ECE 146B: Solutions to Problem Set 4

5.26 $$x(t) = \sum_{n=-\infty}^{\infty} (-1)^n \, p(t-n)$$

$$X(t) = x(t-D). \qquad D \sim \text{uniform } [0,1].$$

(a)
$$R_{XX}(t_1,t_2) = E\{X(t_1)X(t_2)\}$$
$$= E\{x(t_1-D)\,x(t_2-D)\}$$
$$= E\left\{\sum_{n=-\infty}^{\infty}(-1)^n p(t_1-n-D) \cdot \sum_{m=-\infty}^{\infty}(-1)^m p(t_2-m-D)\right\}$$

$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}(-1)^{n+m} E\{p(t_1-n-D)\,p(t_2-m-D)\}.$$

$$E\{p(t_1-n-D)\,p(t_2-m-D)\} = \int_0^1 p(t_1-n-a)\,p(t_2-m-a)\,da$$

let $t_1-n-a = u \Rightarrow -da = du$

$\Rightarrow t_2-m-a$
$= t_2-m-(t_1-n-u)$
$= (t_2-t_1)-(m-n)+u$

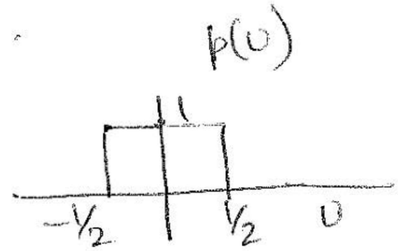
$$= -\int_{t_1-n}^{t_1-n-1} p(u)\,p((t_2-t_1)-(m-n)+u)\,du$$

$$= \int_{t_1-n-1}^{t_1-n} p(u)\,p(u+\underbrace{(t_2-t_1)}_{\tau}-(m-n))\,du$$

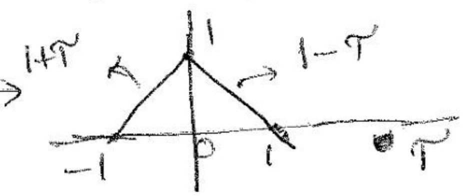
$$\therefore R_{XX}(t_1,t_2) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}(-1)^{m+n}\int_{t_1-n-1}^{t_1-n} p(u)\,p(u+\tau-(m-n))\,du$$

for $m=n$

$$R_{XX}(t_1,t_2) = \sum_{m=-\infty}^{\infty} \int_{t_1-m-1}^{t_1-m} p(u)\, p(u+\tau)\, du$$



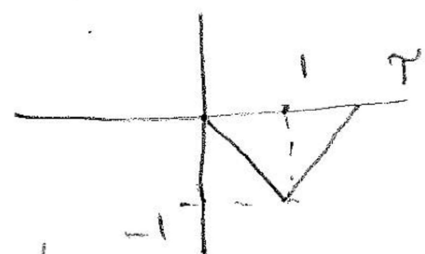
$$= \int_{-\infty}^{\infty} p(u)\, p(u+\tau)\, du \longrightarrow$$



for $(m-n)=1$

$$R_{XX}(t_1,t_2) = \sum_{m=-\infty}^{\infty} (-1)^1 \int_{t_1-(m-1)-1}^{t_1-(m-1)} p(u)\, p(u+\tau-1)\, du$$

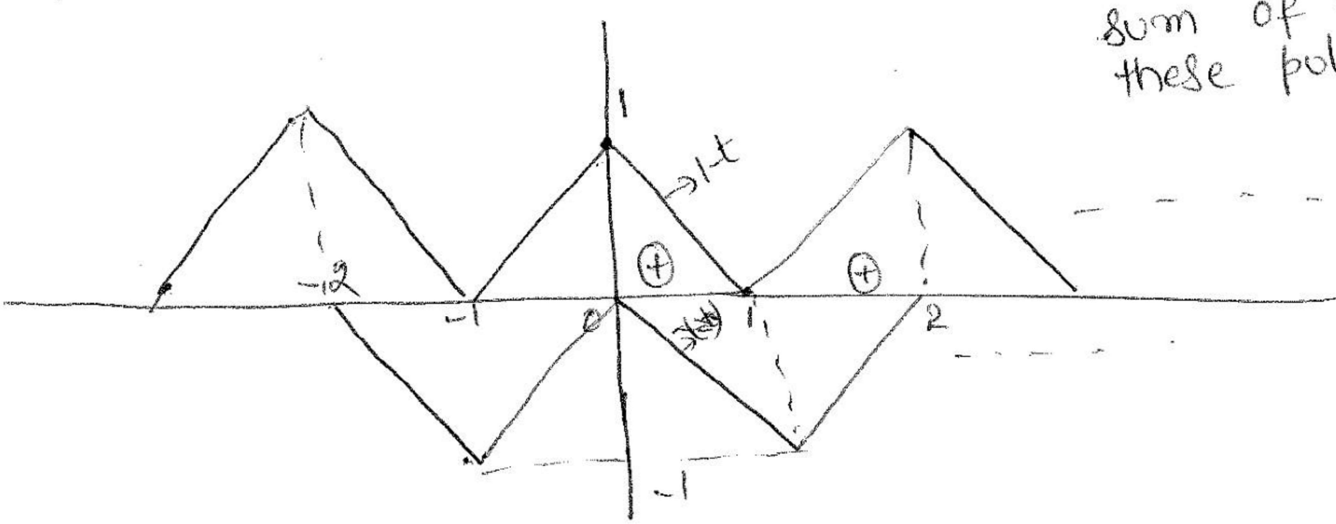
$\longrightarrow$



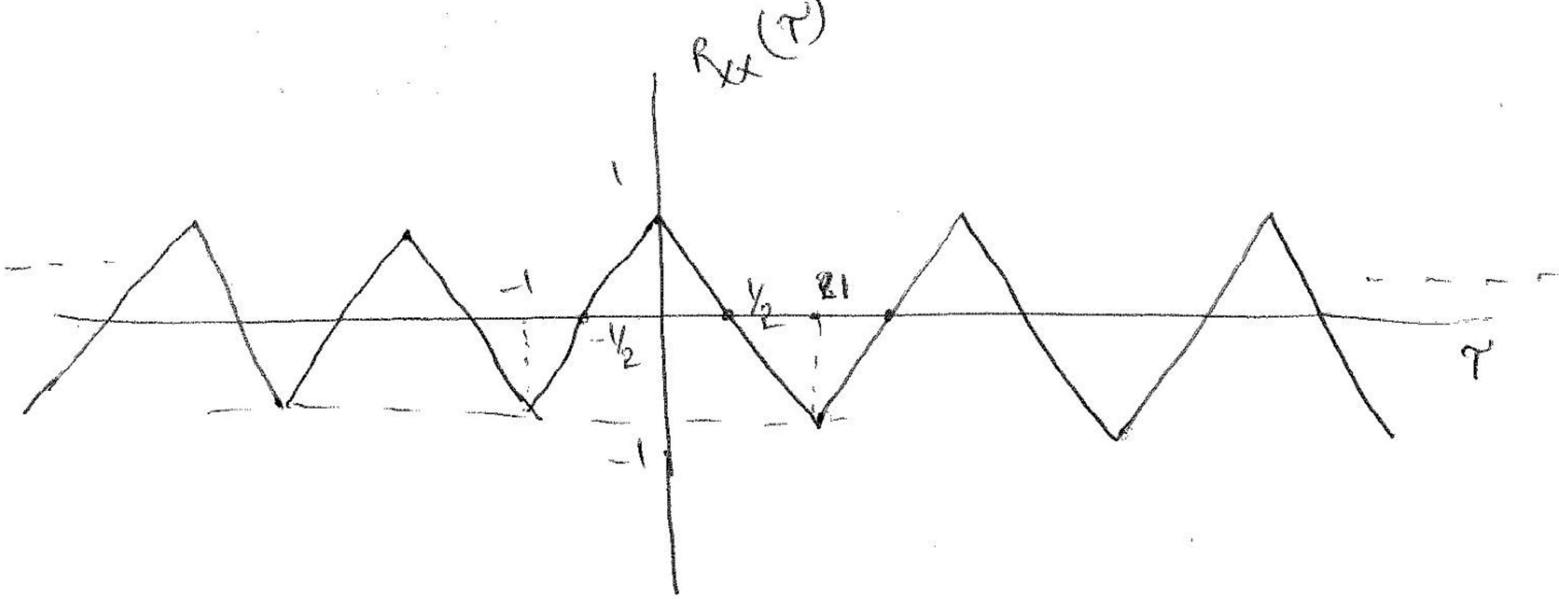
proceeding in a similar way we get

$R_{xx}(t_1,t_2) = R_{xx}(\tau)$

sum of all these pulses



This results in



(b).

$$E\{X(t)\} = E\{x(t-D)\}$$

$$= E\left\{ \sum_{n=-\infty}^{\infty} (-1)^n \, p(t-n-D) \right\}$$

$$= \sum_{n=-\infty}^{\infty} (-1)^n \, E\{p(t-n-D)\}$$

$$= \sum_{n=-\infty}^{\infty} (-1)^n \cdot \int_0^1 p(t-n-a) \cdot 1 \cdot da$$

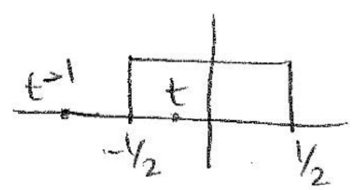
$$= \sum_{n=-\infty}^{\infty} (-1)^n \cdot \int_0^1 I_{[-1/2 \;\; 1/2]}(t-n-a) \; da.$$

let $t-n-a = u$

$$= \sum_{n=-\infty}^{\infty} (-1)^n \int_{t-n}^{t-n-1} I_{[-1/2 \;\; 1/2]}(u) \; (-du)$$

$$= \sum_{n=-\infty}^{\infty} (-1)^n \int_{t-(n+1)}^{t-n} I_{[-1/2 \;\; 1/2]}(u) \; du$$

for $n=0$



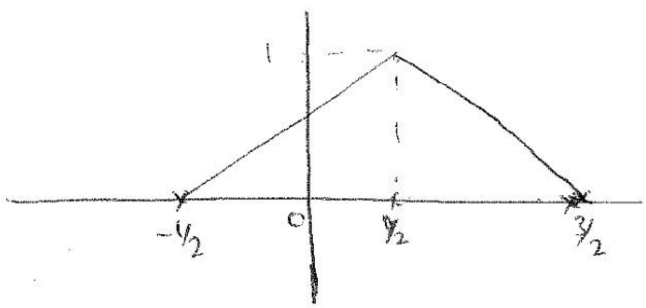
$$= \int_{t-1}^{t} I_{[-1/2 \;\; 1/2]}(u) \; du$$

$$= \int_{-1/2}^{t} du = t + 1/2 \quad \text{if } -1/2 < t < 1/2$$

$$= \int_{t-1}^{1/2} du \qquad \text{if} \qquad 1/2 < t < 3/2$$

$$= 1/2 - (t-1)$$

$$= 3/2 - t$$



(b) $n = 1$

$$\int_{-1/2}^{t-1} dt \qquad -1/2 < t-1 < 1/2$$

$$\qquad 1/2 < t < 3/2$$

$$= t - 1 + 1/2$$

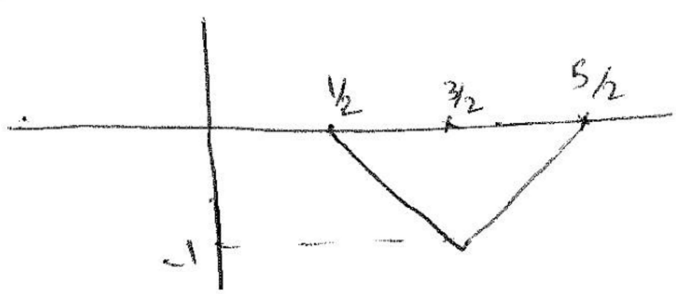
$$= t - 1/2.$$

$$= \int_{t-2}^{1/2} dt \quad . \qquad \text{if} \qquad -1/2 < t-2 < 1/2$$

$$3/2 < t < 5/2$$

$$= 1/2 - (t-2)$$

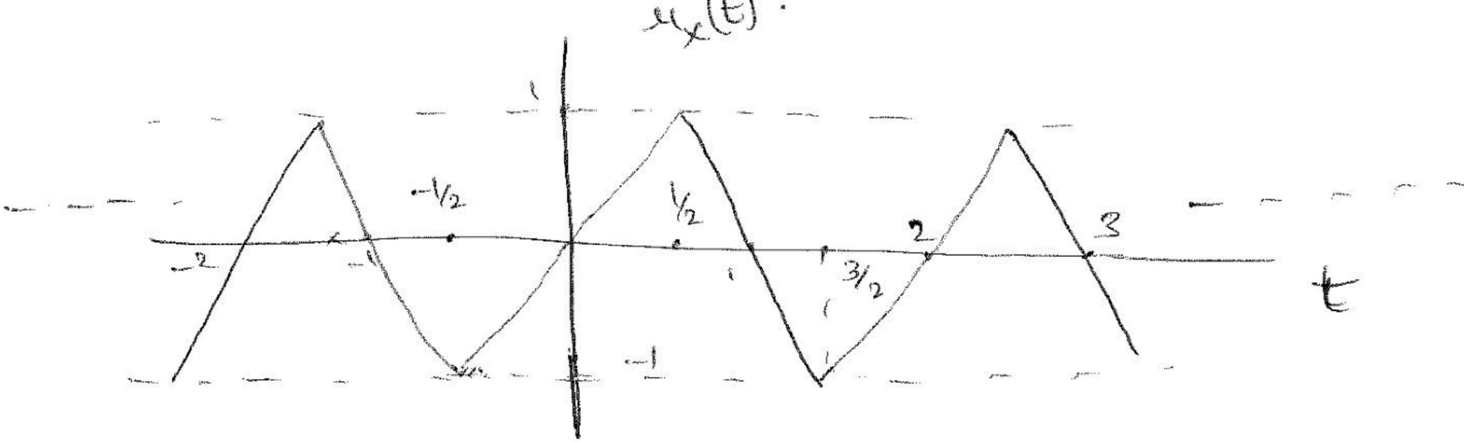
$$= -t + 3/2 \qquad \text{(simply shifts by 1)}$$



(sign reversed because of $(-1)^n$).

III'y for other n

by adding all these we get

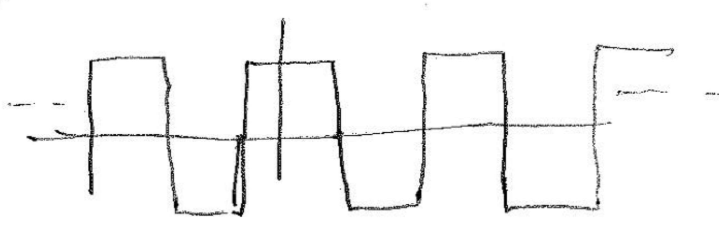


(b) $\because$ $\mu_x(t)$ depends on t it is not WSS

(c) $\because$ X is not WSS it is not stationary

(d) consider time average of mean

$$\bar{\mu}_x(t) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t-d)\, dt$$



over infinite intervals mean = 0

$$\bar{\mu}_x(t) = 0$$

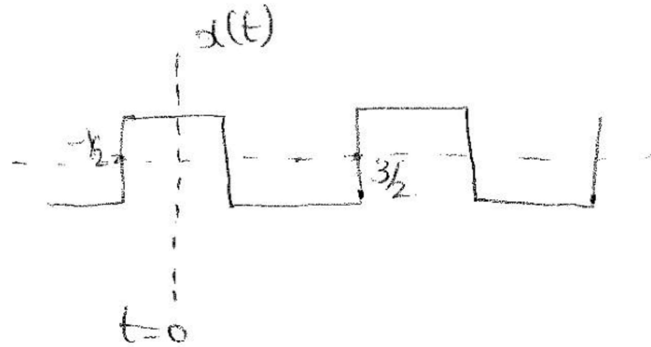
$\therefore$ X is not ergodic in mean.

from sdn to 5.205 and 5.206(a)

we see that ensemble autocorrelation is same as time averaged autocorrelation.

So x is ergodic in autocorrelation.

Pdf approach to find mean



$x(t)$ at $t=t_1$ can take either $+1$ or -1.

$P\{x(t_1) = 1\}$

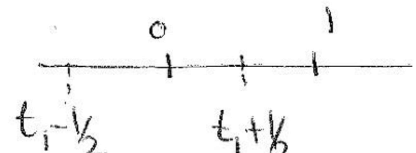
$= P\{x(t_1-D) = 1\}$ consider $-\frac{1}{2} < t_1 \le \frac{3}{2}$

$x(t)$ is periodic with period 2.

$= P\{-\frac{1}{2} \le t_1 - D \le \frac{1}{2}\}$

$= P\{-\frac{1}{2} \le D - t_1 \le \frac{1}{2}\}$

$= P\{t_1 - \frac{1}{2} \le D \le t_1 + \frac{1}{2}\}$



if $0 \le t_1 + \frac{1}{2} \le 1$

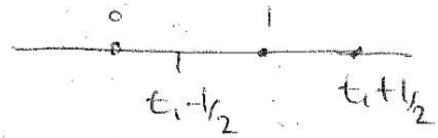
$-\frac{1}{2} \le t_1 \le \frac{1}{2}$

then $P\{t_1 - \frac{1}{2} \le D \le t_1 + \frac{1}{2}\} = \left(t_1 + \frac{1}{2} - 0\right) \cdot 1$

$= t_1 + \frac{1}{2}$

if $0 \le t_1 - \frac{1}{2} \le 1$

$\Rightarrow \frac{1}{2} \le t_1 \le \frac{3}{2}$



$$P\left\{ t_1 - \tfrac{1}{2} \le D \le t_1 + \tfrac{1}{2} \right\} = \left[1 - \left(t_1 - \tfrac{1}{2}\right)\right] \cdot 1$$

$$= \tfrac{3}{2} - t_1$$

$$\therefore P\{x(t_1) = 1\} = \begin{cases} t_1 + \frac{1}{2} & -\frac{1}{2} \le t_1 \le \frac{1}{2} \\ \frac{3}{2} - t_1 & \frac{1}{2} \le t_1 \le \frac{3}{2} \end{cases}$$

$$E\{x(t_1)\} = 1 \cdot P\{x(t_1) = 1\} - 1 \; P\{x(t_1) = -1\}$$

$$= P\{x(t_1) = 1\} - \{1 - P\{x(t_1) = 1\}\}$$

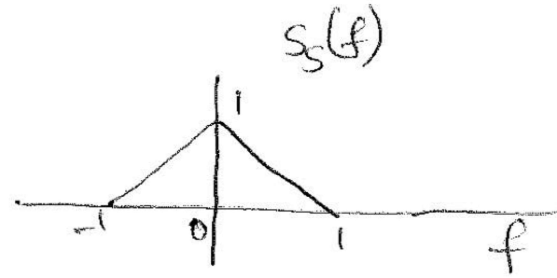
$$= 2P\{x(t_1) = 1\} - 1$$

$$\therefore E\{x(t_1)\} = \begin{cases} 2t_1 - 1 & -\frac{1}{2} \le t_1 \le \frac{1}{2} \\ 3 - 2t_1 & \frac{1}{2} \le t_1 \le \frac{3}{2} \end{cases}$$

$\because$ $x(t)$ is periodic this results in same answer as previous.

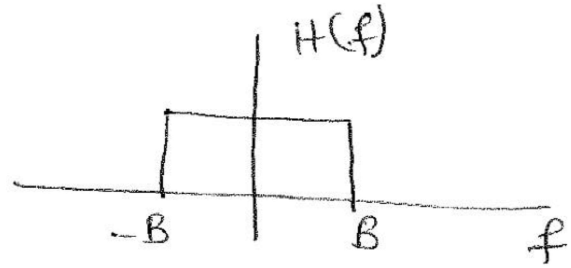
5.31

$$S_s(f) = (1-|f|)\, I_{[-1,1]}(f)$$



$$y(t) = s(t) + n(t)$$

$$S_n(f) \equiv 0.001.$$



(a) SNR at filter i/p

$$= \frac{\text{signal power}}{\text{noise power}}$$

$$\text{signal power} = \int_{-1}^{1} S_s(f)\, df = \frac{1}{2} \times 2 \times 1 = 1$$

$$\text{noise power} = 2 \times 0.001 = 0.002$$

$$SNR = \frac{1}{2\times 10^{-3}} = 500.$$

(b) for $B=1$

$$SNR_{o/p} = SNR_{i/p}$$

for $B = \frac{1}{2}$

$$\text{Signal power at the o/p} = 2\int_{0}^{1/2} (1-f)\, df$$

$$= \frac{3}{4}$$

noise power = 10^{-3}

$\therefore$ $SNR_{o/p} = \frac{3}{4} \times 1000$

$= 750$

$\therefore$ SNR is better for $B = 1/2$.

Problem 6.1
(a) Let $Z = (y * h)(t_0)$, where $h(t) = s(-t)$. Then $Z \sim N(m, v^2)$ if 1 sent, and $Z \sim N(0, v^2)$ if 0 sent, where

$$v^2 = \sigma^2 ||h||^2 = \sigma^2 4 \int_0^1 t^2 dt = \frac{4}{3}\sigma^2$$

$$m = (s * h)(t_0) = \int s(t)s(t - t_0)dt = \int_0^1 t(1-t)dt = \frac{1}{6}$$

Thus, for the ML decision rule

$$P_e = Q\left(\frac{|m|}{2v}\right) = Q\left(\frac{1}{8}\sqrt{\frac{E_b}{N_0}}\right)$$

using the fact that we must have

$$\frac{|m|}{2v} = a\sqrt{\frac{E_b}{N_0}}$$

(why?) where a is a constant determined by substituting $E_b = \frac{1}{2}||s||^2 = \frac{2}{3}$ and $N_0 = 2\sigma^2$.
(b) We can improve the error probability by sampling at $t_0 = 0$. We then have $m = ||s||^2 = \frac{4}{3}$, while v^2 is as before. This gives, reasoning as in (a),

$$P_e = Q\left(\frac{|m|}{2v}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

the usual formula for the performance of optimal reception of on-off keying in AWGN.
(c) For $h(t) = I_{[0,2]}$, we again have the same model for the decision statistic $Z = y * h(t_0)$, but with $v^2 = \sigma^2 ||h||^2 = 2\sigma^2$, and $m = (s * h)(t_0)$. The performance improves with $|m|$, which is maximized at $t_0 = 2$ ($m = 1$) or $t_0 = 4$ ($m = -1$). We therefore get that, for the ML decision rule,

$$P_e = Q\left(\frac{|m|}{2v}\right) = Q\left(\sqrt{\frac{3E_b}{8N_0}}\right)$$

(d) Note that we can approximate the matched filter $s(-t)$ using linear combinations of two shifted versions of $h(t) = I_{[0,2]}$, by approximating triangles by rectangles. That is, the matched filter shape is approximated as $\tilde{h}(t) = h(t+2) - h(t+4)$. Thus, we can use the decision statistic

$$Z = (y * \tilde{h})(0) = (y * h)(2) - (y * h)(4)$$

We now have $Z \sim N(m, v^2)$ if 1 sent, and $Z \sim N(0, v^2)$ if 0 sent, where

$$v^2 = \sigma^2 ||\tilde{h}||^2 = 4\sigma^2$$

$$m = (s * \tilde{h})(0) = 2$$

As before, we can enforce the scaling with $\frac{E_b}{N_0}$ to get

$$P_e = Q\left(\frac{|m|}{2v}\right) = Q\left(\sqrt{\frac{3E_b}{4N_0}}\right)$$