Homework 2

Due: **MONDAY**, January 30, 2011, **at 5 pm**, in the homework boxes outside Harold Frank Hall 3120.

Notes:
1. As you see above, the homework deadline is delayed this week.
2. I will be at a DARPA Principal Investigator meeting Mon. (1/23) through early Thurs. (1/26).
3. We have a guest lecturer (Petar Kokotovic) teaching lecture Tuesday (1/24).
4. My regular office hours Monday (1/23) and Wednesday (1/25) are canceled.
5. My NEW OFFICE HOURS this week are Friday (1/27), 10am-noon.
6. I will be checking email nightly while on the road.

Problem 1. Final Value Theorem (FVT).
This is really a "tutorial" problem. i.e., the problem itself should be relatively easy, and the preamble is given to provide a derivation of the FVT, only for your own understanding. (In fact, you could probably just skip to the top of page 2, if you wish…)

[Tutorial] Imagine you are given a discrete-time transfer function, F(z). You wish to calculate what the final value will be, given a unit pulse input, *for a STABLE (all poles in the unit circle) system.*

Intuitive approach.
We anticipate the method will be analogous to what holds for continuous-time (CT) TF's.
CT FVT: $\lim_{t\to\infty} x(t) = \lim_{s\to 0}(s \cdot X(s))$
A CT zero at the origin of the s-plane corresponds to a DT zero at z=1, so instead of (s-0)*X(s), we might expect to evaluate (z-1)*X(z). Similarly, instead of evaluating the limit as s goes to 0 on the s-plane, we might expect to evaluate the DT limit as z goes to +1 on the z-plane. Intuitively, this gives:

DT FVT: $\boxed{\lim_{k\to\infty} x(k) = \lim_{z\to 1}((z-1)\cdot X(z))}$

Mathematical approach.
Let us be more formal. First, note that:

1. $\mathcal{Z}[x(k)] = \boldsymbol{X(z)} = \sum_{k=0}^{\infty} \boldsymbol{x_k z^{-k}} = x_0 z^0 + x_1 z^{-1} + x_2 z^{-2} + \cdots + x_{ss} z^{-\infty}$
2. $\mathcal{Z}[x(k+1)] = \sum_{k=0}^{\infty} \boldsymbol{x_{k+1} z^{-k}} = x_1 z^0 + x_2 z^{-1} + x_3 z^{-2} + \cdots + x_{ss} z^{-\infty}$
   $= \boldsymbol{z \cdot X(z) - x_0 z^{+1}}$

Here, x_{ss} is the final value of x(k), i.e., as k goes to infinity.

Now, take equation 2. minus 1. to get:

3. $\left(\boldsymbol{z \cdot X(z) - x_0 z^{+1}}\right) - \boldsymbol{X(z)} = \sum_{k=0}^{\infty}\left[\boldsymbol{x_{k+1} z^{-k}} - \boldsymbol{x_k z^{-k}}\right]$

Now, imagine that we reach a steady state, where you can write the difference equation such that all x_k equal x_{ss}. If so, there is no point distinguishing among powers of z. (For a CT differential equation, we set derivatives to zero at steady state; here we x_k converge to a single value, at steady states…) This corresponds to the limit as all z^k are simply equal to z^0=1. So, letting z=1, we take equation 3,

$$\left(\boldsymbol{z \cdot X(z) - x_0 z^{+1}}\right) - \boldsymbol{X(z)} = \sum_{k=0}^{\infty}\left[\boldsymbol{x_{k+1} z^{-k} - x_k z^{-k}}\right]$$

Rearranging, and taking the limit as $z \to 1$,

$$\lim_{z\to 1}((\boldsymbol{z-1})\cdot \boldsymbol{X(z)} - \boldsymbol{x_0 z}) = \lim_{z\to 1} \sum_{k=0}^{\infty}\left[\boldsymbol{x_{k+1} z^{-k} - x_k z^{-k}}\right]$$

In the equation above, we will need to check if "(z-1)" can be factored out of the denominator of X(z) before evaluating the limit. (This is analogous to leaving "s*X(s)" in the CT FVT…) Simplifying:

$$\lim_{z\to 1}\big((z-1)\cdot X(z)\big) - x_0 = [x_1 - x_0] + [x_2 - x_1] + \cdots + [x_{\infty-1} - x_{\infty-2}] + [x_\infty - x_{\infty-1}]$$
$$\lim_{z\to 1}\big((z-1)\cdot X(z)\big) - x_0 = [-x_0] + [x_\infty]$$
$$\lim_{z\to 1}\big((z-1)\cdot X(z)\big) = x_\infty = x_{ss} = \lim_{k\to\infty} x(k) \qquad \textbf{DT FINAL VALUE THEOREM}$$

For a UNIT STEP RESPONSE, recall that the transfer function for a unit step is z/(z-1). For a continuous-time system, the final value to a unit step input is:

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0}\left(s \cdot \frac{1}{s} X(s)\right)$$

Similarly, for a discrete-time system, **the final value to a unit step input** is:

4. $\lim_{k\to\infty} x(k) = \lim_{z\to 1}\left((z-1)\cdot\frac{z}{(z-1)}X(z)\right)$

So, the final value to a unit step (above) just amounts to plugging in "z=1" in the transfer function F(z).

Using equation 4, for each system (a-e) below,
i) (Try to) evaluate the final value for a unit step input for the DT TF.
ii) Solve for the pole locations on the z-plane.
iii) Sketch the unit step response by hand. (I suggest you use MATLAB to check the step response, but then sketch a plot by hand and label axes clearly.)

a) $\frac{z}{z^2-0.6z+0.4}$ b) $\frac{z}{z^2-0.6z-0.4}$ c) $\frac{z}{z^2+0.6z+0.4}$ d) $\frac{z}{z^2+0.6z-0.4}$

e) What happens when you try to use the DT FVT for a system with a pole at z=1? (Why?) Mathematically, how can you calculate the steady-state *slope* of the unit step response in this case?

f) What happens for systems with poles elsewhere on the unit circle? (i.e., not at z=1, but at a distance exactly 1 from the origin on the z-plane.) What info does the FVT give in this (marginally stable) case?

Problem 2. Unit step response characteristics for DT TF.

We have mentioned briefly in class that there is a correspondence between poles on the s-plane at a given "s" (where s is a complex number) and poles on the z-plane at "$z = e^{sT}$".

Show that this is the case for each of the three systems (a-c) below. Specifically,

i) Solve for the poles on the z plane (roots of the denominator) for each DT TF.

ii) Find corresponding poles on the s plane, such that $z = e^{sT}$, and derive a CT TF with these poles and with the same DC gain. [Show work!!]

iii) Use MATLAB to plot the DT and (your calculated) CT unit step response for each system. Use one set of axes for each pair (DT and CT) of step responses.

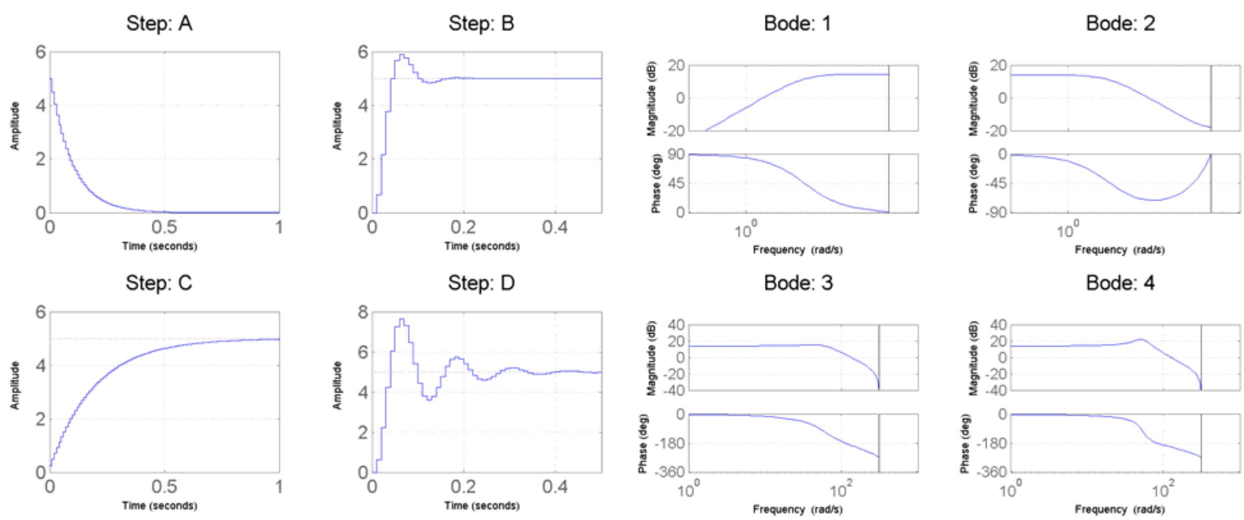
[*Note: Do <u>not</u> worry about the zeros in the problem, just the poles! Recall that z-plane zeros essentially map to time delays in the step response. You can use MATLAB's d2c to verify your answers to part ii. First, create the DT TF; then convert it to a CT TF and find poles of this CT TF. Use "help tf" and "help d2c" for information on usage. You must still show work for credit in ii, however.*]

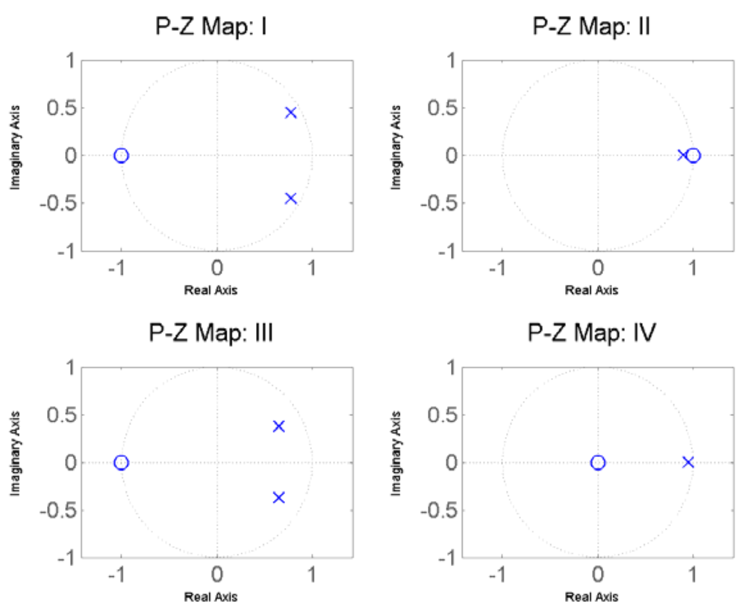
a) $\frac{1}{z-0.9}$ b) $\frac{z+1}{z^2-1.975z+1}$ c) $\frac{z+1}{z^2-1.929z+0.9537}$

Problem 3. Discrete-time dynamics.
Below are 4 step responses (a,b,c,d), 4 Bode plots (1,2,3,4), and 4 pole-zero plots (I,II,III,IV).

a) Match these up. (i.e., list the 4 correctly-matched triplets. e.g., if A, 1, and I match, write “A-1-I”.)

b) Estimate the discrete-time transfer function for each system. If all cases, sampling time is T=.01 sec.





Problem 4. Tustin with pre-warp.

Assume we wish to match both the magnitude and phase of the continuous-time controller below, C(s), when evaluated at the frequency $\omega_o = 100$ rad/sec. Use a sampling time of T = 0.001 seconds.

$$C(s) = 10\frac{(s+50)}{(s+200)}$$

a) By hand, plug in $s = j\omega_o = 100j$ rad/sec, and calculate gain and phase at this frequency.

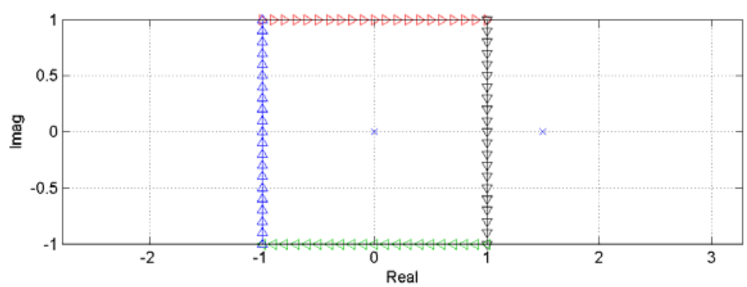
b) By hand, solve for $C_b(z)$ by replacing s with $\frac{2}{T}\frac{(z-1)}{(z+1)}$, which is the standard Tustin transformation. Evaluate magnitude and phase for $z = e^{sT} = e^{j\omega_o T}$.

c) Now, solve for $C_c(z)$ by replacing s with $\alpha\frac{(z-1)}{(z+1)}$, where alpha is selected via the Tustin with pre-warp transformation, for $\omega_o = 100$ rad/sec. By hand, show magnitude and phase do in fact match the result in part a), as desired.

d) At what frequency do both the magnitude and phase of $C_b(z)$ and $C(s)$ match? (i.e., using standard Tustin, without pre-warp. Hint, think about what α=2/T would correspond to…)

Problem 5. Nyquist intuition tutorial.
By now, you should basically know how Nyquist works. This problem is designed in hope of giving a slightly more intuitive about "why" it works.

Using MATLAB, write a script to do the following, toward creating two labeled output plots (a and b):
a-i) Create a set of roughly 100 points, evenly spaced and going clockwise around the box shown below on the s-plane. (Instead of enclosing the righthand plane, our contour encloses some other, arbitrary area… Each point on the contour is a complex number.)



a-ii) Using MATLAB, evaluate the function $G_a(s) = \frac{1}{s}$ for each point from part i above, and plot the resulting closed curve. Clearly label where each of the 4 "sides" (blue/left; red/top; black/right; green/bottom) has been mapped to on your new plot. (Use a different color or line type or set of symbols for each edge.) Explain the direction of encirclement (CW vs CCW) of the origin.

a-iii) For what values of K will all pole(s) of $\frac{KG}{1+KG}$ all be inside the contour above? (Why?) For what values of K will all pole(s) of $\frac{KG}{1+KG}$ all be outside the contour above? (Why?)

b) Repeat steps ii and iii for $G_b(s) = \frac{1}{s(s-1.5)}$ (instead of for $G_a(s)$).