_s(QAM2) = \mathrm{I-channel\ avg\ energy} \; + \; \mathrm{Q-channel\ avg\ energy}$$
$$= [\tfrac{6}{8}(d_2/2)^2 + \tfrac{2}{8}(3d_2/2)^2] + [\tfrac{4}{8}d_2^2 + \tfrac{4}{8}0] = \tfrac{5}{4}d_2^2$$

where $d_1 = d_{min}^{(1)}$ and $d_2 = d_{min}^{(2)}$. Since the number of constellation points is the same for each constellation, equal energy per bit corresponds to equal symbol energy, and occurs when

$$R = \sqrt{3/2}\; d_1, \qquad d_2 = \sqrt{6/5}\; d_1$$

(b) From (a), for the same E_b,

$$d_{min}^{8PSK} = 2R\sin\frac{\pi}{8} = .937d_1 < d_1 < d_2$$

so that, in the high SNR regime, we expect that

$$P_e(8PSK) > P_e(QAM1) > P_e(QAM2)$$

(c) Each symbol is assigned 3 bits. Since 8-PSK and QAM1 are regular constellations with at most 3 nearest neighbors per point, we expect to be able to Gray code. However, QAM2 has some points with 4 nearest neighbors, so we definitely cannot Gray code it. We can, however, try to minimize the number of bit changes between neighbors. Figure **??** shows Gray codes for 8-PSK and QAM1. The labeling for QAM2 is arbitrarily chosen to be such that points with 3 or fewer nearest neighbors are Gray coded.

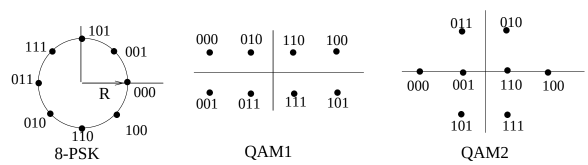

Figure 1: Bit mappings for Problem 6.19.

(d) For Gray coded 8-PSK and QAM1, a symbol error due to decoding to a nearest neighbor causes only 1 out of the 3 bits to be in error. Hence, using the nearest neighbors approximation, $P[bit\ error] \approx \frac{1}{3}P[symbol\ error]$. On the other hand, $P[symbol\ error] \approx \bar{N}_{d_{min}} Q(\frac{d_{min}}{2\sigma})$, where $\bar{N}_{d_{min}}$ is the average number of nearest neighbors. While the latter is actually an upper bound on the symbol error probability (the nearest neighbors approximation coincides with the intelligent union bound in these cases), the corresponding expression for the bit error probability need not be an upper bound (why?).
For 8-PSK, $d_{min} = 2R\sin\frac{\pi}{8}$ and $E_s = 3E_b = R^2$. Plugging in $\sigma^2 = N_0/2$ and $\bar{N}_{d_{min}} = 2$, we obtain

$$P[bit\ error]_{8PSK} \approx \frac{2}{3}Q\left(\sqrt{\frac{3(1-\frac{1}{\sqrt{2}})E_b}{N_0}}\right)$$

For QAM1, $\bar{N}_{d_{min}} = 5/2$ and $E_s = 3E_b = 3d_1^2/2$, so that

$$P[bit\ error]_{QAM1} \approx \frac{5}{6}Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

For QAM2, we need to make nearest neighbors approximation specifically for bit error probability. Let $N(\mathbf{b})$ total number of bit changes due to decoding into nearest neighbors when symbol $\mathbf{b}$ is sent. For the labeling given, these are specified by Table 1. Let $\bar{N}_{bit} = \frac{1}{8}\sum_{\mathbf{b}} N(\mathbf{b}) = 11/4$ denote

b	000	001	010	011	100	101	110	111
$N(\mathbf{b})$	1	6	2	2	1	2	6	2

Table 1: Number of bit changes due to decoding into nearest neighbors for each symbol of QAM2

the average number of bits wrong due to decoding into nearest neighbors. Since each signal point is labeled by 3 bits, the nearest neighbors approximation for the bit error probability is now given by

$$P[bit\ error]_{QAM2} \approx \frac{1}{3}\bar{N}_{bit} Q\left(\frac{d_{min}}{2\sigma}\right) = \frac{11}{12} Q\left(\sqrt{\frac{6E_b}{5N_0}}\right)$$

(We can reduce the factor from 11/12 to 5/6 using an alternative labeling.)

While we have been careful about the factors multiplying the Q function, these are insignificant at high SNR compared to the argument of the Q function, and are often ignored in practice. For $E_b/N_0 = 15dB$, the preceding approximations give the values shown in Table 2. The ordering of error probabilities is as predicted in (b).

Constellation	8PSK	QAM1	QAM2
P[bit error]	4.5×10^{-8}	7.8×10^{-9}	3.3×10^{-10}

Table 2: Bit error probabilities at E_b/N_0 of 15 dB

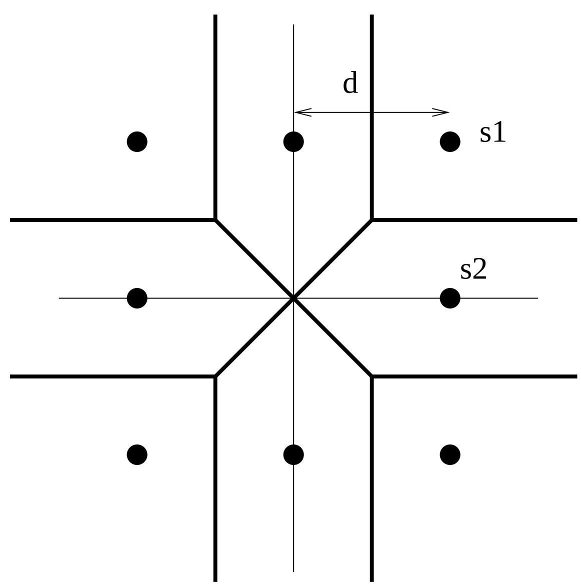

Figure 2: ML decision regions for Problem 6.20(a).

Problem 6.20:
(a) For $R/r = \sqrt{2}$. the constellation takes the rectangular shape shown in Figure 2.
(b) We compute the intelligent union bound on the error probability conditioned on two typical signal points shown in Figure **??**.

$$P(e|s1) \leq 2Q\left(\frac{d}{2\sigma}\right) = Q\left(\sqrt{\frac{d^2}{E_b}}\sqrt{\frac{E_b}{2N_0}}\right)$$

We now compute $\eta = \frac{d^2}{E_b}$ to express the result in terms of E_b/N_0. The energy per symbol is

$$E_s = 2 \times \mathrm{I-channel\ energy} = 2\left(\frac{6}{8} \times d^2 + \frac{2}{8} \times 0\right) = 3d^2/2$$

Since $E_b = E_s/\log_2 8$, we have $\eta = \frac{d^2}{E_b} = 2$, which yields

$$P(e|s1) \leq 2Q\left(\sqrt{E_b/N_0}\right)$$

Similarly,

$$P(e|s2) \leq 2Q\left(\frac{d}{2\sigma}\right) + 2Q\left(\frac{\sqrt{2}d}{2\sigma}\right) = 2Q\left(\sqrt{E_b/N_0}\right) + 2Q\left(\sqrt{2E_b/N_0}\right)$$

The average error probability is given by

$$P_e = \frac{1}{2}\left(P(e|s1) + P(e|s2)\right) \leq 2Q\left(\sqrt{E_b/N_0}\right) + Q\left(\sqrt{2E_b/N_0}\right)$$

(c) We wish to design the parameter $x = R/r \geq 1$ to optimize the power efficiency, which is given by

$$\eta = \min(d_1^2, d_2^2)/E_b$$

where d_1 and d_2 shown in Figure 3 are given by

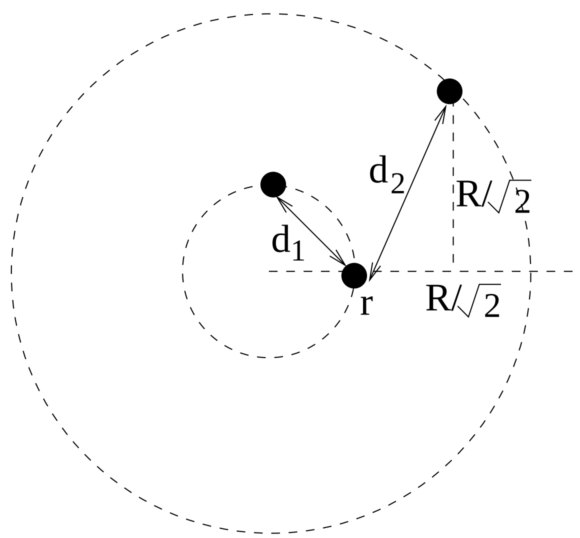

Figure 3: Signal space geometry for Problem 6.20(c).

$$d_1^2 = 2r^2$$

$$d_2^2 = \left(R/\sqrt{2}\right)^2 + \left(R/\sqrt{2} - r\right)^2 = R^2 + r^2 - \sqrt{2}Rr = r^2(1 + x^2 - \sqrt{2}x)$$

The energy per symbol is $E_s = (r^2 + R^2)/2$, so that

$$E_b = E_s/\log_2 8 = (r^2 + R^2)/6 = r^2(1 + x^2)/6$$

It is easy to check the following: $\eta_1 = d_1^2/E_b$ decreases with x, and $\eta_2 = d_2^2/E_b$ increases with x (for $x \geq 1$). Furthermore, at $x = 1$, $\eta_1 > \eta_2$. This shows that the optimal $x \geq 1$ corresponds to $\eta_1 = \eta_2$, i.e., $d_1^2 = d_2^2$. That is,

$$2r^2 = r^2(1 + x^2 - \sqrt{2}x)$$

or

$$x^2 - \sqrt{2}x - 1 = 0$$

The solution to this in the valid range $x \geq 1$ is given by

$$x = R/r = \frac{\sqrt{2} + \sqrt{6}}{2} \approx 1.93$$

The corresponding power efficiency is given by

$$\eta = 2(3 - \sqrt{3}) \approx 2.54$$

which is about 1 dB better than the power efficiency of $\eta = 2$ for $x = \sqrt{2} \approx 1.41$ in part (a).

Problem 6.24:

(a) Letting N_c, N_s denote the iid $N(0, \sigma^2)$ noise components in the I and Q directions. As in the standard QPSK analysis, we may use symmetry to condition on a particular symbol being sent to compute the error and erasure probabilities.

$$p = P[error] = P[error|{-1}{-j}\ sent] \leq P[N_c > (1{+}\alpha)d/2 \text{ or } N_s > (1{+}\alpha)d/2] \leq 2P[N_c > (1{+}\alpha)d/2]$$

so that

$$p \leq 2Q\left((1+\alpha)\sqrt{\frac{2E_b}{N_0}}\right) \tag{1}$$

Similarly,

$$\begin{aligned} q = P[erasure] &= P[erasure| - 1 - jsent] \\ &= P[(1-\alpha)d/2 < N_c < (1+\alpha)d/2 \text{ or } (1-\alpha)d/2 < N_s < (1+\alpha)d/2] \\ &\leq 2P[(1-\alpha)d/2 < N_c < (1+\alpha)d/2] \end{aligned}$$

so that

$$q \leq 2\left\{Q\left((1-\alpha)\sqrt{\frac{2E_b}{N_0}}\right) - Q\left((1+\alpha)\sqrt{\frac{2E_b}{N_0}}\right)\right\} \tag{2}$$

(b) For the exact probabilities, conditioning on $-1-j$ as before, we find

$$\begin{aligned} p &= P[N_c > (1+\alpha)d/2, N_s < (1-\alpha)d/2] \ + \ P[N_c < (1-\alpha)d/2, N_s > (1+\alpha)d/2] \\ &\quad + \ P[N_c > (1+\alpha)d/2, N_s > (1+\alpha)d/2] \\ &= 2Q\left((1+\alpha)\sqrt{\tfrac{2E_b}{N_0}}\right)\left[1 - Q\left((1-\alpha)\sqrt{\tfrac{2E_b}{N_0}}\right)\right] + Q^2\left((1+\alpha)\sqrt{\tfrac{2E_b}{N_0}}\right) \end{aligned}$$

Similarly,

$$\begin{aligned} q &= P[\{(1-\alpha)d/2 < N_c < (1+\alpha)d/2\} \text{ or } \{(1-\alpha)d/2 < N_s < (1+\alpha)d/2\}] \\ &= 2P[\{(1-\alpha)d/2 < N_c < (1+\alpha)d/2\}] - P^2[\{(1-\alpha)d/2 < N_c < (1+\alpha)d/2\}] \end{aligned}$$

That is,

$$q = 2q_1 - q_1^2$$

where

$$q_1 = Q\left((1-\alpha)\sqrt{\frac{2E_b}{N_0}}\right) - Q\left((1+\alpha)\sqrt{\frac{2E_b}{N_0}}\right)$$

(c) Setting $q = 2p$ using the bounds (1)-(2), we obtain

$$Q\left((1-\alpha)\sqrt{\frac{2E_b}{N_0}}\right) = 3Q\left((1+\alpha)\sqrt{\frac{2E_b}{N_0}}\right)$$

Solving numerically for $E_b/N_0 = 10^{4/10}$ (i.e., 4 dB), we obtain $\alpha = .095$.

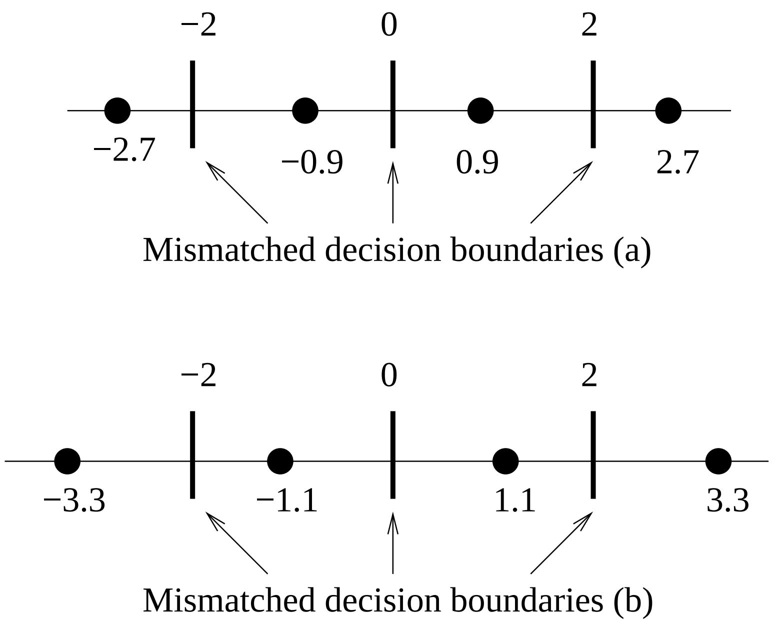

Figure 4: Signal points and decision regions for Problem 6.28

Problem 6.28
(a) From Figure 4, we can evaluate the error probabilities conditioned on the inner and outer points as follows:

$$P(e|inner) = Q(0.9/\sigma) + Q(1.1/\sigma)$$

$$P(e|outer) = Q(0.7/\sigma)$$

Each of the arguments must take the form $a\sqrt{E_b/N_0}$. For the scaling shown in the figure, $E_s = (0.9^2 + 2.7^2)/2$ and $E_b = E_s/\log_2 4 = 0.9^2(5/2)$. This yields that

$$P(e|inner) = Q\left(\sqrt{4E_b/5N_0}\right) + Q\left(\frac{11}{9}\sqrt{4E_b/5N_0}\right)$$

$$P(e|outer) = Q\left(\frac{7}{9}\sqrt{4E_b/5N_0}\right)$$

Averaging, we get

$$P_e = \frac{1}{2}Q\left(\frac{7}{9}\sqrt{4E_b/5N_0}\right) + \frac{1}{2}Q\left(\sqrt{4E_b/5N_0}\right) + \frac{1}{2}Q\left(\frac{11}{9}\sqrt{4E_b/5N_0}\right)$$

At high SNR, we have a degradation of $20\log_{10}(7/9) = -2.2$ dB due to the mismatch.
(b) Proceeding exactly as before, we have

$$P(e|inner) = Q(1.1/\sigma) + Q(0.9/\sigma)$$

$$P(e|outer) = Q(1.3/\sigma)$$

and $E_b = 1.1^2(5/2)$. This yields

$$P(e|inner) = Q\left(\sqrt{4E_b/5N_0}\right) + Q\left(\frac{9}{11}\sqrt{4E_b/5N_0}\right)$$

$$P(e|outer) = Q\left(\frac{13}{11}\sqrt{4E_b/5N_0}\right)$$

with

$$P_e = \frac{1}{2}Q\left(\frac{9}{11}\sqrt{4E_b/5N_0}\right) + \frac{1}{2}Q\left(\sqrt{4E_b/5N_0}\right) + \frac{1}{2}Q\left(\frac{13}{11}\sqrt{4E_b/5N_0}\right)$$

At high SNR, we have a degradation of $20\log_{10}(9/11) = -1.74$ dB due to the mismatch.
(c) The AGC circuit is overestimating the received signal power, and therefore scaling the signal points down excessively.