UCSB

ECE 146B: Solutions to Problem Set 2

Problem 5.7 The random variable $X \sim Exp(\mu = \frac{1}{10})$ (mean $\mathbb{E}[X] = \frac{1}{\mu} = 10$).
(a) $P[X > x] = e^{-\mu x}$ for $x \geq 0$, hence $P[X > 20] = e^{-20/10} = e^{-2} = 0.1353$.
(b) $P[X \leq x] = P[X < x] = 1 - e^{-\mu x}$ for $x \geq 0$, hence $P[X < 5] = 1 - e{-}5/10 = 0.3935$.
(c) By Bayes' rule,

$$P[X > 20|X > 10] = \frac{P[X>20, X>10]}{P[X>10]} = \frac{P[X>20]}{P[X>10]}$$
$$= \frac{e^{-20/10}}{e^{-10/10}} = e^{-1} = 0.3679$$

(d) We have

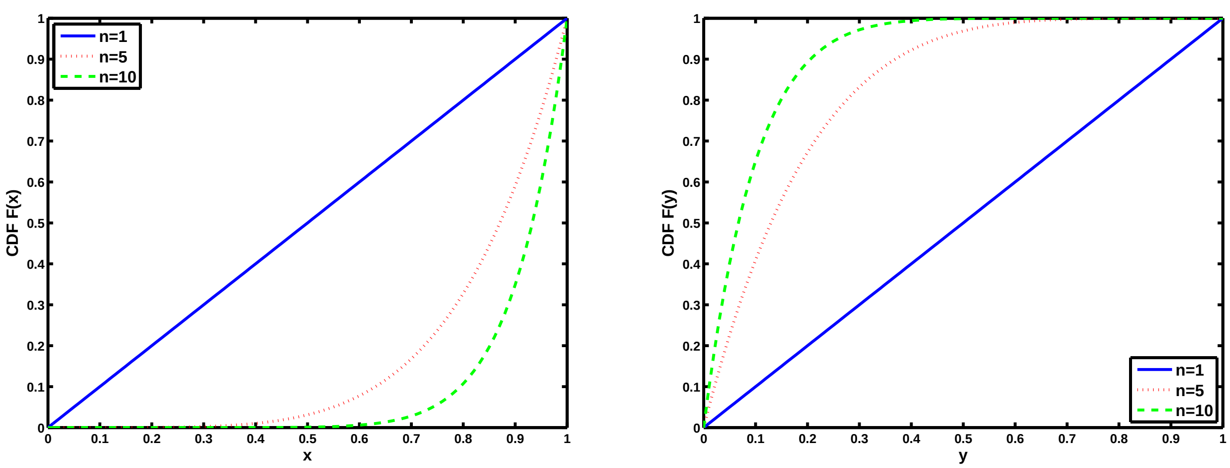
$$\mathbb{E}\left[e^{-X}\right] = \int e^{-x} p(x) dx = \int_0^\infty e^{-x} \mu e^{-\mu x} dx = \frac{e^{-(\mu+1)x}}{-(\mu+1)}\Big|_0^\infty \qquad = \frac{\mu}{\mu+1} = \frac{1}{11}$$

setting $\mu = \frac{1}{10}$. (e) We have

$$\mathbb{E}\left[X^3\right] = \int x^3 p(x) dx = \int_0^\infty x^3 \mu e^{-\mu x} dx = \frac{1}{\mu^3} \int_0^\infty t^3 e^{-t} dt$$

substituting $t = \mu x$. As discussed in the notes, the integral evaluates to $\Gamma(4) = 3! = 6$, so that

$$\mathbb{E}\left[X^3\right] = \frac{6}{\mu^3} = 6000$$



(a) CDF of maximum

(b) CDF of minimum

Figure 1: The CDFs of the maximum and minimum of n i.i.d. uniform random variables.

Problem 5.8 (a) For $X = \max(U_1, ..., U_n)$, we have $X \leq x$ if and only if $U_1 \leq x, ..., U_n \leq x$. The CDF of X is therefore given by

$$F_X(x) = P[X \leq x] = P[U_1 \leq x, ..., U_n \leq x] = P[U_1 \leq x] ... P[U_n \leq x] = F_U^n(x)$$

where we have used the independence of the $\{U_i\}$.
(b) For $Y = \min(U_1, ..., U_n)$, we have $Y > y$ if and only if $U_1 > y, ..., U_n > y$. Thus,

$$P[Y > y] = P[U_1 > y, ..., U_n > y] = P[U_1 > y] ... P[U_n > y] = (1 - F_U(y))^n$$

Thus, the CDF of Y is given by

$$F_Y(y) = P[Y \leq y] = 1 - P[Y > y] = 1 - (1 - F_U(y))^n$$

(c) When the $\{U_i\}$ are uniform over $[0, 1]$, we have $F_U(u) == u$, $0 \leq u \leq 1$, hence

$$F_X(x) = x^n \ , \quad 0 \leq x \leq 1$$

(Of course, $F_X(x) = 0$ for $x < 0$ and $F_X(x) = 0$ for $x \geq 1$, since X lies in $[0, 1]$.) Figure 1(a) plots the CDF of X. The probability mass shifts towards one as n increases. For $0 < x < 1$, $x^n \to 0$ as $n \to \infty$, so that the limiting CDF concentrates all of its probability mass at one:

$$\lim_{n \to \infty} F_X(x) = \begin{cases} 0, & x < 1 \\ 1, & x \geq 1 \end{cases}$$

Thus, we get a discrete random variable putting all its probability mass at one, taking the limit of a large number of continuous random variables.
(d) When the $\{U_i\}$ are uniform over $[0, 1]$, their minimum Y also lies in $[0, 1]$. The CDF is given by

$$F_Y(y) = 1 - (1 - y)^n \ , \quad 0 \leq y \leq 1$$

Figure 1(b) plots the CDF of Y. The probability mass shifts towards zero as n increases. For $0 < y \leq 1$, we have $0 \geq 1 - y < 1$ and $(1 - y)^n \to 0$, so that we get that

$$\lim_{n \to \infty} F_Y(y) = \begin{cases} 0, & y \leq 0 \\ 1, & y > 0 \end{cases}$$

This limit is actually not a valid CDF, since it is not right-continuous at $y = 0$. But it does show a unit jump at $y = 0$, indicating that all the probability mass is concentrated at zero. We can prove the latter rigorously using more sophisticated techniques, but we do not attempt to do this here.

Problem 5.10 (a) This is the same as Problem 5.8(b), but we can see the explicit form here

$$P[min(U, V) < x] = 1 - (1 - x)^2 = 2x - x^2 \ , \quad 0 \leq x \leq 1$$

Sketch omitted. See Figure 1(b) for similar plots.
(b) $Y = V/U$ takes values in $[0, \infty)$ for U, V, taking values in $[0, 1]$.
Method 1: The CDF can be computed by conditioning on $U = u$, and then removing the conditioning, as follows:

$$F_Y(y) = P[Y = V/U \leq y] = P[V \leq Uy] = \int P[V \leq Uy | U = u] p(u) du = \int_0^1 P[V \leq uy] du$$

where we have used the independence of U, V, and the uniform distribution of U, in the last equality. We know that $P[V \leq v] = v$, $0 \leq v \leq 1$, and $P[V \leq v] = 1$ for $v \geq 1$. For $y \leq 1$, $uy \leq 1$, so that

$$F_Y(y) = \int_0^1 uy du = \frac{y}{2} \ , \quad 0 \leq y \leq 1$$

For $y > 1$, $uy > 1$ and $P[V \leq uy] = 1$ for $u > \frac{1}{y}$, so that

$$F_Y(y) = \int_0^{\frac{1}{y}} uy du + \int_{\frac{1}{y}}^1 1 dy = 1 - \frac{1}{2y} \ , \quad y > 1$$

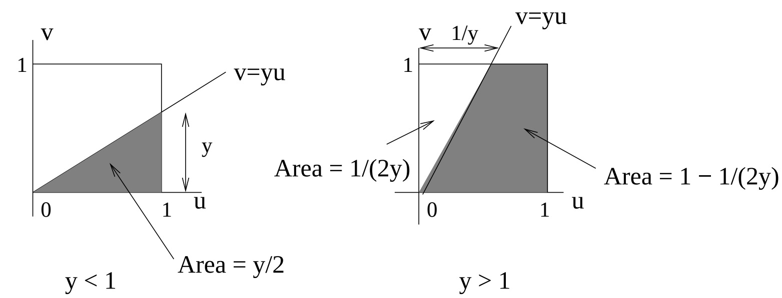


Figure 2: Pictorial computation of CDF of V/U, where U, V are i.i.d. uniform over $[0, 1]$.

Method 2: We can also compute the CDF pictorially as shown in Figure 2. The joint distribution of (U, V) is uniform over the unit square in the (u, v) plane, so that the event $V/U \leq y$ whose probability we desire to find represents the shaded regions shown, and probability itself is simply the area of these shaded regions. For $y > 1$, the region $V/U \leq y$ is triangular, with area $y/2$. For $y > 1$, the region $V/U \leq y$ is the complement of a triangular region of area $1/(2y)$.