ECE 146B: Solutions to Problem set 5

Problem 5.20 (a) We have $\text{cov}(X,Y) = \rho\sigma_X\sigma_Y = -\frac{3}{4}$, so the covariance matrix is given by

$$\mathbf{C_X} = \begin{pmatrix} 1 & -\frac{3}{4} \\ -\frac{3}{4} & 1 \end{pmatrix}$$

where $\mathbf{X} = (X,Y)^T$.
(b) $Z = \mathbf{a}^T\mathbf{X}$, where $\mathbf{a}^T = (2\ 3)$, so that $Z \sim N(\mathbf{a}^T\mathbf{m_Y} = 8, \mathbf{a}^T\mathbf{C_X}\mathbf{a} = 4)$.
(c) In order to compute $P[Z^2 - Z > 6] = P[Z^2 - Z - 6 > 0]$, we factorize

$$Z^2 - Z - 6 = Z^2 - 3Z + 2Z - 6 = (Z+2)(Z-3)$$

This expression is positive if both factors are positive ($Z > -2$ and $Z > 3$, which is equivalent to $Z > 3$), or if both factors are negative ($Z < -2$ and $Z < 3$, which is equivalent to $Z < -2$). These two events are mutually exclusive, hence

$$P\left[\{Z > 3\} \text{ or } \{Z < -2\}\right] = P[Z > 3] + P[Z < -2] = Q\left(\frac{3-8}{\sqrt{4}}\right) + \Phi\left(\frac{-2-8}{\sqrt{4}}\right)$$
$$Q(-5/2) + \Phi(-10/2) = 1 - Q(5/2) + Q(5) = 0.9938$$

Problem 5.35 We can write $Y_1 = \langle y, u_1 \rangle$, $Y_2 = \langle y, u_2 \rangle$, where $u_1(t) = I_{[0,2]}(t)$, $u_2(t) = I_{[1,3]}(t)$. The noisy signal $y(t) = s(t) + n(t)$, where n is WGN and $s(t) = I_{[0,3]}(t)$. Plugging in, we get

$$Y_1 = \langle s + n, u_1 \rangle = \langle s, u_1 \rangle + \langle n, u_1 \rangle = 2 + \langle n, u_1 \rangle$$

$$Y_2 = \langle s + n, u_2 \rangle = \langle s, u_2 \rangle + \langle n, u_2 \rangle = 2 + \langle n, u_2 \rangle$$

are jointly Gaussian with mean vector $\mathbf{m} = (2, 2)^T$, since $\langle n, u_1 \rangle$, $\langle n, u_2 \rangle$ are zero mean (since n has zero mean) and jointly Gaussian (obtained by linear operations on the Gaussian random process n). We have (for $i, j = 1, 2$)

$$\text{cov}(Y_i, Y_j) = \text{cov}\left(2 + \langle n, u_i \rangle, 2 + \langle n, u_j \rangle\right) = \text{cov}\left(\langle n, u_i \rangle, \langle n, u_j \rangle\right)$$

since covariance is not affected by translation by constants. Thus, the means are due to the signal, and the covariances due to the noise. We now use computations similar to those we have done in Chapter 5 and in class, exploiting the impulsive autocorrelation function of white noise:

$$\begin{aligned}
&\text{cov}(Y_i, Y_j) = \text{cov}\left(\langle n, u_i \rangle, \langle n, u_j \rangle\right) = \mathbb{E}\left[\langle n, u_i \rangle, \langle n, u_j \rangle\right] \\
&\quad = \mathbb{E}\left[\int n(t) u_i(t) dt \int n(t') u_j(t') dt'\right] = \int \int \mathbb{E}\left[n(t) n(t')\right] u_i(t) u_j(t') dt dt' \\
&\quad = \int \int \sigma^2 \delta(t - t') u_i(t) u_j(t') dt dt' = \sigma^2 \int u_i(t) u_j(t) dt \\
&\quad = \sigma^2 \langle u_i, u_j \rangle
\end{aligned}$$

We get the diagonal terms in the covariance matrix by setting $i = j$, and the off-diagonal terms for $i \neq j$. The covariance matrix is given by

$$\mathbf{C} = \begin{pmatrix} \sigma^2 ||u_1||^2 & \sigma^2 \langle u_1, u_2 \rangle \\ \sigma^2 \langle u_1, u_2 \rangle & \sigma^2 ||u_2||^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

plugging in $\sigma^2 = \frac{1}{4}$, and the inner products

$$||u_1||^2 = \int_0^2 1^2 dt = 2, \quad ||u_2||^2 = \int_1^3 1^2 dt = 2, \quad \langle u_1, u_2 \rangle = \int_1^2 1^2 dt = 1$$

(b) Since Y_1, Y_2 are jointly Gaussian, their linear combination $Z = Y_1 + Y_2$ is Gaussian, with mean

$$\mathbb{E}[Z] = \mathbb{E}[Y_1 + Y_2] = \mathbb{E}[Y_1] + \mathbb{E}[Y_2] = 2 + 2 = 4$$

and variance

$$\begin{aligned}
&\text{var}(Z) = \text{cov}(Z, Z) = \text{cov}(Y_1 + Y_2, Y_1 + Y_2) = \text{cov}(Y_1, Y_1) + \text{cov}(Y_2, Y_2) + 2\text{cov}(Y_1, Y_2) \\
&\quad = \frac{1}{2} + \frac{1}{2} + 2\left(\frac{1}{4}\right) = \frac{3}{2}
\end{aligned}$$

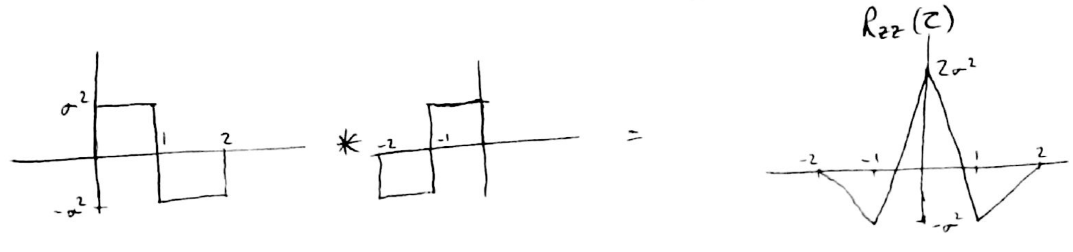
The probability

$$P[Z < 2] = \Phi\left(\frac{2 - \mathbb{E}[Z]}{\sqrt{\text{var}(Z)}}\right) = \Phi\left(\frac{2 - 4}{\sqrt{3/2}}\right) = \Phi\left(-2\sqrt{6}/3\right) = Q\left(2\sqrt{6}/3\right)$$

Problem 5.36 a) Autocorrelation of $z(t)$, $R_{zz}(\tau) = h(\tau) * h_{mf}(\tau) * R_{nn}(\tau)$

where $h_{mf}(\tau) = h(-\tau)$, $R_{nn}(\tau) = \sigma^2 \delta(\tau)$ $\rightarrow n(t)$ is white gaussian noise.

$$\therefore R_{zz}(\tau) = \sigma^2 \int_{-\infty}^{\infty} h(m)\, h(-(\tau - m))\, dm = \sigma^2 \int_{-\infty}^{\infty} h(m) h(m-\tau) = \sigma^2 (h(\tau) * h(-\tau))$$



$$R_{zz}(\tau) = \sigma^2 \left(2\left(1 - \frac{3}{2}|\tau|\right) I_{[-1,1]}(\tau) + (|\tau| - 2) I_{[1,2]}(|\tau|) \right)$$

b) Joint distribution of $\underline{z}_1 = \begin{bmatrix} z(49) \\ z(50) \end{bmatrix}$ will be jointly gaussian

mean $E[\underline{z}_1] = E\begin{bmatrix} z(49) \\ z(50) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Covariance $C_{z_1} = \begin{bmatrix} E(z^2(49)) & E[z(49)z(50)] \\ E[z(50)z(49)] & E[z^2(50)] \end{bmatrix} = \begin{bmatrix} R_z(0) & R_z(1) \\ R_z(1) & R_z(0) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \sigma^2$

$\therefore$ Joint distribution is given by $p(z_{49}, z_{50}) = \frac{1}{\sqrt{2\pi |C_z|}} \exp\left(-\frac{1}{2}(\underline{z}_1 - \mu_{z_1})^H C_z^{-1} (\underline{z}_1 - \mu_{z_1})\right)$

$$= \frac{1}{\sqrt{2\pi(3\sigma^4)}} \exp\left(-\frac{1}{2} \begin{bmatrix} z_{49} & z_{50} \end{bmatrix} \frac{1(\sigma^2)}{3\sigma^4} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} z_{49} \\ z_{50} \end{bmatrix}\right)$$

$$= \frac{1}{2\sqrt{3}\pi\sigma^2} \exp\left(-\frac{1}{6\sigma^2} \begin{bmatrix} 2z_{49} + z_{50} & z_{49} + 2z_{50} \end{bmatrix} \begin{bmatrix} z_{49} \\ z_{50} \end{bmatrix}\right)$$

$$= \frac{1}{2\sqrt{3}\pi\sigma^2} \exp\left(-\frac{(z_{49}^2 + z_{50}^2 + z_{49} z_{50})}{3\sigma^2}\right)$$

$\underline{z}_1 \sim N(0, C_{z_1})$

c) Similarly for $\underline{z}_2 = \begin{bmatrix} z(49) \\ z(52) \end{bmatrix}$ Mean, $\mu_{\underline{z}_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $C_{\underline{z}_2} = \begin{bmatrix} R_z(0) & R_z(3) \\ R_z(3) & R_z(0) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\sigma^2$

$\therefore$ Joint distribution $p(z(49), z(52)) = \frac{1}{\sqrt{2\pi(4\sigma^4)}} \exp\left(-\frac{1}{2\sigma^2} [z_{49} \;\; z_{52}] \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} z_{49} \\ z_{52} \end{bmatrix}\right)$

$= \frac{1}{2\sqrt{2}\sigma^2} \exp\left(-\frac{(z_{49}^2 + z_{52}^2)}{4}\right)$ $\qquad \underline{\underline{z}_2 \sim N(0, C_{\underline{z}_2})}$

d) $P(2z(50) > z(49) + z(51)) = P(2z(50) - z(49) - z(51) > 0)$

$= P\Bigg(\underbrace{[2 \;\; -1 \;\; -1]\begin{bmatrix} z(50) \\ z(49) \\ z(51) \end{bmatrix}}_{K} > 0\Bigg)$

Now, find Joint distribution of $\underline{z}' = \begin{bmatrix} z(50) \\ z(49) \\ z(51) \end{bmatrix}$ Mean, $\mu_{z'} = E[\underline{z}'] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Covariance, $C_{\underline{z}'} = \begin{bmatrix} Cov(z(50)z(50)) & Cov(z(49)z(50)) & Cov(z(51)z(50)) \\ Cov(z(50)z(49)) & Cov(z(49)z(49)) & Cov(z(51)z(49)) \\ Cov(z(50)z(51)) & Cov(z(49)z(51)) & Cov(z(51)z(51)) \end{bmatrix}$

$= \begin{bmatrix} R_z(0) & R_z(-1) & R_z(1) \\ R_z(1) & R_z(0) & R_z(2) \\ R_z(-1) & R_z(-2) & R_z(0) \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}\sigma^2$ $\qquad (\sigma^2 = 1)$

$\det(C_{z'}) = |C_{z'}| = 2(4) + 1(-2) - 1(2) = 4$

Variance of variable $K = [2 \;\; -1 \;\; -1]\underline{z}'$ is $\sigma_K^2 = [2 \;\; -1 \;\; -1] C_{z'} \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$

$\sigma_K^2 = [2 \;\; -1 \;\; -1] \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}\sigma^2 = [6 \;\; -4 \;\; -4]\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}\sigma^2 = 20\sigma^2 = 20$

$(\sigma^2 = 1)$

$\therefore K \sim N(0, 20)$

$P(2z(50) > z(49) + z(51)) = P(K > 0) = \frac{1}{2}$

e) $P(2z(50) > z(49) + z(51) + 2) = P(K > 2) = Q\left(\frac{2-0}{\sqrt{20}}\right) = Q\left(\frac{1}{\sqrt{5}}\right)$

Problem 6.12 (a) The conditional densities and ML regions are sketched in Figure 6. We have $p(y|0) = \frac{1}{5}e^{-y/5}I_{[0,\infty)}(y)$ and $p(y|1) = \frac{1}{10}I_{[0,10]}(y)$. The ML decision regions are given by $\Gamma_1 = \{y : p(y|1) > p(y|0)\}$ and $\Gamma_0 = \{y : p(y|1) < p(y|0)\}$, with crossover point given by $\frac{1}{5}e^{-y/5} = \frac{1}{10}$, which gives $y = 5\log 2 \approx 3.47$. We can also write the ML rule as

$$\delta_{ML}(y) = \begin{cases} 1, & 5\log 2 < y < 10 \\ 0, & else \end{cases}$$

(b) The conditional error probability is given by

$$P_{e|0} = P[Y \in \Gamma_1|H_0] = P[5\log 2 < Y < 10|H_0] = e^{-5\log 2/5} - e^{-10/2} = \frac{1}{2} - e^{-2} = 0.3647$$

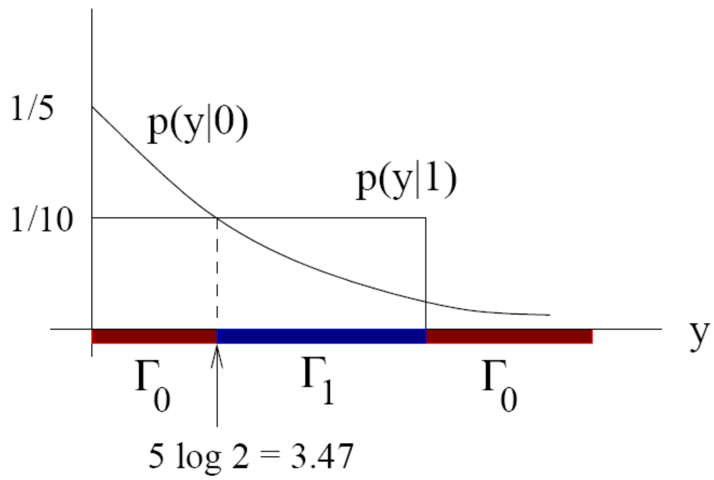


Figure 6: Conditional densities and ML decision regions for Problem 6.12.

(c) The posterior probability for $Y = 4$ is given by

$$P[H_0|Y = y] = \frac{\pi_0 p(y|0)}{\pi_0 p(y|0) + (1-\pi_0)p(y|1)} = \frac{\frac{1}{3}\ \frac{1}{5}e^{-4/5}}{\frac{1}{3}\ \frac{1}{5}e^{-4/5} + \frac{2}{3}\frac{1}{10}}$$
$$= 0.31$$

Problem 6.14 (a) The ML rule

$$\langle y, s_1\rangle - ||s_1||^2/2 \overset{H_1}{\underset{H_0}{\gtrless}} \langle y, s_0\rangle - ||s_0||^2/2$$

specializes to

$$\langle y, s_1\rangle \overset{H_1}{\underset{H_0}{\gtrless}} 0$$

for $s_0 = -s_1$. Thus, it can be implemented by correlating the received signal against $s_1(t)$ and then deciding based on the sign of the output. The conditional error probability is given by $P_{e|0} = Q\left(\frac{||s_1 - s_0||}{2\sigma}\right)$. Now, $||s_1 - s_0|| = ||2s_1|| = 2||s_1||$, $||s_1||^2 = 2\int_0^1 (1-t)^2 dt = 2/3$, and $\sigma^2 = 0.1$, so that

$$P_{e|0} = Q\left(\frac{2\sqrt{2/3}}{2\sqrt{0.1}}\right) = Q\left(\sqrt{20/3}\right) = 0.0049$$

(b) Conditioned on s_0 sent, $y(t) = s_0(t) + n(t)$. Let us compute the signal and noise contributions separately. The noise output $n_0 = \int_{-1}^{-0.5} n(t)dt$ has variance $\sigma^2 \int_{-1}^{-0.5} 1^2 dt = \sigma^2/2$. The other noise components are independent and have identical variance, since they are computed over disjoint intervals of equal length. Thus, n_i are i.i.d., $N(0, \sigma^2/2)$. The signal contributions are given by integrating $s_0(t)$ over these intervals. It is easy to see, therefore, that $\mathbf{y} \sim N(\mathrm{s}_0, \frac{\sigma^2}{2}\mathbf{I})$, where $\mathrm{s}_0 = -(1/8, 3/8, 3/8, 1/8)^T$.
(c) Since the signals under the two hypotheses are negatives of each other, we infer from (b) that the signal model after discretization is $\mathbf{y} \sim N(\mathrm{s}_1, \frac{\sigma^2}{2}\mathbf{I})$ conditioned on s_1 sent, and $\mathbf{y} \sim N(\mathrm{s}_0 = -\mathrm{s}_1, \frac{\sigma^2}{2}\mathbf{I})$, where $\mathrm{s}_1 = (1/8, 3/8, 3/8, 1/8)^T$. Thus, we are hypothesis testing in discrete time WGN with variance $\tilde{\sigma}^2 = \sigma^2/2$. Thus, the ML rule is given by

$$\langle \mathbf{y}, \mathrm{s}_1\rangle - ||\mathrm{s}_1||^2/2 \overset{H_1}{\underset{H_0}{\gtrless}} \langle \mathbf{y}, \mathrm{s}_0\rangle - ||\mathrm{s}_0||^2/2$$

which specializes to

$$\langle \mathbf{y}, \mathrm{s}_1\rangle = \frac{1}{8}(y_0 + 3y_1 + 3y_2 + y_3) \overset{H_1}{\underset{H_0}{\gtrless}} 0$$

Simplifying, our ML rule is

$$y_0 + 3y_1 + 3y_2 + y_3 \overset{H_1}{\underset{H_0}{\gtrless}} 0$$

The conditional error probability is given by the standard expression for WGN: $P_{e|0} = Q\left(\frac{||\mathbf{s}_1 - \mathbf{s}_0||}{2\tilde{\sigma}}\right)$, where we must use the variance of the discrete time WGN obtained after discretization. As before, $||\mathbf{s}_1 - \mathbf{s}_0|| = ||2\mathbf{s}_1|| = 2||\mathbf{s}_1||$. We have $||\mathbf{s}_1||^2 = \frac{1}{64}(1+9+9+1) = \frac{5}{16}$, so that

$$P_{e|0} = Q\left(\frac{2\sqrt{5/16}}{2\sqrt{0.1/2}}\right) = Q(2.5) = 0.0062$$

(only a small degradation relative to ML performance for the original continuous time system).
(d) For observation $Z = y_0 + y_1 + y_2 + y_3$, adding the signal components and the noise variances, we obtain that $Z \sim N(1, 4\tilde{\sigma}^2 = 2\sigma^2)$ if s_1 is sent, and $Z \sim N(-1, 2\sigma^2)$ if s_0 is sent. This falls within the basic Gaussian format, and the conditional error probability is given by

$$P_{e|0} = Q\left(\frac{|m_1 - m_0|}{2v}\right) = Q\left(\frac{1-(-1)}{2\sqrt{2\sigma^2}}\right) = Q(\sqrt{5}) = 0.0127$$

(e) Smallest error probability is in (a), largest is in (d). We could have rank ordered without explicit computation. The probability in (a) is the best achievable, since we are implementing the ML rule for the original continuous time observation. We lose some performance in discretizing as in (b), because we are not doing optimal processing of the continuous time signal (which we know is to correlate with s_1). We lose further performance in (d), because we are not implementing the ML rule for the discretized signal $\mathbf{y}$.

Problem 6.16 (a) The ML rule

$$\langle y, s_1 \rangle - ||s_1||^2/2 \underset{H_0}{\overset{H_1}{\gtrless}} \langle y, s_0 \rangle - ||s_0||^2/2$$

can be implemented by a single correlator as follows:

$$\langle y, s_1 - s_0 \rangle \underset{H_0}{\overset{H_1}{\gtrless}} \frac{||s_1||^2 - ||s_0||^2}{2}$$

The correlator $a(t) = s_1(t) - s_0(t)$ is sketched in Figure 8. We have $||s_1||^2 = 2$ and $||s_0||^2 = 2\int_0^1 t^2 dt = 2/3$, so that the threshold $\gamma_{ML} = \frac{||s_1||^2 - ||s_0||^2}{2} = (2-2/3)/2 = 2/3$.
(b) Note that $E_b = \frac{1}{2}\left(||s_1||^2 + ||s_0||^2\right) = \frac{1}{2}\left(2 + 2/3\right) = 4/3$ and

$$d^2 = ||s_1 - s_0||^2 = ||s_1||^2 + ||s_0||^2 - 2\langle s_1, s_0 \rangle = 2 + 2/3 - 2(-1/2) = 5/3$$

so the power efficiency $\frac{d^2}{E_b} = \frac{5}{4}$. The ML rule has error probability

$$P_e = Q\left(\sqrt{\frac{d^2}{E_b}\frac{E_b}{2N_0}}\right) = Q\left(\sqrt{\frac{5E_b}{6N_0}}\right)$$

(c) The MPE rule is given by

$$\langle y, s_1 \rangle - ||s_1||^2/2 + \sigma^2 \log \pi_1 \underset{H_0}{\overset{H_1}{\gtrless}} \langle y, s_0 \rangle - ||s_0||^2/2 + \sigma^2 \log \pi_0$$

and can be implemented using the same correlator as the ML rule, but with the threshold adjusted to account for the priors:

$$\langle y, s_1 - s_0 \rangle \underset{H_0}{\overset{H_1}{\gtrless}} \gamma_{MPE} = \frac{||s_1||^2 - ||s_0||^2}{2} + \sigma^2 \log \frac{\pi_0}{\pi_1} = \gamma_{ML} + \sigma^2 \log \frac{\pi_0}{\pi_1}$$

For $\pi_0 = \frac{1}{3}$, the threshold is therefore given by $\gamma_{MPE} = \gamma_{ML} - \sigma^2 \log 2 = 2/3 - 0.693\sigma^2$.
(d) For $h(t) = I_{[0,1]}(t)$, filter-and-sample is the same as integrate and dump: $(y * h)(t) = \int_{t-1} t y(\tau) d\tau$. The samples at $t = 1, 2, 3$ correspond to different intervals of unit length, hence the noise outputs are i.i.d. $N(0, \sigma^2)$. The signal outputs $(s_i * h)(t) = \int_{t-1} t s(\tau) d\tau$ for $t = 1, 2, 3$ are easily computed and form the vectors: $\mathrm{s}_1 = (0, 1, 1)^T$ and $\mathrm{s}_0 = (-1/2, -1/2, 0)^T$. The conditional distribution of $\mathbf{Z}$ given 0 sent is therefore $N(\mathrm{s}_0, \sigma^2 \mathbf{I})$.
(e) Actually, since the noise is uncorrelated, this is not really challenging as promised. This is simply signaling in discrete WGN, and the ML rule is given by

$$\langle \mathbf{Z}, \mathrm{s}_1 \rangle - ||\mathrm{s}_1||^2/2 \underset{H_0}{\overset{H_1}{\gtrless}} \langle \mathbf{Z}, \mathrm{s}_0 \rangle - ||\mathrm{s}_0||^2/2$$

which simplifies to

$$\langle \mathbf{Z}, \mathrm{s}_1 - \mathrm{s}_0 \rangle \underset{H_0}{\overset{H_1}{\gtrless}} \langle (||\mathrm{s}_1||^2 - ||\mathrm{s}_0||^2)/2$$

Plugging in the numbers, we obtain

$$\frac{1}{2}Z_1 + \frac{3}{2}Z_2 + Z_3 \underset{H_0}{\overset{H_1}{\gtrless}} 3/4$$

The error probability for ML reception is discrete WGN is given by

$$P_e = Q\left(\frac{||\mathbf{s}_1 - \mathbf{s}_0||}{2\sigma}\right) = Q\left(\sqrt{\frac{d^2}{E_b}\frac{E_b}{2N_0}}\right)$$

where $d^2 = ||\mathbf{s}_1 - \mathbf{s}_0||^2 = 7/2$ is for the discretized system, but $E_b = 4/3$ for the original continuous time system, computed in (a). (Note that the scaling is such that the noise variance per dimension is the same in both systems, otherwise we would have to account for that.) We have $d^2/E_b = 21/8$, so that

$$P_e = Q\left(\sqrt{\frac{21E_b}{16N_0}}\right)$$