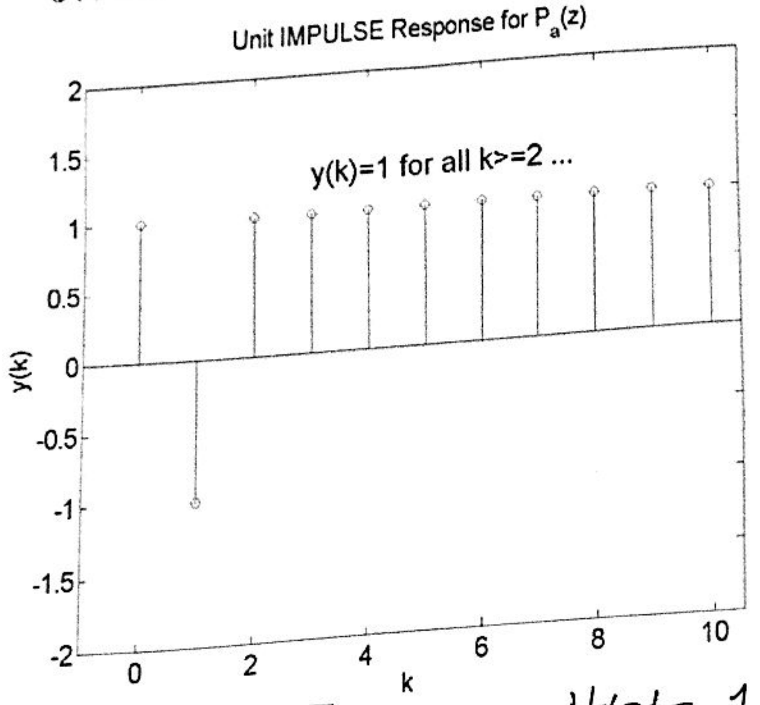
1) Estimating a discrete-time transfer function from a time response.
Please note whether each shows an **IMPULSE *(part a)*** *or* **STEP (part b)** *response!!*

a) For the **unit impulse** response shown below, estimate the corresponding plant, $P_a(z)$.
($y(k)=0$ for all $k<0$ here.)



$h(n) = \delta(k) \Rightarrow H(z) = 1$

$\Rightarrow Y(z) = \cancel{H(z)}\, P_a(z)$

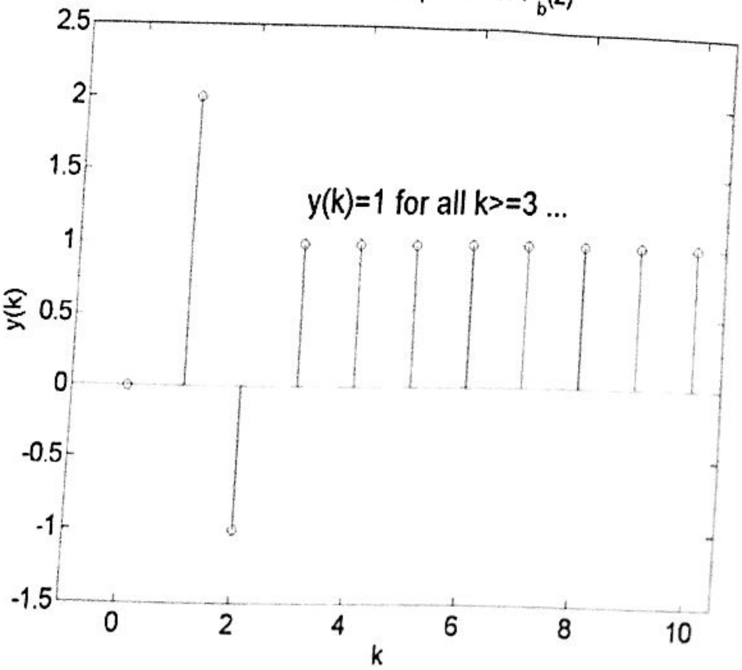
$Y(z) = P_a(z) = 1 - 1z^{-1} + \sum_{n=2}^{\infty} z^{-n}$

$= 1 - z^{-1} + z^{-2} \sum_{n=0}^{\infty} z^{-n} = 1 - z^{-1} + \frac{z^{-2}}{1-z^{-1}}$

$= \frac{z-1}{z} + \frac{z^{-1}}{z-1} = \frac{z^2 - 2z + 1 + z\,z^{-1}}{z(z-1)}$

$$\boxed{P_a(z) = \frac{z^2 - 2z + 2}{z^2 - z}}$$

b) For the **unit step** response shown below, estimate the corresponding plant, $P_b(z)$. ($y(k)=0$ for all $k<=0$ here.)



$$u(k) = \begin{cases} 1 & k \geq 0 \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow U(z) = \frac{z}{z-1}$$

$$Y(z) = 2z^{-1} - z^{-2} + \frac{z^{-3}}{1-z^{-1}} = \frac{2(z-\frac{1}{2})}{z^2} + \frac{z^{-2}}{z-1}$$

$$P_b(z) = Y(z)/U(z) = \left(\frac{2z-1}{z^2} + \frac{z^{-2}}{z-1}\right)\frac{z-1}{z}$$

$$= \frac{(2z-1)(z-1)}{z^3} + \frac{1}{z^3} = \boxed{\frac{2z^2-3z+2}{z^3} = P_b(z)}$$

Similarly

FIR sysytem => Deadbeat

steady state @ k=3 => $z^3 = 0$

K	u	y
-3	0	0
-2	0	0
-1	0	0
0	1	0
1	1	2
2	1	-1
3	1	1
4	1	1

You can find difference eqn. from here as well.

2) Below is the transfer function for a continuous-time controller:

$$C(s) = \frac{s+1}{s+4}$$

a) Convert $C(s)$ to a discrete-time transfer function, $C(z)$, using the Tustin transformation (no "pre-warp"). Use a sampling time of T=0.01 seconds. $T = 0.01$

$$s = \frac{2}{T}\ \frac{z-1}{z+1}$$

$$C(z) = \frac{\frac{2}{T}\left(\frac{z-1}{z+1}\right) + 1}{\frac{2}{T}\left(\frac{z-1}{z+1}\right) + 4} = \frac{2(z-1) + T(z+1)}{2(z-1) + 4T(z+1)}$$

$$= \frac{(2+T)z \quad -2+T}{(2+4T)z \quad -2+4T} = \boxed{\frac{2.01z - 1.99}{2.04z - 1.96} = C(z)}$$

b) *Rapidly* estimate the gain and phase of C(z) at 2 rad/sec. (Hint: evaluate C(s), using appendix to estimate phase of C(s) quickly! C(s) and C(z) should be close here.)

$$C(s)\Big|_{j2} = \frac{j2+1}{j2+4} = \frac{j2+1}{2(j+2)}$$

$$|C(j2)| \approx \frac{1}{2}\ \frac{|j2+1|}{|j+2|} = \boxed{\frac{1}{2}} \leftarrow \text{Gain}$$

$$\angle C(j2) = \angle(j2+1) - \angle(j+2) = \text{atan}(2) - \text{atan}\left(\tfrac{1}{2}\right)$$

$$\approx 63^\circ - 26^\circ = \boxed{37^\circ}$$

$\uparrow$ Phase

c) To evaluate the *exact* frequency response of C(z) at 2 rad/sec requires plugging in some value for z. Estimate numerically what this value of "z" is, and SKETCH this value of z on the z-plane. (Show all or a blown-up subsection of the unit circle in this plot. Recall, T=0.01 sec.)

$$z = e^{sT} = e^{0.02} = \cos(0.02) + j\sin(0.02)$$

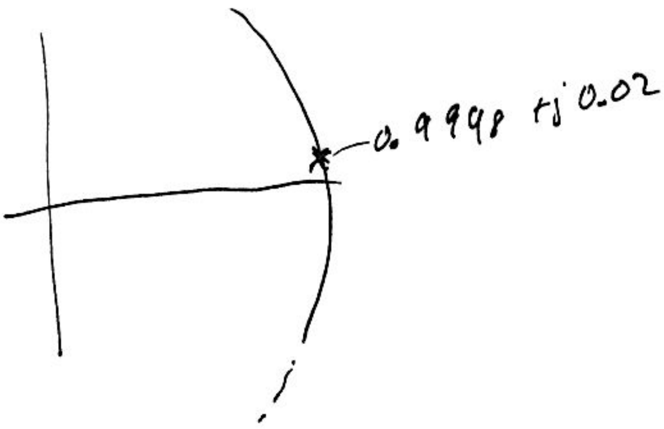
$$\approx \cos(0.02) + j\,\cancel{\sin}\,0.02$$

$$\cos^2(0.02) + (0.02)^2 \approx 1$$

$$\Rightarrow \cos^2(0.02) = 1 - 0.0004 = 0.9996$$

$$\cos(0.02) = \sqrt{0.9996} = 0.9998$$

$$\boxed{z \approx 0.9998 + j0.02}$$



3) Discrete-time system time response analysis. Given a second-order system, P(z), with sampling time T=0.1 seconds and poles on the z-plane at

$$z = 0.99 \pm 0.03j$$

answer the following questions about a ***unit step*** response for the system:

a) Approximately how many samples are there in one oscillation, and what is the time for one full cycle (peak-to-peak)?

$z \approx 1 + sT \Rightarrow s \approx \frac{-0.01 \pm j\,0.03}{T}$

$s \approx -0.1 \pm j\,0.3$

$\omega_d \approx 0.3$

$T_{pp} = T_{cycle} = \frac{2\pi}{\omega_d} = \frac{2\pi}{0.3} \approx \frac{6.28}{0.3} \approx 20.9333$ seconds

$\Rightarrow$ samples $K = \frac{T_{pp}}{T} = \frac{20.933}{0.01} = 209.33 \approx 210$ samples

b) What is the time constant of the decay envelope? $\tau = \frac{-1}{Re\{s\}}$

$Re\{s\} \approx -0.1$

$\Rightarrow \tau = \frac{1}{0.1} = 10$ sec.

c) Assume the system has a single zero at $z = -1$. Either write an expression for (or estimate numerically) the percent overshoot of a unit step response. (i.e., the percentage by which the response exceeds the final value.)

$t_{peak} = \frac{1}{2} T_{pp} \approx 10.465$

$PO = 100\% * \exp\left\{-\frac{t_{peak}}{\tau}\right\} = 100 \exp\{-1.0465\}$

4) For the discrete-time transfer function with T=0.01 seconds, shown below:

$$G(z) = \frac{4}{(z-0.9)(z-0.96)}$$

a) What are the DC gain and phase? $\Rightarrow z=1$

$$|G(z)|\Big|_{z=1} = \frac{4}{(1-0.9)(1-0.96)} = \frac{4}{(0.1)(0.04)} = 1000$$

$$\phi(G(1)) = 0 \qquad \Rightarrow 60\,dB$$

b) What is the Nyquist frequency?

$$\omega_{Nyq} = \frac{\pi}{T} \approx \frac{3.14}{0.01} = 314 \text{ rad/sec}$$

c) What are the gain (approximately, to within +/-3dB) and phase (exact value) at the Nyquist frequency?

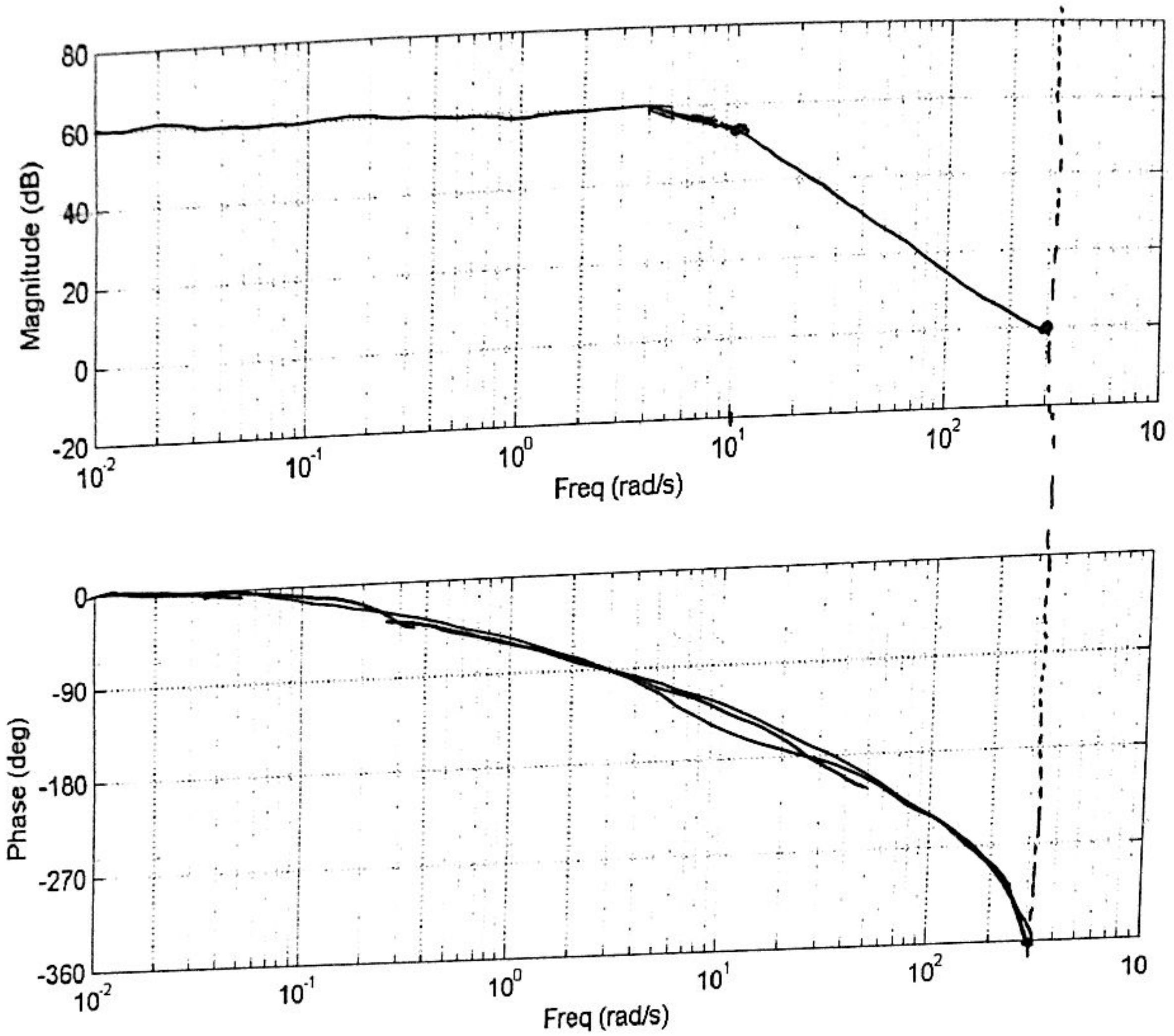
$Nyq \Rightarrow z=-1$

$$|G(z)|\Big|_{z=-1} = \frac{4}{(-1-0.9)(-1-0.96)} = \frac{4}{(-1.9)(-1.96)} \approx \frac{4}{4} = 1 \Rightarrow 0\,dB$$

2 poles & ~~[illegible]~~ phase delay from sampling

$$\phi(G(z))\Big|_{z=-1} = -360^\circ$$

d) Sketch a Bode plot for G(z) on the axes provided. (Hint, you can break the TF into two parts, if helpful. Multiplying two transfer functions corresponds to adding their respective Bode plots.) Carefully label breakpoints for all poles and/or zeros, and show the Nyquist frequency (e.g., as a dashed vertical line).



$$z \approx 1 + sT \Rightarrow s = \frac{z-1}{T}$$

$$s_1 = \frac{0.96-1}{0.01} = -4$$

$$s_2 = \frac{0.9-1}{0.01} = -10$$

5) Consider the discrete-time state space system,

$$x_{k+1} = A_d x_k + B_d u_k$$
$$y_k = C_d x_k + D_d u_k$$

with the following matrices:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad A_d = \begin{bmatrix} 0.99 & -1 \\ 0.01 & 1 \end{bmatrix} \quad B_d = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C_d = [0 \quad 1] \quad D_d = [0]$$

a) What is the characteristic equation for the DT SS system shown? Is the system stable, unstable, or marginally stable – and why?

$$|zI_2 - A_d| = \begin{vmatrix} (z-0.99) & 1 \\ -0.01 & z-1 \end{vmatrix} = (z-0.99)(z-1) + 0.01$$
$$= z^2 - 1.99z + 0.99 + 0.01$$
$$\boxed{= z^2 - 1.99z + 1}$$

complex poles $\ni |z_p| = 1$

=> marginally stable

poles lie on unit circle

b) Find the four matrices A, B, C, and D for a continuous-time state space representation that maps to this DT SS system via the forward Euler transformation, with T=.01 seconds. (As partial hints, recall for this transformation that: $A_d = I + AT$, $B_d = BT$, ...)

$$A_d = I + AT \Rightarrow A = (A_d - I)\frac{1}{T} = \begin{bmatrix} -0.01 & -1 \\ 0.01 & 0 \end{bmatrix} \frac{1}{0.01}$$
$$= \boxed{\begin{bmatrix} -1 & -100 \\ 1 & 0 \end{bmatrix} = A}$$

$$B_d = BT \Rightarrow B = B_d/T = \boxed{\begin{bmatrix} 100 \\ 0 \end{bmatrix} = B}$$

$$\boxed{\begin{matrix} C = C_d \\ D = D_d \end{matrix}}$$

[continues next page...]

c) Solve for the control gain matrix, K, to place all poles at the origin (i.e., deadbeat control) for the original **discrete-time** state space system from part a.

Want char. eqn $= 0$

i.e. $\left| zI_2 - (A_d - B_d * K) \right| = 0$

$$\left| zI_2 - \begin{bmatrix} 0.99 - K_1 & -1 - K_2 \\ 0.01 & 1 \end{bmatrix} \right|$$

$$= \begin{vmatrix} z - (0.99 - K_1) & (1 + K_2) \\ -0.01 & z - 1 \end{vmatrix}$$

$$= (z-1)(z - 0.99 + K_1) + 0.01 + 0.01 K_2$$

$$= z^2 - 0.99z + K_1 z - z + 0.99 - K_1 + 0.01 K_2 = 0$$

$$\Rightarrow z[-0.99 - 1 + K_1] = 0 \Rightarrow K_1 = 1.99$$

$$\Rightarrow z^0 [0.99 - K_1 + 0.01 K_2] = 0$$

$$\Rightarrow 0.99 - 1.99 + 0.01 K_2 = 0$$

$$\Rightarrow .01 K_2 = 0.99$$

$$\Rightarrow K_2 = 99$$

$$\boxed{K = [1.99, 99]}$$

Verify..

$$\begin{vmatrix} z - (0.99 - 1.99) & (1 + 99) \\ -0.01 & z - 1 \end{vmatrix} = \begin{vmatrix} z + 1 & 100 \\ -0.01 & z - 1 \end{vmatrix}$$

$$= z^2 - 1 + 1 = z^2 \checkmark$$

Works! :)