6.10

(a). $p(y|0) \underset{H_1}{\overset{H_0}{\gtrless}} p(y|1)$ $\qquad P\{X=K\} = \frac{\lambda^K}{K!} e^{-\lambda}$ $\quad K=0,1,2$

$\uparrow$ Poisson $\quad$ mean $= \lambda$.

$$\frac{1}{K!} e^{-1} \underset{1}{\overset{0}{\gtrless}} \frac{10^K}{K!} e^{-10}$$

$$e^{9} \underset{1}{\overset{0}{\gtrless}} 10^K$$

$$\frac{9}{\ln(10)} \underset{1}{\overset{0}{\gtrless}} K$$

$3.91 \underset{1}{\overset{0}{\gtrless}} K$ $\quad \therefore$ Threshold $= 4$ ($\because$ K can take only integer values).

(b) $P_{e|0 \text{ sent}} = P\{K \geq 4\}$ $\quad$ K is poisson with mean 1.

$$= 1 - P\{K < 4\}$$

$$= 1 - \left(P\{K=0\} + P(K=1) + P(K=2) + P(K=3)\right)$$

$$= 1 - \left(e^{-1}\left(1 + 1 + \tfrac{1}{2} + \tfrac{1}{6}\right)\right)$$

$$= \underline{1 - \frac{8}{3e}}$$

$P_{e|1 \text{ sent}} = P\{K < 4\}$ $\quad$ K is poisson with mean 10

$$= e^{-10}\left(1 + 10 + \frac{10^2}{2} + \frac{10^3}{6}\right).$$

(c)

$$\pi_0\, p(y|0) \overset{0}{\underset{1}{\gtrless}} \pi_1\, p(y|1) \qquad \pi_0 = 0.9 \quad \pi_1 = 0.1$$

$$\pi_0 \cdot \frac{1}{K!} e^{-1} \overset{0}{\underset{1}{\gtrless}} \pi_1 \frac{10^K}{K!} e^{-10}$$

$$9 \cdot e^{-1} \overset{0}{\underset{1}{\gtrless}} 10^K e^{-10}$$

$$\ln(9) + 9 \overset{0}{\underset{1}{\gtrless}} K \ln(10).$$

$$4.86 \overset{0}{\underset{1}{\gtrless}} K$$

(d)

$$P(\text{error} \mid 0 \text{ sent}) = P\{K \geq 5\} \qquad K \sim \text{poisson}(1)$$

$$= 1 - P\{K \leq 4\}$$

$$= 1 - e^{-1}\left(1 + 1 + \tfrac{1}{2} + \tfrac{1}{6} + \frac{1}{24}\right)$$

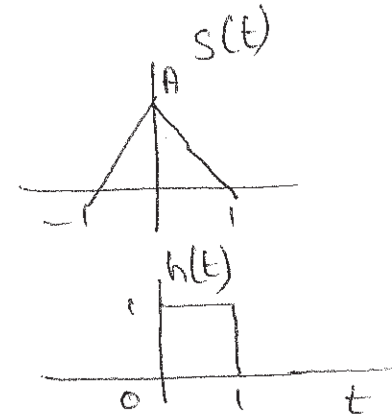
$$P(\text{error} \mid 1 \text{ sent}) = P\{K \leq 4\} \qquad K \sim \text{poisson}(10).$$

$$= e^{-10}\left(1 + 10 + \frac{10^2}{2} + \frac{10^3}{6} + \frac{10^4}{24}\right).$$

$$P_e = \pi_0 \cdot P_{e|0 \text{ sent}} + \pi_1 P_{e|1 \text{ sent}}.$$

6.15

$$y(t) = \begin{cases} s(t) + n(t) & 1 \text{ sent} \\ n(t) & 0 \text{ sent} \end{cases}$$



(a) $z(t) = y(t) * h(t)$

$Z = z(0) + Z(1)$.

when 0 is sent

$y(t) = n(t)$

$z(t) = (n * h)(t) = \int n(x)\, h(t-x)\, dx$

$z(0)$, $z(1)$ are both Gaussian

and since $h(t)$ and $h(1-t)$ are orthogonal

$z(0)$ and $z(1)$ are independent

$E\{z(t)\} = \int E\{n(x)\}\, h(t-x)\, dx = 0$

$\therefore E\{z(0)\} = E\{z(1)\} = 0$. $\quad \therefore E\{z\} = E\{z(0) + z(1)\}$
$= 0$.

$\text{var}(z) = \text{var}(z(0)) + \text{var}(z(1))$ $\because$ they are independent

$z(0) = \int n(x)\, h(-x)\, dx = \langle n(x)\, h(-x) \rangle$.

$$\mathrm{var}(z(0)) = \sigma^2 \|h(-x)\|^2 = \sigma^2 \|h(x)\|^2$$

$$z(1) = \int n(x)\, h(1-x)\, dx = \langle n(x)\ h(1-x) \rangle$$

$$\mathrm{var}(z(1)) = \sigma^2 \|h(1-x)\|^2 = \sigma^2 \|h(x)\|^2$$

$$\therefore \quad \mathrm{var}(z) = 2\sigma^2 \|h(x)\|^2$$

$$= 2\sigma^2.$$

$$\therefore \quad z \sim N(0, 2\sigma^2).$$

<u>when 1 is sent</u>

$$z(t) = (s(t) + n(t)) * h(t)$$

$$= s(t) * h(t) + n(t) * h(t).$$

observe that only mean changes.

$$\therefore \quad E\{z\} = E\{z(0) + z(1)\}$$

$$= (s * h)(0) + (s * h)(1)$$

$$= \frac{A}{2} + \frac{A}{2}$$

$$= A$$

$$\therefore \quad z \sim N(A, 2\sigma^2).$$

(b) The model in (a) falls within the standard Gaussian format, and the ML rule is given by

$$Z \underset{H_0}{\overset{H_1}{\substack{> \\ <}}} \frac{A}{2}$$

The error probability is

$$P_e = P_{e|0} = P_{e|1} = Q\left(\frac{A-0}{2\sqrt{2\sigma^2}}\right)$$

Noting that $E_b = \frac{1}{2}||s||^2 = \frac{A^2}{3}$, we obtain

$$P_e = Q\left(\frac{\sqrt{3E_b}}{2\sqrt{N_0}}\right) = Q\left(\sqrt{\frac{3E_b}{4N_0}}\right)$$

This is a factor of 3/4 (or about 1.25 dB) worse than optimal demodulation of OOK.
(c) The decision statistic

$$Z_2 = z(0) + z(0.5) + z(1) = \int_{-1}^{0} y(\tau)d\tau + \int_{-0.5}^{0.5} y(\tau)d\tau + \int_{0}^{1} y(\tau)d\tau = \langle y, v\rangle$$

where the effective correlator $v(t) = I_{[-1.-0.5]}(t) + 2I_{[-0.5,0.5]}(t) + I_{[0.5,1]}(t)$ (draw this to see how the shape compares with the original signal $s(t)$). We leave it as an exercise to show that $||v||^2 = 5$ and $\langle s, v\rangle = \frac{7}{4}A$. Following the same steps as in (b), we have $Z_2 \sim N(0, \sigma^2||v||^2 = 5\sigma^2)$ if 0 is sent, and $Z_2 \sim N(\langle s, v\rangle = \frac{7}{4}A, \sigma^2||v||^2 = 5\sigma^2)$ if 1 is sent. This again falls within the standard Gaussian format, and the error probability is given by

$$P_e = P_e = P_{e|0} = P_{e|1} = Q\left(\frac{\frac{7}{4}A - 0}{2\sqrt{5\sigma^2}}\right) = Q\left(\frac{\frac{7}{4}\sqrt{3E_b}}{2\sqrt{5N_0/2}}\right)$$

which simplifies to

$$P_e = Q\left(\sqrt{\frac{147E_b}{160N_0}}\right)$$

This is a factor of 147/160 (or 0.37 dB) worse than OOK with optimal demodulation. Thus, the degradation is smaller than in (b), which is to be expected since the shape of the effective correlator $v(t)$ here is a better approximation for the signal $s(t)$ than the effective correlator $u(t)$ in (b).

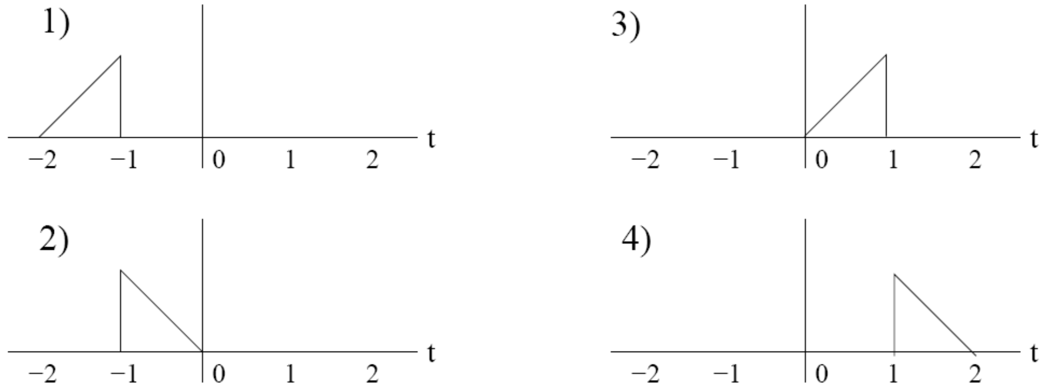


Figure 9: Natural basis for signal set in Problem 6.18, labeled in the order of the coordinates they correspond to in signal space.

Problem 6.18 Using the natural basis shown in Figure 9, the 4 signals map (not worrying about scale factors to the vectors $\mathbf{a} = (1,1,1,1)^T$, $\mathbf{b} = (1,1,-1,-1)^T$, $\mathbf{c} = (1,-1,1,-1)^T$, $\mathbf{d} = (1,-1,-1,1)^T$. It is easy to check that these vectors are all orthogonal, so this is 4-ary equal energy, orthogonal signaling. Thus, $E_s \equiv 4$ (hence $E_b = \frac{E_s}{\log_2 4} = 2$) and all pairwise distances are given by $d^2 = 2E_s = 8$. The power efficiency $d^2/E_b = 4$.
(a) The union bound is given by

$$P_{e|c} \leq Q(\tfrac{d_{ca}}{2\sigma}) + Q(\tfrac{d_{cb}}{2\sigma}) + Q(\tfrac{d_{cd}}{2\sigma}) = 3Q(\tfrac{d}{2\sigma})$$
$$= 3Q\left(\sqrt{\frac{d^2}{E_b}\frac{E_b}{2N_0}}\right) = 3Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

(b) False: not more power-efficient than QPSK. The power efficiency equals 4, which is the same as that of QPSK.

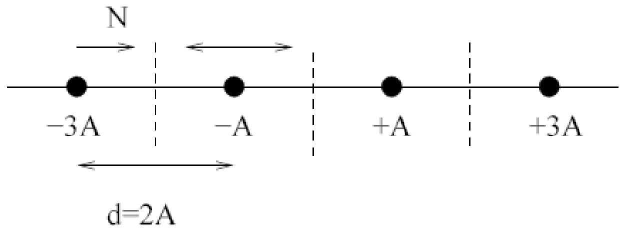


Figure 13: 4PAM constellation and decision boundaries for Problem 6.21.

Problem 6.21: The 4PAM constellation is shown in Figure 13. Let $N \sim N(0, \sigma^2)$ denote the noise sample corrupting the received signal. Conditioned on an outer signal point being sent (consider the leftmost point without loss of generality), the error probability is

$$P(e|outer) = P[N > d/2] = Q\left(\frac{d}{2\sigma}\right)$$

where $d = 2A$. Conditioned on one of the two inner points being sent, the error probability is

$$P(e|inner) = P[|N| > d/2] = 2Q\left(\frac{d}{2\sigma}\right)$$

Note that $E_s = \frac{1}{2}\left(A^2 + (3A)^2\right) = 5A^2$ and $E_b = E_s/\log_2 4 = 5A^2/2$, so that $d^2/E_b = (2A)^2/(5A^2/2)$ $8/5$. We therefore can write

$$P_e = \frac{1}{2}\left(P(e|outer) + P(e|inner)\right) = \frac{3}{2}Q\left(\frac{d}{2\sigma}\right) \quad = \frac{3}{2}Q\left(\sqrt{\frac{d^2}{E_b}\frac{E_b}{2N_0}}\right) = \frac{3}{2}Q\left(\sqrt{\frac{4E_b}{5N_0}}\right)$$

16QAM can be viewed as a product of 2 4PAM constellations sent in parallel over independent WGN channels, so that a symbol error occurs if either of the two 4PAM symbols are received incorrectly. Thus,

$$P_e(16QAM) = 1 - (1 - P_e(4PAM))^2 = 2P_e(4PAM) - P_e^2(4PAM)$$

Using the 4PAM error probability bound yields the following bound on the error probability of 16QAM:

$$P_e(16QAM) = 3Q\left(\sqrt{\frac{4E_b}{5N_0}}\right) - \frac{9}{4}Q^2\left(\sqrt{\frac{4E_b}{5N_0}}\right)$$

Problem 6.31
(a) For $b = 0, 1$, the received signal is given by

$$y = (-1)^b A + N(0, \sigma^2)$$

Thus,

$$p(y|0) = \frac{1}{2\pi\sigma^2} e^{-(y-A)^2/2\sigma^2}, \qquad p(y|1) = \frac{1}{2\pi\sigma^2} e^{-(y-A)^2/2\sigma^2}$$

The LLR is given by

$$LLR(b) = \log \frac{P(b=0|y)}{P(b=1|y)} = \log \frac{p(y|0)P[b=0]}{p(y|1)P[b=1]} = \log \frac{P[b=0]}{P[b=1]} + \log \frac{p(y|0)}{p(y|1)}$$

For equiprobable bits, the first term on the extreme right-hand side is zero. Upon simplifying, we get

$$LLR(b) = \log \frac{p(y|0)}{p(y|1)} = \frac{2Ay}{\sigma^2}$$

Since y is conditionally Gaussian, conditioned on the bit sent, so is the LLR. The conditional mean and variance are easily seen to be $(-1)^b \frac{2A^2}{2\sigma^2}$ and $\frac{4A^2}{\sigma^2}$, respectively. These depend only on the SNR $\frac{A^2}{\sigma^2}$. (For uncoded transmission, $E_b = A^2$ and $\frac{A^2}{\sigma^2} = \frac{2E_b}{N_0}$.)
(b) For uncoded transmission,

$$P_e = Q\left(\frac{A}{\sigma}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

Numerical search shows that $Q(1.2815) \approx 0.1$, so that an error probability of 10% corresponds to $\frac{A}{\sigma} = 1.2815$. From (a), this implies that the conditional distribution of the LLR, conditioned on 0 sent, is $N(3.285, 6.57)$.