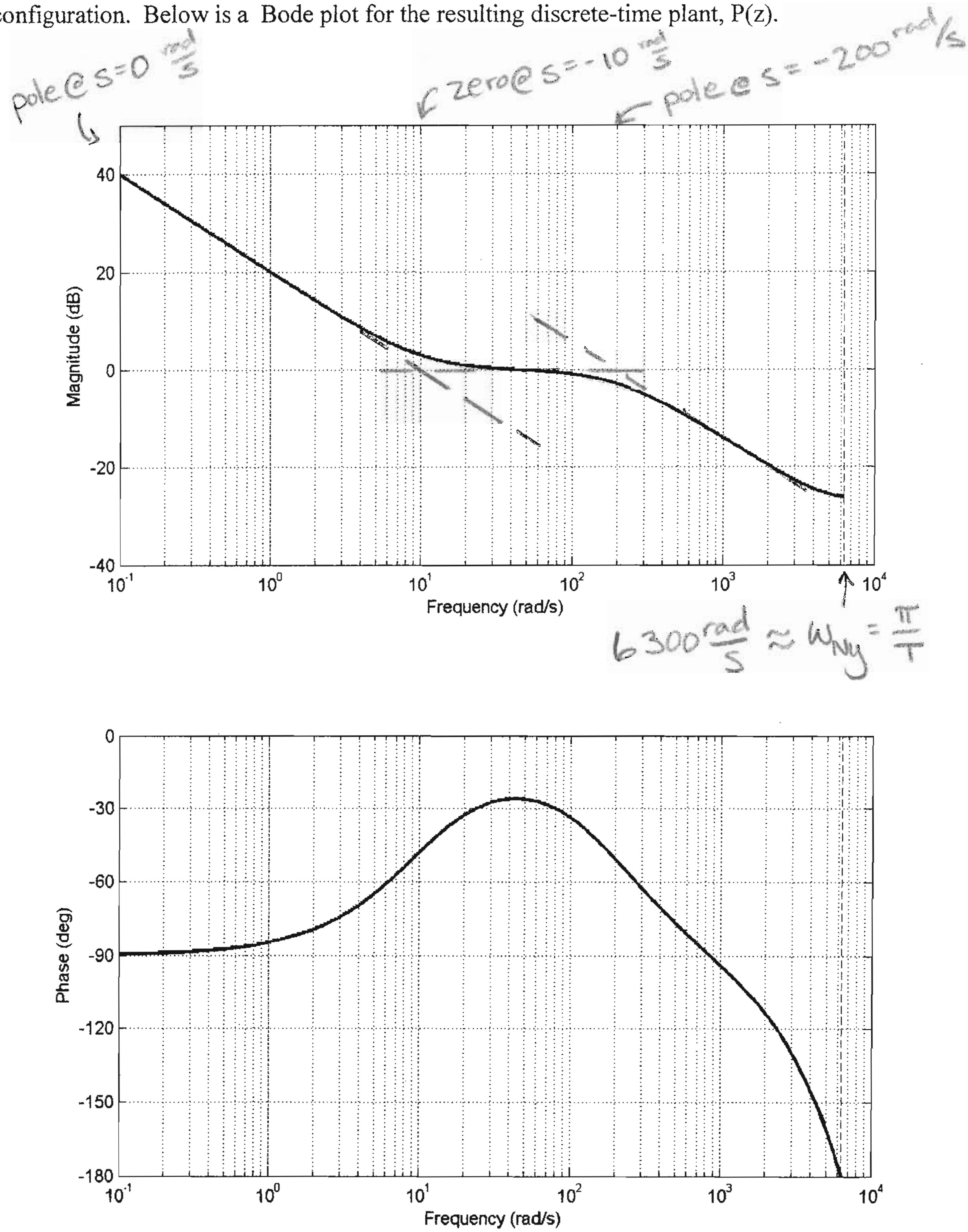
1) In this problem, a continuous-time plant, P(s), is in the standard ZOH (zero-order hold) configuration. Below is a Bode plot for the resulting discrete-time plant, P(z).



(Problem 1, continued…)

4 pts a) The thin, dashed line toward the right edge of the plot indicates the Nyquist frequency for the system. What is the sample time, T (approximately)?

$$\omega_{Ny} = 6300 \frac{rad}{s} \approx \frac{\pi\,(rad)}{T\,(s)}, \quad \therefore T = \frac{\pi}{6.3e3}(s) = 0.5\times10^{-3}\,s$$

$$\boxed{T \approx .0005 \text{ sec}}$$

6 b) Estimate the poles and zeros (s plane values) for the CONTINUOUS TIME system, G(s).

poles: $\begin{cases} s=0 \ (rad/s) \\ s=-200\,(rad/s) \end{cases}$

zero: $\{ s = -10 \ (rad/s)$

6 c) Estimate the poles and zeros (z plane values) for the actual DISCRETE-TIME system, G(z).

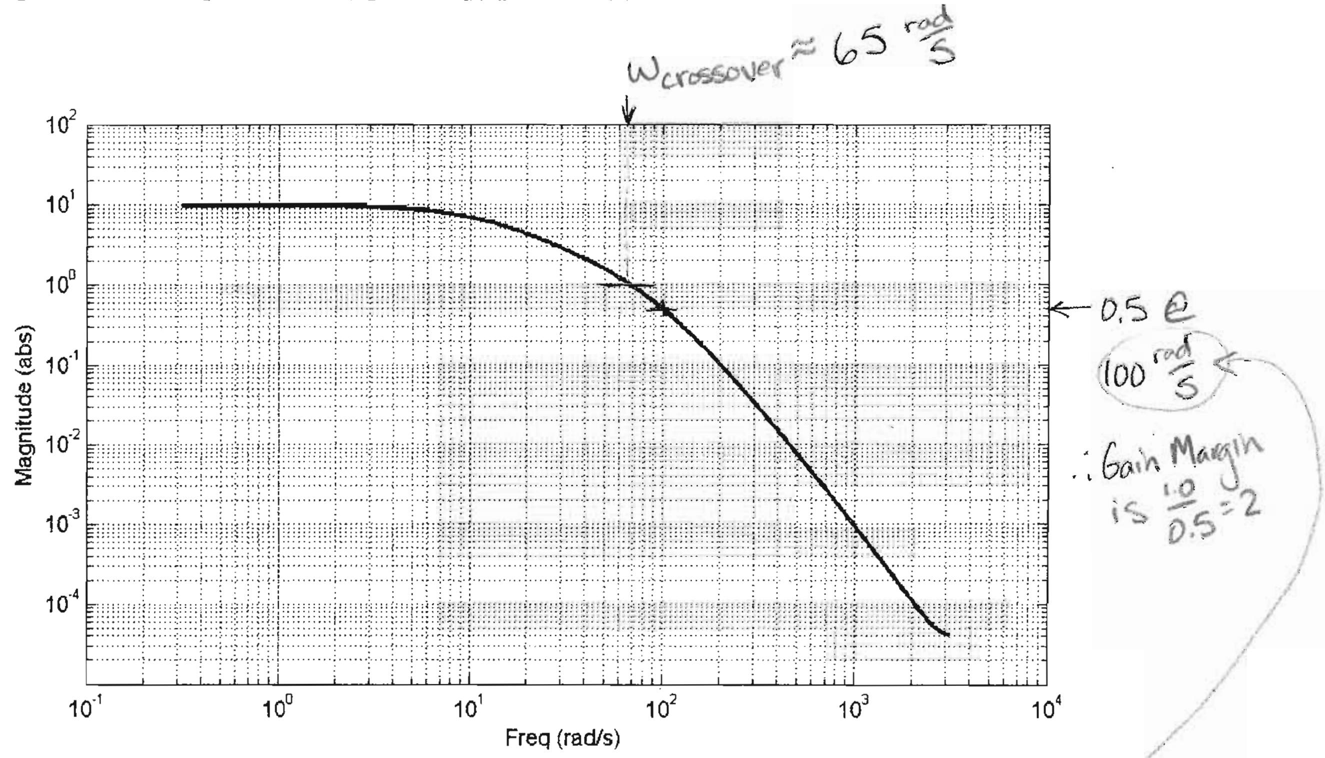
$$z = e^{sT}$$

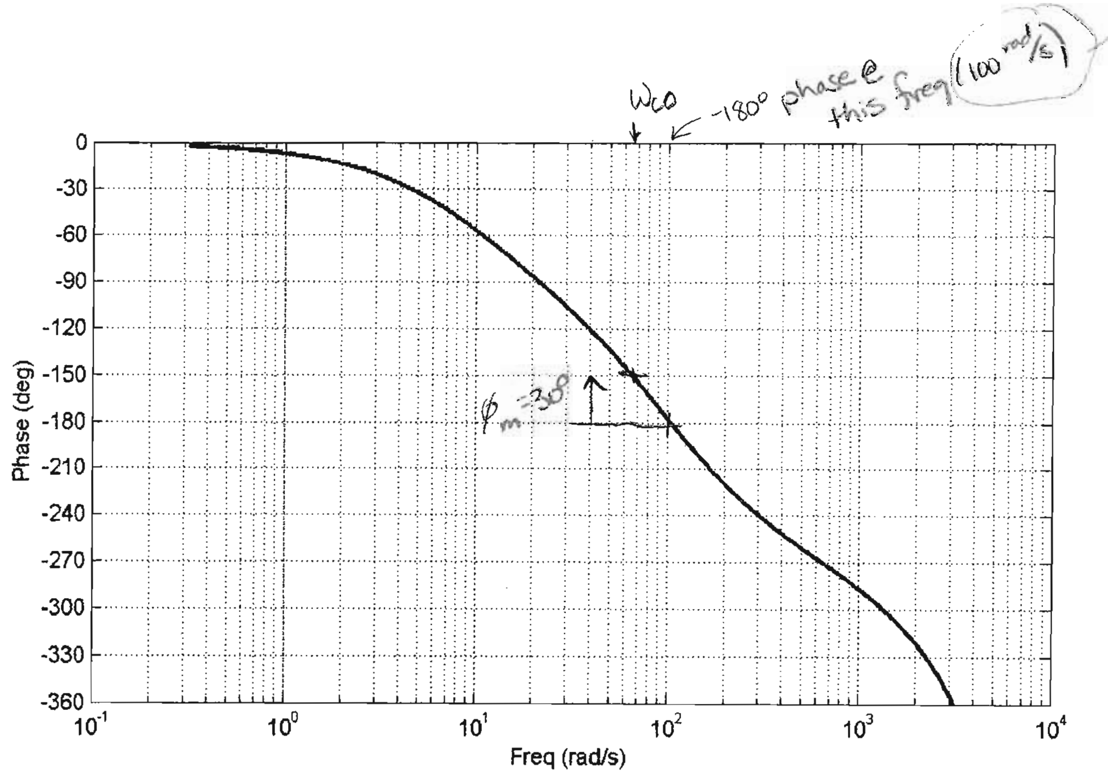
Ans: $\boxed{\text{poles} \begin{cases} z=1 \\ z\approx .9 \end{cases} \quad \text{zero} \{ z \approx .995}$

poles $\begin{cases} z = e^{0} = 1 \\ z = e^{-200\times5\times10^{-4}} = e^{-10^{3}\times10^{-4}} = e^{-0.1} \approx 0.9 \end{cases}$

zero $\begin{cases} z = e^{-10\times5\times10^{-4}} \\ = e^{-50\times10^{-4}} \\ = e^{-.005} \approx .995 \end{cases}$

2) In this problem, you are asked to analyze proposed discrete-time controllers on a given plant. A Bode plot of the (open-loop) plant, G(z), is shown below. T=.001 seconds.





(Problem 2, continued…)

/6 total, see below

a) First, assume we plan to close the loop using a proportional controller with unity gain: $C(z)=1$. From the plot on the previous page, estimate the following. (You must also label the plot to illustrate each value, too!)

2 i) Crossover frequency : $\boxed{65 \text{ rad/s}}$ } where is gain @ 1.0?

2 ii) Gain margin : $\boxed{2.0 \approx 6\text{dB}} \leftarrow = \frac{1.0}{0.5}$ } @ 100 rad/s

2 iii) Phase margin : $\boxed{30^\circ} \leftarrow (-150^\circ)-(-180^\circ)=30^\circ$ } @ $\omega_{co}=65\frac{\text{rad}}{s}$

2 b) Would the resulting closed loop system (from part a) be stable? (Why or why not?)

$\boxed{\text{Yes! } \phi_m > 0.}$

4 c) Suppose your lab partner proposes the following continuous-time controller:

$$C(s) = K\frac{(s+\frac{100}{2.5})}{(s+2.5\cdot 100)} = K\frac{(s+40)}{(s+250)}$$

What is the peak phase this CT controller adds, and at what frequency does this occur? (Hint: See Appendix B!)

From App. B, $\boxed{+47^\circ \text{ phase @ } 100 \text{ rad/s}}$

To calculate K_{DT}, first find K_{CT}: $|C(s)|$ is ⇒

250 rad/s

$(K_{CT}\frac{40}{25} = \frac{K}{(2.5)^2})$ 40 rad/s $(K_{CT}(\frac{1}{2.5}))$ @ 100 rad/s (K_{CT})

We require Gain @ 100 rad/s $= 1 = (\frac{1}{2})(\frac{K_{CT}}{2.5})$

8 total, see below

• Value of K worth 4 pts.

• a & b worth 2 pts each

d) Design a DT controller, C(z), with a crossover frequency of 100 rad/s that provides same phase at crossover as the CT design in part c. Your controller should have the form:

$$C(z) = K\frac{(z-a)}{(z-b)}$$ (Recall T=.001 sec.)

∴ Prewarp not required!!

Note, for "prewarp", $\alpha = \frac{\omega_0}{\tan(\frac{\omega_0 T}{2})}$, where $\frac{\omega_0 T}{2} = \frac{(100\frac{\text{rad}}{s})(.001s)}{2} = .05 \text{ rad} = $ "small"

$\therefore \tan(\frac{\omega_0 T}{2}) \approx \frac{\omega_0 T}{2} \rightarrow \boxed{\alpha \approx \frac{\omega_0}{\omega_0(\frac{T}{2})} = \frac{2}{T}}$

{ $\tan(\Delta)\approx\Delta$, Δ "small" }

We need an additional gain of 2.0 @ 100 rad/s to set crossover to 100 rad/s,

$\therefore 2 = \frac{K_{CT}}{2.5} \rightarrow \boxed{K_{CT}=5} \leftarrow$ CT, NOT DT!

$\therefore \boxed{K_{DT} \approx 4.5}$ $\boxed{a \approx .96}$

$\boxed{\text{Tustin: } s = \alpha\frac{z-1}{z+1} \approx \frac{2}{T}\left(\frac{z-1}{z+1}\right)}$ (w/ pre-warp)

$40T=.04$

$$\therefore C(z) = 5\left[\frac{2(z-1)+40T(z+1)}{2(z-1)+250T(z+1)}\right] = 5\left[\frac{(2.04)z-(2-.04)}{(2.25)z-(2-.25)}\right] = 5\left(\frac{2.04}{2.25}\right)\left[\frac{z-\frac{1.96}{2.04}}{z-\frac{1.75}{2.25}}\right]$$

$250T=.25$

$\boxed{b \approx .78}$

3) Discrete-time system time response analysis.

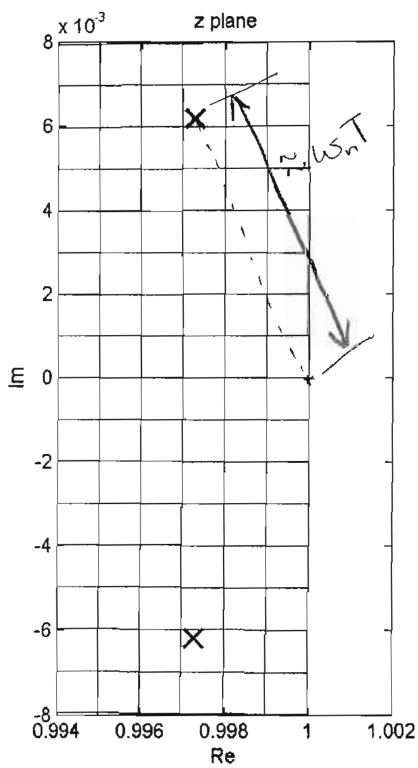
$\boxed{6}$ a) Given $P(z) = \frac{5z^2 - z + k}{z^2}$, solve for the value of k such that the steady state response to a unit step input is zero.

We want: $5z^2 - z + K = 0$ for $z = 1$,

$5 - 1 + K = 0$

$\boxed{K = -4}$

$\boxed{10}$ b) Assume P(z) has poles on the z-plane as shown below (plus a zero at the origin), and that the sampling time is T=.001 sec. Estimate t_{peak} and the percent overshoot for a unit step response. (Tip, if you cannot calculate these values, then instead estimate ω_n, ω_d, and ζ for a system with matching poles on the s plane…)



$z = e^{sT} \approx 1 + sT$ for $|sT| << 1$

So, $sT \approx z - 1 \approx -.0027 \pm .0062j$

$T = .001$, ∴ $\boxed{s \approx -2.7 \pm 6.2j}$

$+6.2 = \omega_d$

$-2.7 = -\zeta\omega_n$

$\boxed{\omega_n \approx 6.8 \frac{rad}{s}}$ (measuring the diagonal by hand…)

$\boxed{\omega_d \approx 6.2 \frac{rad}{s}}$

$\boxed{\zeta = \frac{\zeta\omega_n}{\omega_n} = \frac{2.7}{6.8} \approx 0.4}$

$t_{peak} = \frac{\pi}{\omega_d} \approx \frac{3.14}{6.2} \approx \boxed{0.5 \text{ sec}}$ ← t peak

% overshoot $= e^{-t_{peak}/\tau}$, where $\tau = \frac{1}{\zeta\omega_n}$

$= e^{-t_p \zeta \omega_n}$

$= e^{-\frac{1}{2}2.7} = \boxed{e^{-1.35}}$ ← fractional overshoot

$\boxed{\%\text{ overshoot} = 100 \times e^{-1.35} \approx 26\%}$

4) For this discrete-time controller: $C(z) = \frac{z-.96}{z-.9975}$, with sample time T=.005 s.

4

a) At approximately what frequencies (rad/s) are the equivalent CT pole and zero?

zero: $sT \approx -.04$, $\therefore s \approx \frac{-.04}{.005} = 8$ rad/s — zero

pole: $sT \approx -.0025$, $\therefore s \approx \frac{.0025}{.005} = 0.5$ rad/s — pole

2

b) What is the DC gain of C(z)? for z=1, (steady state),

$$\frac{1-.96}{1-.9975} = \frac{.04}{.0025} = \frac{400}{25} = 16$$

9

c) Somewhere between the frequency of the equivalent CT pole and zero (from part a), the phase of C(z) has its greatest deviation from 0 degrees. Is the phase change positive or negative? At what frequency (rad/s) on a Bode plot does this occur? And what is (approximately) the peak deviation in phase for C(z)?

pole < zero, $\therefore$ phase is Negative.

Freq. of peak: $\sqrt{8 \cdot .5} = \sqrt{4} = 2$ rad/s

Phase change @ 2 rad/s : $\alpha = 4$, neg phase,

$\therefore$ (App. B), $\boxed{-62^\circ}$

3

d) What type of controller is this, and what is its function? (i.e., in general, why would one use this type of controller?) lag, which increases gain @ low freq, to reduce steady state error.

5) For a continuous-time state space system,

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

with the following A,B,C, and D matrices:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad A = \begin{bmatrix} -15 & 100 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C = [0 \quad 2] \qquad D = [0]$$

6

a) Write a continuous-time transfer function from input U(s) to output Y(s).

$$\begin{cases} \dot{x}_1 = -15x_1 + 100x_2 + u \\ \dot{x}_2 = x_1 \end{cases}$$
$$y = 2x_2$$

$$\therefore \ddot{x}_2 + 15\dot{x}_2 - 100x_2 = u, \quad \& \quad x_2 = \tfrac{1}{2}y$$

$$\tfrac{1}{2}(\ddot{y} + 15\dot{y} - 100y) = u \rightarrow \frac{Y(s)}{U(s)} = \frac{2}{s^2 + 15s - 100}$$

4

b) Based on the A matrix, is this system stable? (What can you say about the poles…?)

No! Unstable. A leads to a characteristic equation where not all coefficients have the same sign:

$$s^2 + 15s - 100 = 0$$

$\therefore$ There is a LHP pole → unstable. ↑ unstable "spring stiffness".

6

c) Solve for the vector K for a state space controller, $u=-BK$, such that the resulting poles are the solution to the characteristic equation: $s^2 + 25s + 400 = 0$.

$$K = [K_1 \quad K_2], \quad -BK = -\begin{bmatrix} 1 \\ 0 \end{bmatrix}[K_1 \quad K_2] = \begin{bmatrix} -K_1 & -K_2 \\ 0 & 0 \end{bmatrix}$$

$$A - BK = \begin{bmatrix} -15-K_1, & 100-K_2 \\ 1, & 0 \end{bmatrix}$$ ← From figuring out pt a, you should tell by inspection that,

[..continues next page!]

"$s^2 + (15+K_1)s + (-100+K_2) = 0$" is the char. eq., $\therefore$ $\boxed{K_1 = 10}$, $\boxed{K_2 = 500}$

4

5-d) Assume you now wish to solve for a vector L for a state space estimator. Write out a MATLAB command that will place the estimator poles at pdes=[-30+30j, -30-30j].

L= place(A', C', pdes)'

(+5)

e) EXTRA CREDIT: Assume all estimator gains in L are equal to 50 (where L is of appropriate size, from part d). The poles are solutions to what characteristic equation??

$$L = \begin{bmatrix} 50 \\ 50 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2 \end{bmatrix}, \quad LC = \begin{bmatrix} 0 & 100 \\ 0 & 100 \end{bmatrix}$$

$$A - LC = \begin{bmatrix} -15 & 100-100 \\ 1 & 0-100 \end{bmatrix} = \begin{bmatrix} -15 & 0 \\ 1 & -100 \end{bmatrix}$$

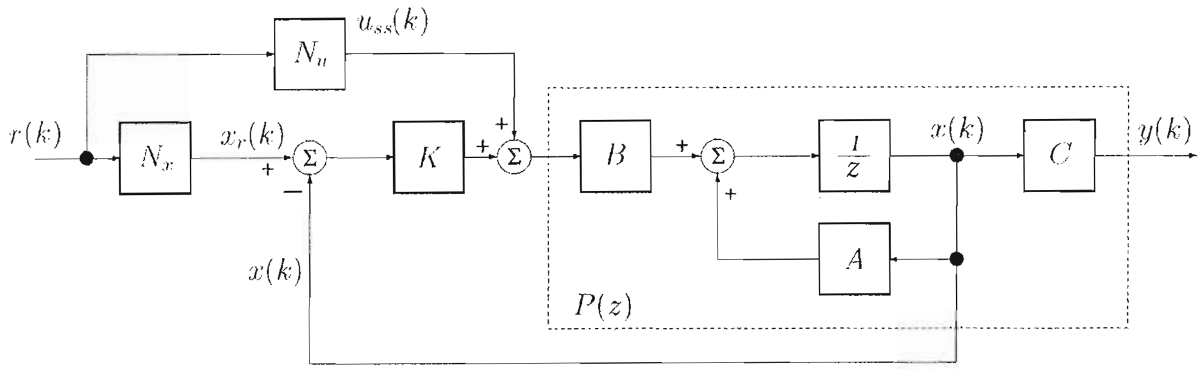
$$\det(sI-(A-LC)) = \begin{vmatrix} (s+15) & 0 \\ 1 & (s+100) \end{vmatrix} = (s+15)(s+100)$$

$\therefore$ Char. eqn. is: $\boxed{s^2 + 115s + 1500 = 0}$

6) Below is a block diagram for a state space feedback configuration we have discussed in class this year. Assume the state space matrices have the following dimensionalities:

A (nxn) ; B (nx1) ; C (mxn) ; [D is just all zeros…]

Also assume *r(k)* and *y(k)* match one another in size. (i.e., that *r(k)* gives a desired set of references for the outputs, *y(k)*.)



3

a) What are the size and function of N_x?

$N_x r(k) = x_r(k)$

$\therefore (? \times ?)(m \times 1)$ is of size $(n \times 1)$

↑ m rows in C matrix

$\therefore$ N_x is $(n \times m)$. N_x "scales" the reference input, $r(k)$, to match the units & dimensionality of $x_r(k)$

3

b) What are the size and function of N_u?

$N_u r(k) = u_{ss}(k)$

$(? \times ?)(m \times 1)$ is of size (1×1)

↑ 1 column in B matrix

$\therefore$ N_u is $(1 \times m)$. N_u sets the feedforward actuation value(s).

4

c) Note that *r(k)*, the reference input, gives a set of desired outputs for *y(k)*. Write an expression that solves (generally) for N_x to scale a desired reference *r(k)* appropriately. (Do NOT assume any of the matrices are scalars here!) For full credit, show work deriving your answer.

$Cx_{ss} = y$, $r \approx y$ (desired)

$x_r \approx x_{ss}$, $x_r = N_x r$

so, $Cx_{ss} \approx Cx_r = \boxed{CN_x r} = y \approx r$

$\therefore \boxed{CN_x = I} \leftarrow I$ is $(m \times m)$

@SS, $x_{ss} = Ax_{ss} + Bu_{ss}$ $\quad x_{ss} = N_x r \quad u_{ss} = N_u r$

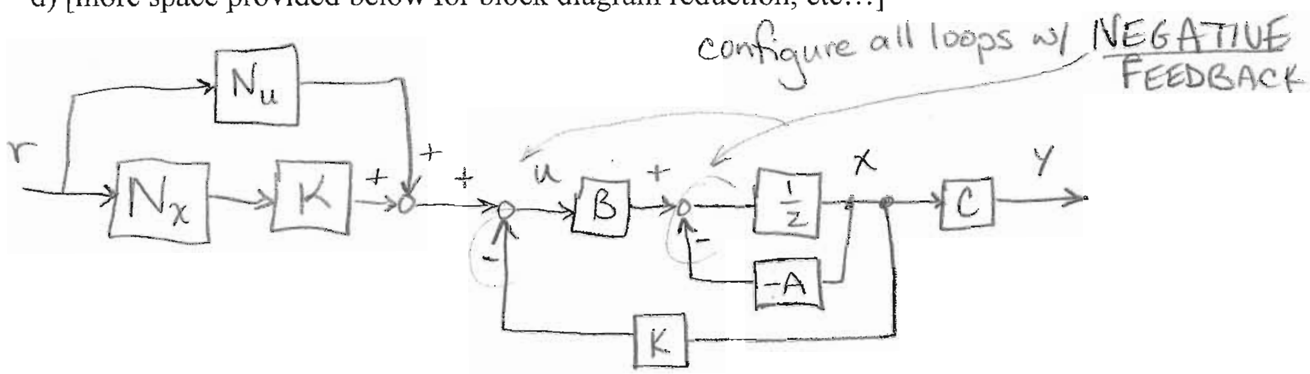
$(A-I)x_{ss} + Bu_{ss} = 0 \rightarrow (A-I)N_x r + BN_u r = 0$

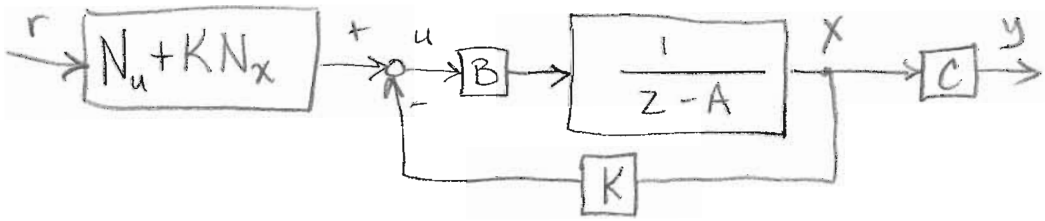
$$\begin{bmatrix} (A-I) & B \\ C & 0 \end{bmatrix} \begin{bmatrix} N_x \\ N_u \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

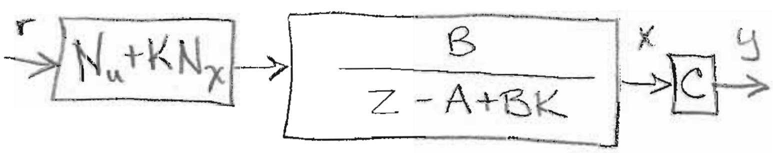
10

d) Now assume that all the state space matrices are simply scalar values. Reduce the block diagram shown to a single input single output (SISO) transfer function from input *R(z)* to output *Y(z)*. (Hint: use block diagram algebra, or write algebraic equations by hand… More space provided on next page!)

d) [more space provided below for block diagram reduction, etc…]







$$\therefore \boxed{\frac{Y(z)}{R(z)} = \frac{BC(N_u + KN_x)}{(Z-(A-BK))}}$$

(+5) e) EXTRA CREDIT: Solve for N_u to obtain zero steady state error, in terms of the variables in your transfer function from part d.

For $z=1$, we desire $\frac{Y(z=1)}{R(z=1)} = 1 = \frac{BC(N_u + KN_x)}{(1-(A-BK))}$, where $N_x = \frac{1}{C}$

So, $BC(N_u + \frac{K}{C}) = 1-(A-BK)$

$$N_u = \left(\frac{1-(A-BK)}{BC}\right) - \frac{K}{C}$$

$$N_u = \left(\frac{1-A+BK-BK}{BC}\right)$$

$$\boxed{N_u = \left(\frac{1-A}{BC}\right)}$$

End of exam. Have a great Spring Break!