ECE 146B: Solutions to Problem Set 3

Problem 5.15(a) The joint density must integrate to one, hence we must have

$$1 = K \int_0^\infty \int_0^\infty e^{-(2x^2+y^2)/2} dxdy + K \int_{-\infty}^0 \int_{-\infty}^0 e^{-(2x^2+y^2)/2} dxdy = 2K \int_0^\infty \int_0^\infty e^{-(2x^2+y^2)/2} dxdy$$

where we have used symmetry. The integrals in x and y separate out, and we have

$$\int_0^\infty e^{-(2x^2)/2} dx = \sqrt{2\pi v_1^2} \int_0^\infty \frac{e^{-x^2/(2v_1^2)}}{2\pi v_1^2} dx = \sqrt{2\pi v_1^2}\, \frac{1}{2} = \frac{\sqrt{\pi}}{2}$$

massaging the x integrand into an $N(0, v_1^2)$ density, with $v_1^2 = \frac{1}{2}$. Similarly, we can massage the y integrand into an $N(0, v_2^2)$ density with $v_2^2 = 1$ to get

$$\int_0^\infty e^{-y^2/2} dy = \sqrt{2\pi v_2^2}\frac{1}{2} = \frac{\sqrt{2\pi}}{2}$$

We therefore have $1 = 2K\frac{\sqrt{\pi}}{2}\frac{\sqrt{2\pi}}{2}$, or $K = \sqrt{2}/\pi$.
(b) The marginal density of X is

$$p(x) = \int p(x, y) dy = \begin{cases} \int_0^\infty e^{-(2x^2+y^2)/2} dy, & x \geq 0 \\ \int_{-\infty}^0 e^{-(2x^2+y^2)/2} dy, & x < 0 \end{cases}$$

By symmetry, the y integrals evaluate to the same answer for the two cases above, so that $p(x) \sim e^{-x^2}$. Thus, $X \sim N(0, \frac{1}{2})$ (the constant must evaluate out to whatever is needed for $p(x)$ to integrate to one. A similar reasoning shows that $Y \sim N(0, 1)$.
(c) The event $X^2 + X > 2$ can be written as

$$X^2 + X - 2 = (X + 2)(X - 1) > 0$$

which happens if $X + 2 > 0, X - 1 > 0$, or $X + 2 < 0, X - 1 < 0$. That is, it happens if $X > 1$ or $X < -2$. Thus,

$$P[X^2 + X > 2] = P[X > 1] + P[X < -2] = Q\left(\frac{1-0}{\sqrt{1/2}}\right) + \Phi\left(\frac{-2-0}{\sqrt{1/2}}\right) = Q(\sqrt{2}) + Q(2\sqrt{2})$$

where we have used $X \sim N(0, 1/2)$ and $\Phi(-x) = Q(x)$.
(d) X, Y are not jointly Gaussian, since the probability mass is constrained to two of the four quadrants, unlike the joint Gaussian density, for which the probability mass is spread over the entire plane.
(e) If $X > 0$, then $Y > 0$ (even though Y can take both positive and negative values). Hence X and Y cannot be independent.
(f) From the marginals in (b), we know that $\mathbb{E}[X] = \mathbb{E}[Y] = 0$. However, $\mathbb{E}[XY] > 0$, since all the probability mass falls in the region $xy > 0$. Thus, $\mathrm{cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] > 0$. So X, Y are not uncorrelated.
(g) The conditional density is $p(x|y) = p(x, y)/p(y)$. If $y > 0$, this evaluates to $p(x|y) = k_1 e^{-x^2} I_{x\geq 0}$. If $y < 0$, it evaluates to $p(x|y) = k_1 e^{-x^2} I_{x<0}$. Since the probability mass of the conditional density is constrained to part of the real line, it is not Gaussian.

5.19

(a) (1) $\sigma_x^2 = 1 \quad \sigma_y^2 = 1 \quad \rho = 0$

$\because$ x and y are jointly Gaussian, any linear combination of x and y is Gaussian. we can specify it completely by specifying its mean and variance.

$$x - 2y = z \text{ (say)}$$

$$E\{z\} = E\{x\} - 2E\{y\}$$

$$= m_x - 2m_y$$

$$\text{var}(z) = \text{var}(x) + 4\,\text{var}(y)$$ ($\because$ x and y are uncorrelated hence independent)

$$= 1+4$$

$$= 5$$

$$\therefore z \sim N(m_x - 2m_y,\ 5).$$

(2) $\sigma_x^2 = 1 \quad \sigma_y^2 = 1 \quad \rho = 0.5$

$$\text{var}(z) = a^T C a$$

here $z = [1 \ \ -2] \begin{bmatrix} x \\ y \end{bmatrix}$ $\quad a = [1 \ \ -2]^T$

C is covariance matrix of vector $\begin{bmatrix} x \\ y \end{bmatrix}$.

$\therefore$ ~~Eq~~ $\text{var}(Z) = \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$= \underline{\underline{3}}$ $\quad Z \sim N(m_x - 2m_y, 3)$.

(3) $\sigma_x^2 = 4 \quad \sigma_y^2 = 1 \quad \rho = 0.5$

$\Rightarrow C = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}$

$\Rightarrow \text{var}(Z) = \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 4$

$Z \sim N(m_x - 2m_y, 4)$.

(b)

(i) X, Y are uncorrelated

$\text{cov}(Z, X) = \text{cov}(X - 2Y, X)$

$= \text{var}(X)$

$\neq 0$

$\therefore X, Z$ are not independent

(2) $\sigma_x^2 = 1, \quad \sigma_y^2 = 1 \quad \rho = 0.5$

$\text{cov}(X - 2Y, X) = \text{var}(X) - 2\,\text{cov}(X, Y)$

$= 1 - 2(0.5)$

$= \underline{0}$ $\quad \therefore X, Z$ are independent.

(3) $\operatorname{cov}(x-2y, x)$

$= \operatorname{var}(x) - 2\operatorname{cov}(x,y)$

$= 4 - 2(\overset{1}{\cancel{0.5}}) = 2$

$\neq 0$

x and z are not independent.

5.21

$x_1 = R\cos\phi \qquad x_2 = R\sin\phi$

(a) $\underline{x} = [R, \phi] \qquad \underline{y} = [x_1, x_2]$

$$J(\underline{y};\underline{x}) = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & -r\cos\phi \end{bmatrix}$$

$$p_{R,\phi}(r,\phi) = \left. \frac{p_{x_1,x_2}(x_1,x_2)}{|\det(J(y;x))|} \right|_{\substack{x_1 = r\cos\phi \\ x_2 = r\sin\phi}}$$

$$p_{x_1,x_2} = \frac{1}{\sqrt{2\pi}\,v} e^{-x_1^2/2v^2} \frac{1}{\sqrt{2\pi}\,v} e^{-x_2^2/2v^2}$$

$$\Rightarrow p_{R,\phi}(r,\phi) = \frac{1}{2\pi v^2} \cdot e^{-r^2/2v^2} \cdot \frac{1}{(r^{-1})}$$

$$= \frac{r}{2\pi v^2} e^{-r^2/2v^2} \cdot I_{[0,\infty)}(r)$$

(b) $$p_{R,\phi}(r,\phi) = \frac{1}{2\pi} \cdot \frac{r}{v^2} e^{-r^2/2v^2}$$

$$= p_R(r)\, p_\phi(\phi)$$

$\therefore$ R and ϕ are independent.

$$p_R(r) = \frac{r}{v^2} e^{-r^2/2v^2} \qquad p_\phi(\phi) = \frac{1}{2\pi}.$$

(c) $y = R^2$

$\Rightarrow R = \pm\sqrt{y}$ since R is +ve

$R = \sqrt{y}$.

$$\Rightarrow p_y(y) = \frac{p_R(\sqrt{y})}{|2\sqrt{y}|} \qquad p_R(\sqrt{y}) = \frac{\sqrt{y}}{v^2} e^{-y/2v^2}$$

$$p_y(y) = \frac{1}{2 v^2} e^{-y/2v^2}, \qquad y = R^2.$$

(d) let $\lambda = \frac{1}{2v^2}$

$$\Rightarrow p_y(y) = \lambda e^{-\lambda y} \qquad \mu = E\{y\} = \frac{1}{\lambda}.$$

we need $-10\log(y) + 10\log(\mu) > 20$

$$\Rightarrow \log\left(\frac{\mu}{y}\right) > 2$$

$$\Rightarrow \mu/y > 100$$

$\Rightarrow \; y < \frac{u}{100}$

$$= \int_0^{\frac{u}{100}} \lambda e^{-\lambda y} \, dy$$

$$= e^{-\lambda y} \Big|_{\frac{\mu}{100}}^{0}$$

$$= 1 - e^{-\lambda \cdot \frac{u}{100}} = 1 - e^{-1/100}$$

it doesn't depend on v^2.

Solution 5.43

$$Q(x) = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt \quad \text{——} \quad (1)$$

$$t^2 = (t-x)^2 + 2tx - x^2$$

$$\geq (t-x)^2 + 2x^2 - x^2$$

$$= (t-x)^2 + x^2 \qquad \text{substituting this in (1)}$$

$$\therefore Q(x) \leq \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{((t-x)^2+x^2)}{2}} \, dt$$

$$= e^{-\frac{x^2}{2}} \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(t-x)^2}{2}} \, dt$$

→ half area under Gaussian density

$$= \frac{1}{2} e^{-x^2/2}$$

$$\therefore Q(x) \leq \frac{1}{2} e^{-x^2/2}.$$

(b) $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}\, t} \cdot t \; e^{-t^2/2} \, dt$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{t} \int t e^{-t^2/2} dt \Big|_x^\infty - \int_0^\infty \frac{-1}{t^2} \left(\int t e^{-t^2/2} dt \right) dt \right]$$

consider $\int t e^{-t^2/2} \, dt$ $\qquad t^2/2 = u$

$\qquad \Rightarrow t\,dt = du$

$$= \int e^{-u} du = -e^{-u}$$

$$= -e^{-t^2/2}.$$

$$\therefore Q(x) = \frac{1}{\sqrt{2\pi}} \left[-\frac{1}{t} e^{-t^2/2} \Big|_x^\infty - \int_0^\infty \frac{1}{t^2} \cdot e^{-t^2/2} dt \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{x} e^{-x^2/2} - \int_0^\infty \frac{1}{t^2} e^{-t^2/2} dt \right]$$

$$\therefore Q(x) \geq \frac{1}{\sqrt{2\pi}\, x} e^{-x^2/2}.$$

for upper bound

$$Q(x) = \frac{1}{\sqrt{2\pi}}\left[\frac{1}{x}e^{-x^2/2} - \int_0^\infty \frac{1}{t^2} e^{-t^2/2}\,dt\right]$$

$$\leq \frac{1}{\sqrt{2\pi}}\left[\frac{1}{x}e^{-x^2/2} - \int_x^\infty \frac{1}{t^2} e^{-t^2/2}\,dt\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{x}e^{-\frac{x^2}{2}} - \int_x^\infty \frac{1}{t^3}\cdot t e^{-t^2/2}\,dt\right]$$

consider $\int_x^\infty \frac{1}{t^3}\cdot t e^{-t^2/2}\,dt$

$$= \frac{1}{t^3}\cdot\int t e^{-t^2/2}\Big|_x^\infty - \underbrace{\int(\cdots)}_{K} \qquad K > 0$$

$$\therefore\ Q(x) \leq \frac{1}{\sqrt{2\pi}}\left(\frac{1}{x}e^{-x^2/2} - \frac{1}{x^3}e^{-x^2/2} + K\right)$$

$$\leq \frac{1}{\sqrt{2\pi}}\left(\frac{1}{x}e^{-x^2/2} - \frac{1}{x^3}e^{-x^2/2}\right)$$

$$\leq \frac{1}{\sqrt{2\pi}\,x}\left(1 - \frac{1}{x^2}\right)\cdot e^{-x^2/2}$$