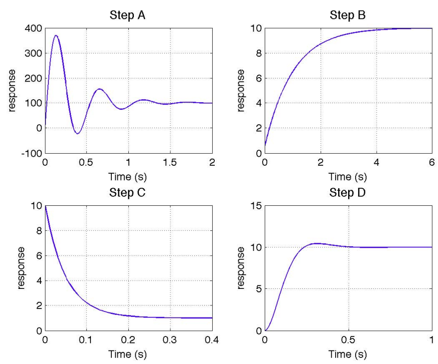
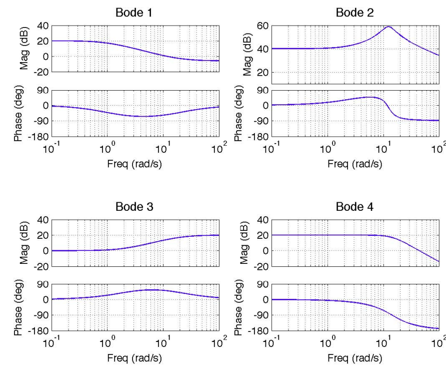
Homework 1 Solutions

Problem 1

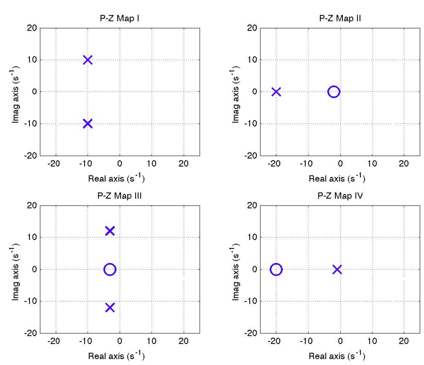
(a)



(a) Step responses A, B, C, and D.



(b) Bode plots 1, 2, 3, and 4.



(c) P-Z maps I, II, III, IV.

Figure 1: Step responses, Bode plots, and P-Z maps to match up.

Using the Final and Initial Value Theorems, we can match the Step Response plots with the Bode plots.

The Final Value Theorem (FVT) states that

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

This implies that the final value of the step response of the time-domain signal is equal to the initial value of the frequency-domain signal since the step input would cancel the s in the limit on the right-hand side of the above expression.

The Initial Value Theorem (IVT) states that

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

This implies that the initial value of the step response of the time-domain signal is equal to the final value of the frequency-domain signal.

The following table summarizes the final and initial values using the above theorems for the four systems.

Step Response	Final Value	Initial Value	Bode Plot
A	$100 = 40$ dB	?	2
B	$10 = 20$ dB	$\approx 0.5 \approx -6$ dB	1
C	$1 = 0$ dB	$10 = 20$ dB	3
D	$10 = 20$ dB	$0 = -\infty$ dB	4

We can then match the P-Z maps to the Bode plots.

Complex conjugate pair poles correspond to a flat magnitude response until the break point at which there is a decrease in magnitude at a rate of -40 dB per decade. Also, the corresponding phase plot starts of at zero until the break point at which it drops to $-180°$. This corresponds to Bode plot 4.

A real pole corresponds to a flat magnitude until the break point at which there is a decrease in magnitude at a rate of -20 dB per decade. A real zero corresponds to a flat magnitude until the break point at which there is an increase in magnitude at a rate of 20 dB per decade. The break point is equal to the value of the pole or zero. If a transfer function has both, and the pole is larger than the zero, the magnitude plot will increase first at a rate of 20 dB per decade and then flatten out once the decrease from the pole cancels out the increase from the zero. If the zero is larger than the pole, the magnitude plot will decrease first at a rate of -20 dB per decade and then flatten out once the increase from the zero cancels out the decrease from the pole. Therefore, we can reason that P-Z Map II corresponds to Bode Plot 3, and P-Z Map IV corresponds to Bode Plot 1.

By process of elimination, P-Z Map III corresponds to Bode Plot 2. This is also supported by the phase plot of Bode Plot 2 which starts of at zero and drops to $-90°$. P-Z Map III has two complex conjugate poles and a zero that have the same break points. This combination would result in the phase plot of Bode Plot 2.

In summary,

Step Response	Bode Plot	P-Z Map
A	2	III
B	1	IV
C	3	II
D	4	I

(b)

We can develop the expressions for the transfer functions from the P-Z maps and apply the FVT and IVT. We can then use the final and initial values from the matched step responses and Bode plots to determine the gain values, K. Note that since we are matching with step responses, the s is canceled by the $\frac{1}{s}$ from the step input. The following table summarizes these expressions and the corresponding initial and final values.

P-Z Map	Transfer Function	Final Value	Initial Value
I	$K\frac{1}{s^2+20s+200}$	$\frac{K}{200} = 10$	0
II	$K\frac{(s+2)}{(s+20)}$	$\frac{K}{10} = 1$	$K = 10$
III	$K\frac{(s+3)}{(s^2+6s+153)}$	$\frac{3K}{153} = \frac{K}{51} = 100$	0
IV	$K\frac{(s+20)}{(s+1)}$	$20K = 10$	$K = 0.5$

Therefore, the transfer functions are as follows:

System	Transfer Function
A-2-III	$5100\frac{(s+3)}{(s^2+6s+153)}$
B-1-IV	$\frac{1}{2}\frac{(s+20)}{(s+1)}$
C-3-II	$10\frac{(s+2)}{(s+20)}$
D-4-I	$2000\frac{1}{s^2+20s+200}$

Problem 2

$$G(s) = \frac{2s-12}{s+3}$$

(a)

Let $T(s)$ be the closed-loop transfer function for the given system under negative unity feedback with a proportional controller $C(s) = K$.

$$\begin{aligned}
T(s) &= \frac{C(s)G(s)}{1+C(s)G(s)} \\
&= \frac{K(2s-12)}{(s+3)+K(2s-12)} \\
&= \frac{K(2s-12)}{(2K+1)s+(3-12K)} \\
T(s) &= \frac{2K}{2K+1} \cdot \frac{s-6}{s+\frac{3-12K}{2K+1}}
\end{aligned}$$

The closed-loop system will be stable if its pole,

$$s = -\frac{3-12K}{2K+1},$$

is strictly negative.

$$-\frac{3-12K}{2K+1} < 0$$

if, for $2K+1 > 0$, or $K > -\frac{1}{2}$,

$$-(3-12K) < 0 \implies K < \frac{1}{4}$$

and, for $2K+1 > 0$, or $K < -\frac{1}{2}$,

$$-(3-12K) > 0 \implies K > \frac{1}{4}$$

The second condition can never be satisfied, therefore, the range of K for stability is

$$-\frac{1}{2} < K < \frac{1}{4}$$

(b)

There are no unstable poles in the transfer function, $G(s)$, therefore $P = 0$. The number of unstable poles in the closed-loop system is given by $Z = N + P$, where N is the number of encirclements of -1. Therefore, we would like no encirclements of -1 in order for the closed-system to be stable.

The Nyquist plot for $G(s)$ is

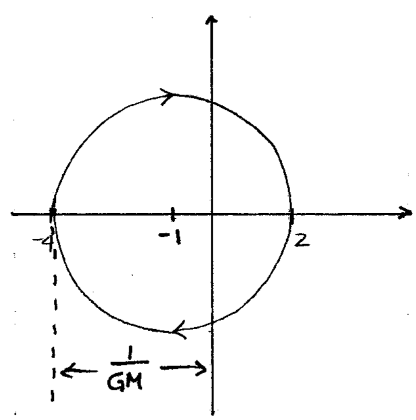


Figure 2: Nyquist plot for G(s)

There is currently one encirclement of -1, which means we want to reduce the gain. We would need to reduce the gain by a factor of at least four in order to achieve closed-loop stability. This implies that, for positive values of K,

$$K < \frac{1}{4}$$

for closed-loop stability.

We can plot the Nyquist plot for $-G(s)$ in order to determine the range of K for stability for negative values of K. The Nyquist plot for $-G(s)$ is

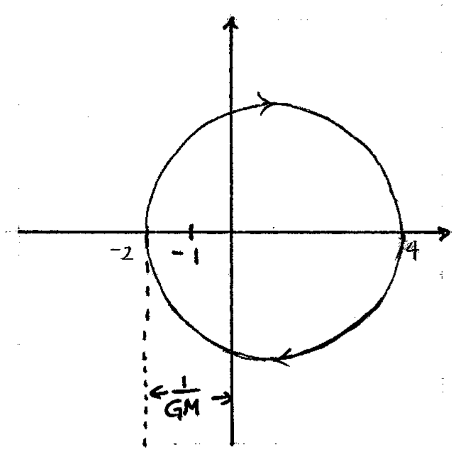


Figure 3: Nyquist plot for -G(s)

There is currently one encirclement of -1, which means we want to reduce the gain. We would need to reduce the gain by a factor of at least two in order to achieve closed-loop stability. This implies that, for negative values of K,

$$-K < \frac{1}{2} \implies K > -\frac{1}{2}$$

for closed-loop stability.

Therefore, for closed-loop stability,

$$-\frac{1}{2} < K < \frac{1}{4}$$

which matches the range we attained in Part (a).

Problem 3

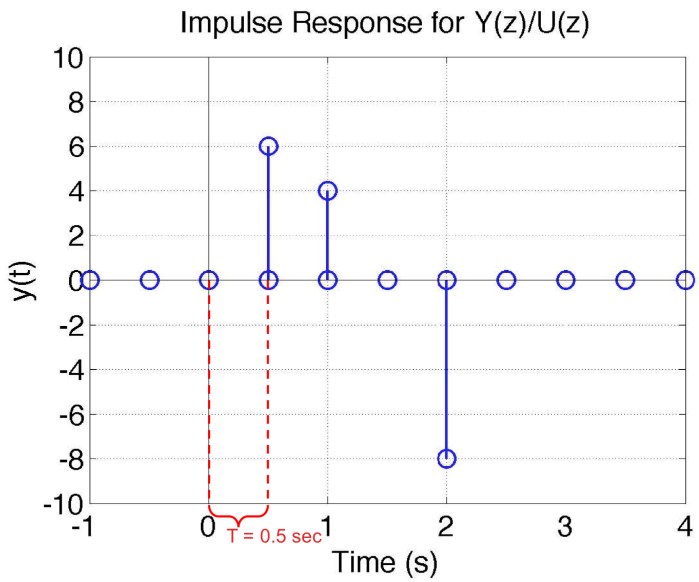


Figure 4: Impulse response of $P(z) = \frac{Y(z)}{U(z)}$.

(a)

The sample time is

$$T = 0.5 \text{ sec}$$

$$P(z) = \frac{Y(z)}{U(z)} \implies Y(z) = P(z) \cdot U(z)$$

When the input is an impulse,

$$Y(z) = P(z) \cdot 1 = P(z)$$

Also,

$$P(z) = Y(z) = \sum_{k=-\infty}^{\infty} y(k) \cdot z^{-k}$$

The signal $y(t)$, given by the Figure 4, is zero everywhere except $t = 0.5$, $t = 1$, and $t = 2$. The sample time is 0.5 seconds, therefore, these correspond to $k = 1$, $k = 2$, and $k = 4$. Thus, we can find $P(z)$ as follows:

$$\begin{aligned} P(z) &= \sum_{k=-\infty}^{\infty} y(k) * z^{-k} \\ &= 6 \cdot z^{-1} + 4 \cdot z^{-2} + (-8) \cdot z^{-4} \\ P(z) &= \frac{6z^3 + 4z^2 - 8}{z^4} \end{aligned}$$

(b)

Similar to the continuous time case, in the discrete time domain, a unit step corresponds to accumulation. This can be seen from the convolution of a signal with a unit step.

$$y(k) * \text{step}(k) = \sum_{n=-\infty}^{\infty} y(n) \cdot \text{step}(k-n) = \sum_{n=-\infty}^{k} y(n)$$

where

$$\text{step}(k) = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

This implies that a unit step input would result in the accumulation of values up to time k of the impulse response of the system.

Therefore, the step response is

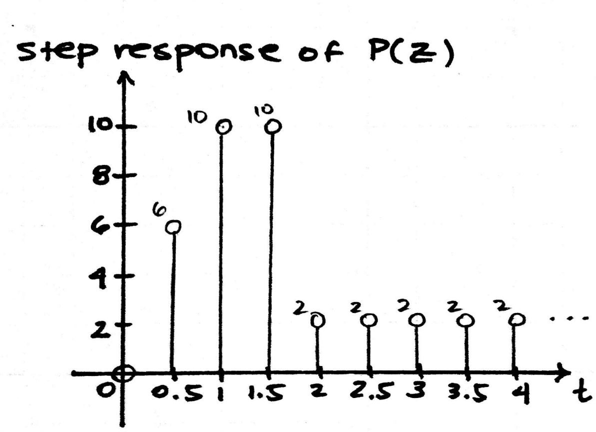


Figure 5: Step response of $P(z)$.

Problem 4

$$C(s) = \frac{s+1}{s+16}$$

(a)

The DC gain is given by the magnitude of the transfer function when $s = 0$.

$$|C(0)| = \left|\frac{0+1}{0+16}\right| = \frac{1}{16}$$

The DC gain of $C(s)P(s)$

$$|C(0)P(0)| = \left|\frac{1}{16}P(0)\right| = \frac{1}{16}|P(0)|$$

Therefore, the controller would decrease the DC gain of the loop transmission by a factor of 16.

(b)

The phase of the controller is given by

$$\angle C(j\omega) = \phi_C = \phi_z - \phi_p$$

where ϕ_z is the phase from the zero and ϕ_p is the phase from the pole. The phase is given by the inverse tangent of the imaginary part over the real part of the zero or pole. Thus,

$$\phi_C = \tan^{-1}(\omega) - \tan^{-1}\left(\frac{\omega}{16}\right)$$

Taking the derivative of the above expression and setting it to zero, we get

$$\frac{\mathrm{d}\phi_C}{\mathrm{d}\omega} = \frac{1}{1+\omega^2} - \frac{1}{16}\frac{1}{1+\left(\frac{\omega}{16}\right)^2} = 0$$

when

$$\begin{aligned} 1+\left(\frac{\omega}{16}\right)^2 - \frac{1}{16}(1+\omega^2) &= 0 \\ 16^2 + \omega^2 - 16 - 16\omega^2 &= 0 \\ -15\omega^2 + 16(16-1) &= 0 \\ 15\omega^2 &= 16(15) \\ \omega^2 &= 16 \\ \omega &= \pm 4 \end{aligned}$$

Since we want a positive value for the frequency,

$$\omega_a = 4 \text{ rad/sec}$$

Note: Taking the second derivative shows that this value is a local maximum.

$$\begin{aligned} \frac{\mathrm{d}^2\phi_C}{\mathrm{d}\omega^2} &= \frac{-2\omega}{(1+\omega^2)^2} + \frac{\frac{2\omega}{16^2\cdot 16}}{(1+\left(\frac{\omega}{16}\right)^2)^2} \\ &= \frac{32\omega}{(16^2+\omega^2)^2} - \frac{2\omega}{(1+\omega^2)^2} \end{aligned}$$

For $\omega = 4$,

$$\frac{\mathrm{d}^2\phi_C}{\mathrm{d}\omega^2}(4) = \frac{128}{(16^2+16)^2} - \frac{8}{(1+16)62} = \frac{0.5-8}{1+16^2} < 0$$

Since the second derivative is negative, there is a local maximum at $\omega = 4$.

Plugging ω_a back into the equation for the phase, we find the maximum phase.

$$\phi_m ax = \phi_C(4) = \tan^{-1}(4) - \tan^{-1}\left(\frac{4}{16}\right) \approx 61.9^\circ$$

(c)

$$T = 0.03 \text{ sec}$$
$$z_a = e^{j\omega_a T} = e^{j4\cdot 0.03} = e^{j0.12} = \cos(0.12) + j\sin(0.12) = 0.9928 + j0.1197$$

(d)

$$\begin{aligned}
C_{\text{forward}}(z) &= \frac{\left(\frac{z-1}{T}\right) + 1}{\frac{z-1}{T} + 16} \\
&= \frac{z - 1 + T}{z - 1 + 16T} \\
&= \frac{z - 1 + 0.03}{z - 1 + 16\cdot 0.03} \\
C_{\text{forward}}(z) &= \frac{z - 0.97}{z - 0.52}
\end{aligned}$$

The phase at z_a is thus

$$\angle C_{\text{forward}}(z_a) = \tan^{-1}\left(\frac{0.1197}{0.9928 - 0.97}\right) - \tan^{-1}\left(\frac{0.1197}{0.9928 - 0.52}\right) \approx 65.0044^\circ$$

(e)

$$\begin{aligned}
C_{\text{forward}}(z) &= \frac{\left(\frac{z-1}{zT}\right) + 1}{\frac{z-1}{zT} + 16} \\
&= \frac{z - 1 + zT}{z - 1 + 16zT} \\
&= \frac{z - 1 + 0.03z}{z - 1 + 16\cdot 0.03z} \\
C_{\text{forward}}(z) &= \frac{1.03z - 1}{1.48z - 1}
\end{aligned}$$

The phase at z_a is thus

$$\angle C_{\text{forward}}(z_a) = \tan^{-1}\left(\frac{0.1197}{0.9928 - \frac{1}{1.03}}\right) - \tan^{-1}\left(\frac{0.1197}{0.9928 - \frac{1}{1.48}}\right) \approx 58.9362^\circ$$

(f)

$$
\begin{aligned}
C_{\text{forward}}(z) &= \frac{\left(\frac{2(z-1)}{T(z+1)}\right)+1}{\frac{2(z-1)}{T(z+1)}+16} \\
&= \frac{2(z-1)+T(z+1)}{2(z-1)+16T(z+1)} \\
&= \frac{2(z-1)+0.03(z+1)}{2(z-1)+16\cdot 0.03(z+1)} \\
C_{\text{forward}}(z) &= \frac{2.03z-1.97}{2.48z-1.52}
\end{aligned}
$$

The phase at z_a is thus

$$
\angle C_{\text{forward}}(z_a) = \tan^{-1}\left(\frac{0.1197}{0.9928-\frac{1.97}{2.03}}\right) - \tan^{-1}\left(\frac{0.1197}{0.9928-\frac{1.52}{2.48}}\right) \approx 61.9275^\circ
$$

(g)

The Nyquist frequency is given by

$$
\omega_N = \frac{pi}{T} = \frac{pi}{0.03} \approx 104.72 \approx 10^{2.02}
$$

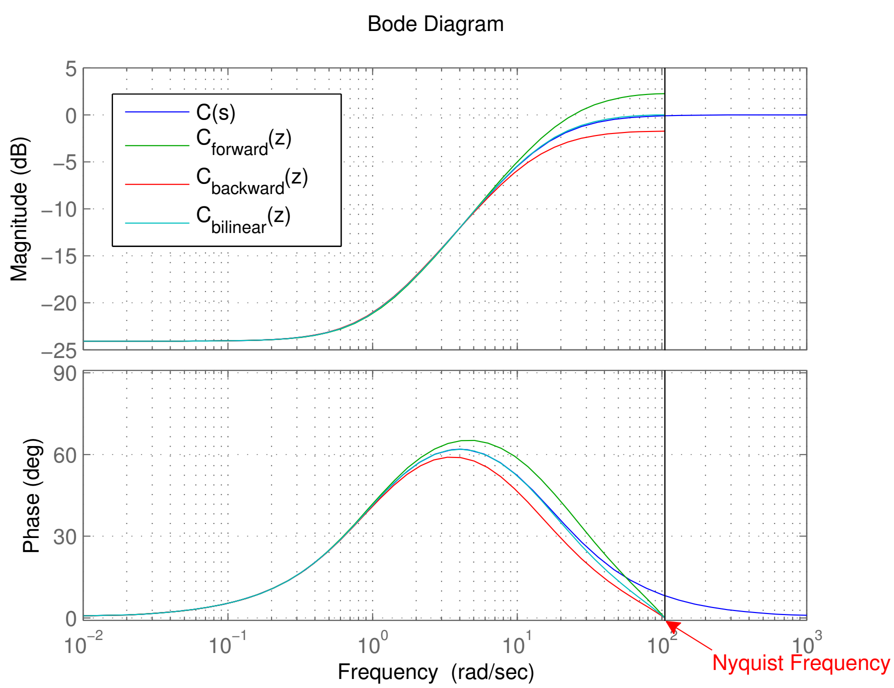


Figure 6: Bode plot for $C(s)$, $C_{\text{forward}}(z)$, $C_{\text{backward}}(z)$, and $C_{\text{bilinear}}(z)$.

In general, the forward difference approximation has a larger magnitude and phase than the original system, especially near the Nyquist frequency. The backward difference approximation has, in general, a smaller magnitude and phase than the original system, especially near the Nyquist frequency. The bilinear approximation is very close in magnitude to the original system while the phase is very close until frequency very close to the Nyquist frequency. For this reason, the bilinear approximation is the best of the three.

Problem 5

$$P(s) = \frac{5}{s^2}$$

(a)

An overshoot of 10% to 20% corresponds to a damping factor, ζ, of approximately 0.6 to 0.46. A damping factor of 0.46 to 0.6 corresponds to approximately a 45° to 60° phase margin. The phase margin for $P(s)$ (a double integrator) is zero, so we need to introduce 45° to 60° of phase at the crossover frequency. This can be done with a lead controller. A lead controller has the following form:

$$C(s) = \frac{Ts+1}{\alpha Ts+1}$$

where

$$\alpha = \frac{1-\sin(\phi_{\max})}{1+\sin(\phi_{\max})},$$

$$T = \frac{1}{\omega_{\max}\sqrt{\alpha}},$$

$\phi_{\max}$ is the desired maximum phase, and $\omega_{\max}$ is frequency at which the desired maximum phase occurs.

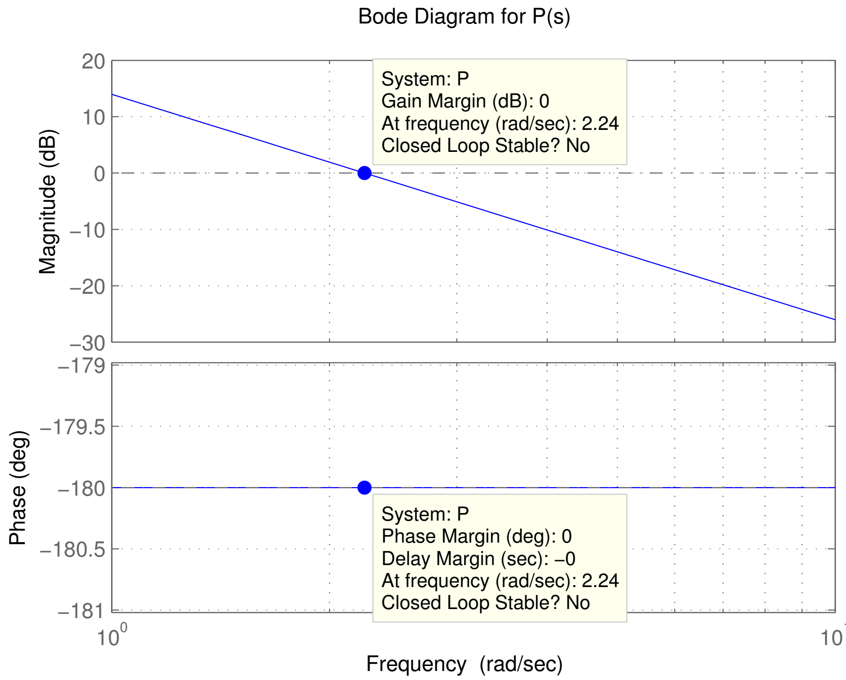


Figure 7: Bode plot for P(s)

For an overshoot of 10%, we want to introduce 60° of phase into the system, we will set

$$\phi_{\max} = 60^\circ$$

and we want this phase to occur at the crossover frequency, so

$$\omega_{\max} = \omega_G$$

We would like a peak time, t_p, of approximately 0.1 seconds. This corresponds to a bandwidth of about 45. The bandwidth is the frequency at which the magnitude of the closed-loop transfer function falls to -3 dB. Therefore, we want our crossover frequency to be somewhere between half the bandwidth and the bandwidth, say 40. Adjusting the location of the crossover frequency will affect how quickly the step responses reaches its peak value. A higher crossover frequency corresponds to a faster peak time, while a lower crossover frequency corresponds to a slower peak time. So,

$$\omega_{\max} = \omega_G = 40$$

Therefore,

$$\alpha = \frac{1 - \sin(\phi_{\max})}{1 + \sin(\phi_{\max})} = \frac{1 - \sin(60^\circ)}{1 + \sin(60^\circ)} \approx 0.07$$

and

$$T = \frac{1}{\omega_{\max}\sqrt{\alpha}} = \frac{1}{40\sqrt{0.07}} \approx 0.09$$

Therefore,

$$C(s) = \frac{Ts + 1}{\alpha Ts + 1} = \frac{0.09s + 1}{\alpha 0.0063s + 1} \approx 14\frac{s + 10.72}{s + 149.3}$$

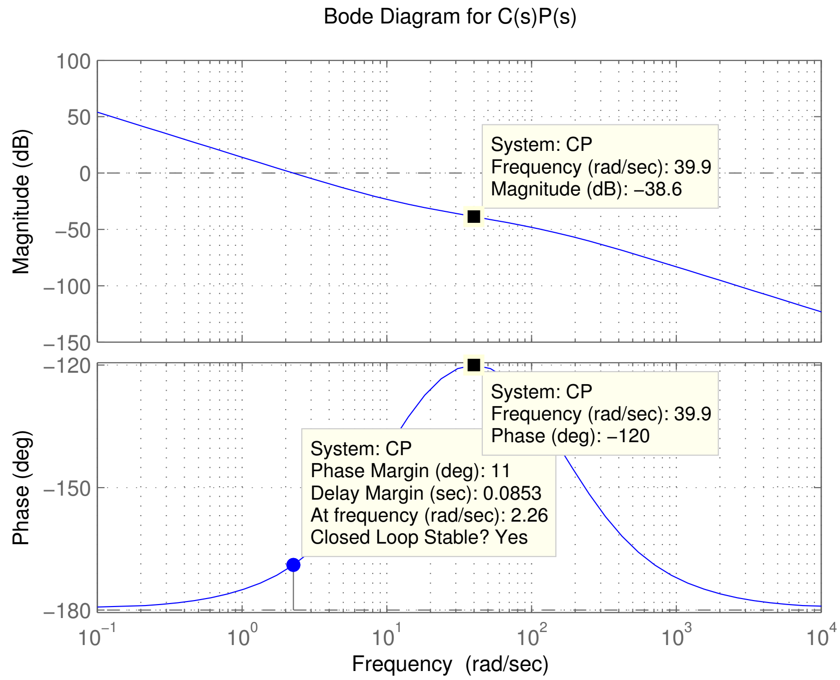


Figure 8: Bode plot for $C(s) \approx 14\frac{s+10.72}{s+149.3}$

However, the desired maximum phase does not occur at the crossover frequency, as seen in Figure 8. In order to achieve this, we need to add a gain of about 85 in order to have a crossover frequency at 40 rad/sec.

Thus,

$$C(s) \approx 1184\frac{s+10.72}{s+149.3}$$

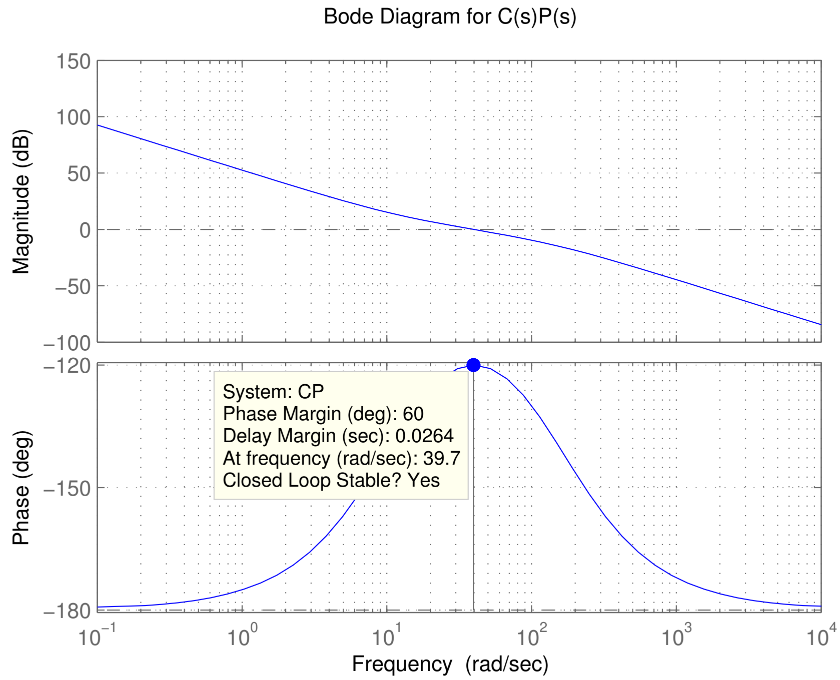


Figure 9: Bode plot for $C(s) \approx 5920\frac{s+10.72}{s+149.3}$

The step response of the closed-loop system is thus

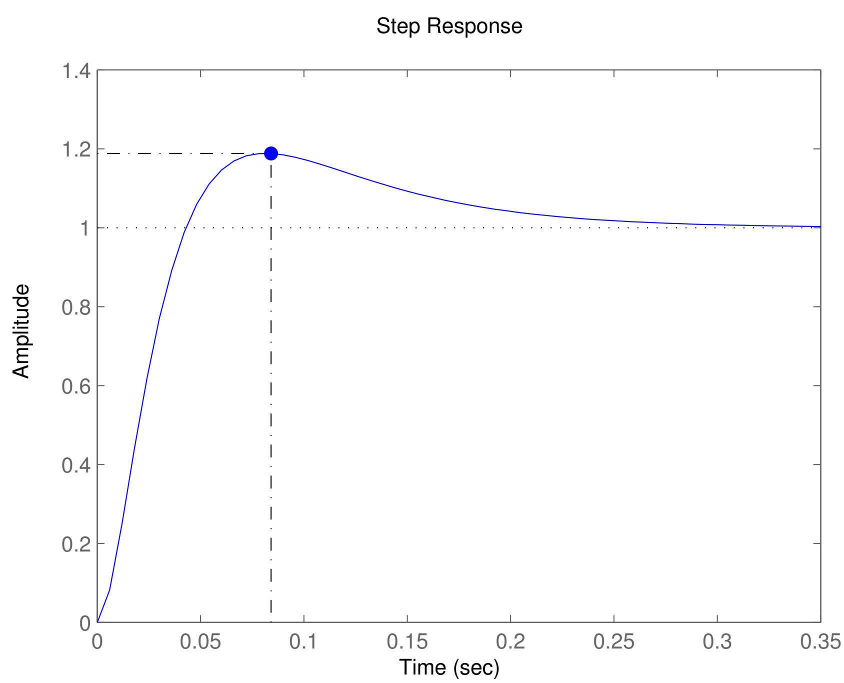


Figure 10: Step response for the closed-loop system for $C(s)P(s)$, where $C(s) \approx 1184\frac{s+10.72}{s+149.3}$

The step response has a percent overshoot of approximately 18.8%, which is the desired range, and a peak time of approximately 0.08 seconds, which is also in the desired range for peak time.

(b)

The closed-loop system is given by

$$G(s) = \frac{C(s)P(s)}{1 + C(s)P(s)} \approx 5920 \frac{s + 10.72}{(s + 92.93)(s + 38.72)(s + 17.63)}$$

Thus, the closed-loop poles are

$$p_1 = -92.93, \qquad p_2 = -38.72, \qquad \text{and} \qquad p_3 = -17.63$$

and the closed-loop zero is

$$z = -10.72$$

While complex poles generally cause overshoot, in this case, the zero causes the overshoot since it is dominant (i.e. closest to the $j\omega$-axis). The closed-loop transfer function can be considered as the sum of two terms:

$$G(s) \approx 5920 \frac{s}{(s + 92.93)(s + 38.72)(s + 17.63)} + 5920 \frac{10.72}{(s + 92.93)(s + 38.72)(s + 17.63)}$$

In the time domain, the first term is the (scaled) derivative of the second term. When the derivative is positive, amplitude is being added to the second term. When the derivative is negative, amplitude is being subtracted from the second term. Note that for any zero, when the derivative is zero, the transfer function will all have the same amplitude. For larger zeros, the derivative will have a relatively small effect on the step response. For smaller zeros, the derivative will have a relatively significant effect on the step response. The overshoot is directly proportional to the effect the derivative has on the step response.

(c)

Let T_s be the sampling rate of the discrete system.

$$P_{\text{ZOH}}(z) = (1 - z^{-1}) \cdot \mathcal{Z}\left\{\frac{P(s)}{s}\right\}$$

For $P(s) = \frac{5}{s^2}$,

$$
\begin{aligned}
P(z) &= (1 - z^{-1}) \cdot \mathcal{Z}\left\{\frac{P(s)}{s}\right\} \\
&= (1 - z^{-1}) \cdot \mathcal{Z}\left\{\frac{5}{s^3}\right\} \\
&= (1 - z^{-1}) \cdot \mathcal{Z}\left\{\frac{5}{2!}(kT_s)^2 u(kT_s)\right\} \\
&= (1 - z^{-1}) \cdot \frac{5T_s^2}{2} \mathcal{Z}\left\{k^2 u(k)\right\} \\
&= \frac{5T_s^2}{2}(1 - z^{-1}) \frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3} \\
&= \frac{5T_s^2}{2} \frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^2} \\
P(z) &= \frac{5T_s^2}{2} \frac{z + 1}{(z - 1)^2}
\end{aligned}
$$

Using the Tustin approximation, $s \leftrightarrow \frac{2(z-1)}{T_s(z+1)}$ for $C(z)$, we get

$$
\begin{aligned}
C(z) &= 1184 \frac{\frac{2(z-1)}{T_s(z+1)} + 10.72}{\frac{2(z-1)}{T_s(z+1)} + 149.3} \\
&= 1184 \frac{2(z - 1) + 10.72T_s(z + 1)}{2(z - 1) + 149.3T_s(z + 1)} \\
&= 1184 \frac{(10.62T_s + 2)z + (10.72T_s - 2)}{(149.3T_s + 2)z + (149.3T_s - 2)} \\
C(z) &= \frac{1184 \cdot (10.62T_s + 2)}{(149.3T_s + 2)} \cdot \frac{z + \frac{(10.72T_s - 2)}{10.72T_s + 2)}}{z + \frac{(149.3T_s - 2)}{(149.3T_s + 2)}}
\end{aligned}
$$

The discrete approximations for $P(s)$ and $C(s)$ are summarized in the following table for the three sampling rates.

Sampling Rate	$P(z)$	$C(z)$
$T_s = 0.01$	$P(z) = \frac{1}{4000} \frac{z+1}{(z-1)^2}$	$714 \frac{z-0.899}{z-0.145}$
$T_s = 0.05$	$P(z) = \frac{1}{160} \frac{z+1}{(z-1)^2}$	$317 \frac{z-0.580}{z+0.577}$
$T_s = 0.06$	$P(z) = \frac{9}{1000} \frac{z+1}{(z-1)^2}$	$285 \frac{z-0.517}{z+0.635}$

(d)

The following table summarizes the closed-loop transfer function, $G(z)$, for the discrete-time system.

Sampling Rate	$G(z)$
$T_s = 0.01$	$G(z) = 0.179\frac{(z+1)(z-0.898)}{(z-0.851)(z^2-1.116z+0.359)}$
$T_s = 0.05$	$G(z) = 1.983\frac{(z+1)(z-0.5774)}{(z-0.4799)(z^2+1.04z+1.182)}$
$T_s = 0.06$	$G(z) = 2.570\frac{(z+1)(z-0.513)}{(z-0.414)(z_1^2.62z+1.652)}$

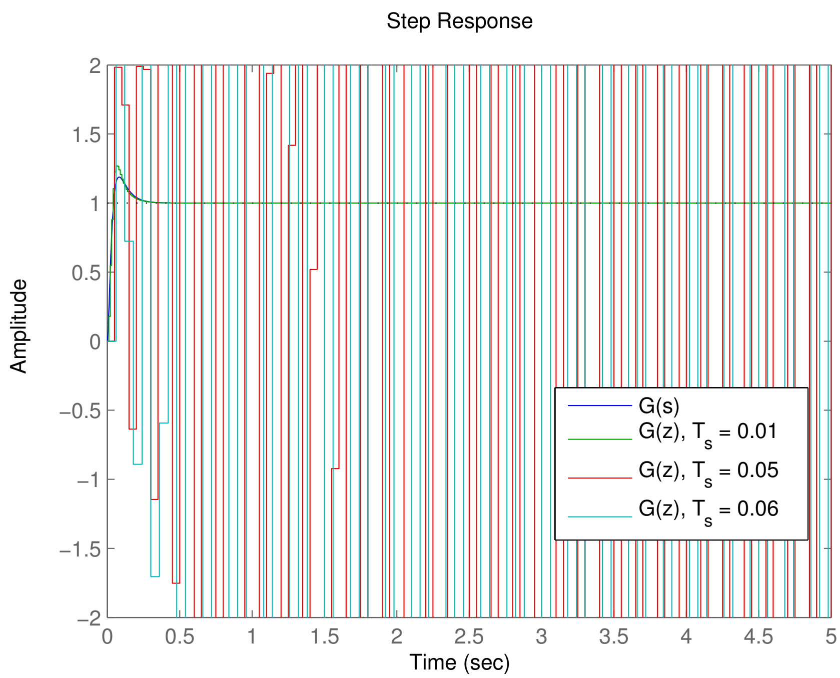


Figure 11: Step responses for the continuous system and discrete systems for $T_s = 0.01$, $T_s = 0.05$, and $T_s = 0.06$.

For $T_s = 0.01$, the peak time is 0.06 seconds and the percent overshoot is 26.9%. For $T_s = 0.05$ and $T_s = 0.06$, the discrete-time systems are unstable, and thus the peak time and percent overshoot are not applicable.

(e)

The phase margins, percent overshoots, and stability are summarized in the following table.

System	Phase Margin	Percent Overshoot	Stable?
$G(s)$	$60°$	18.8%	Yes
$G(z)$, $T = 0.01$	$49°$	26.9%	Yes
$G(z)$, $T = 0.05$	$-7.96°$	N/A	No
$G(z)$, $T = 0.06$	$-25.54°$	N/A	No

Note that larger percent overshoots will result in instability for larger sampling times. Systems with small percent overshoots (closer to 10%) will be stable for both $T = 0.05$ and $T = 0.06$.

Problem 6

(a)

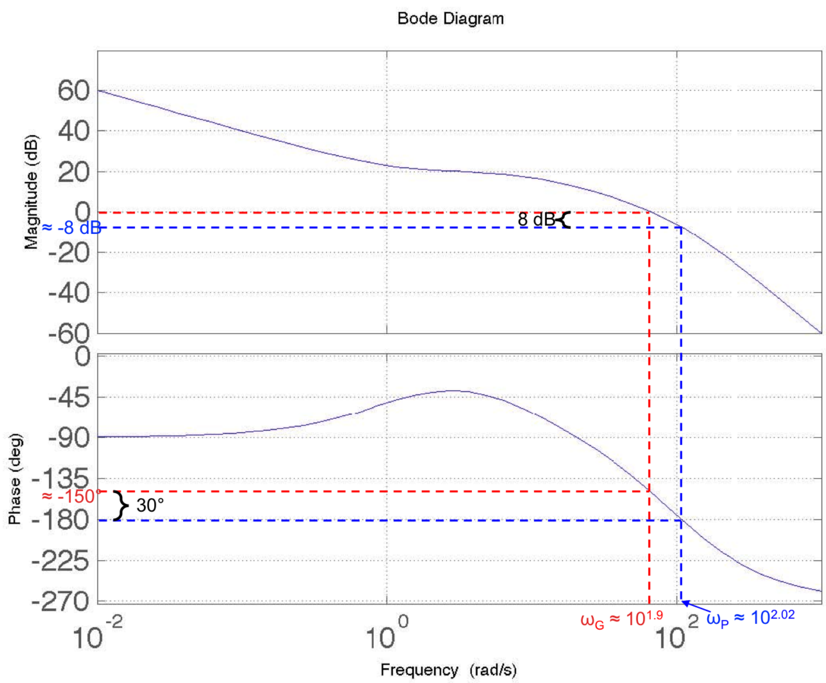


Figure 12: Bode plot for the given system with the margins and crossover frequencies labeled.

When determining stability using Bode plots, we look for a negative magnitude (in dB) when the phase is $-180°$. The gain margin is the value by which the magnitude can be increased before becoming positive (in dB) when the phase is $-180°$. The phase margin is the value by which the phase can be decreased when the magnitude is 0 dB.

Therefore,

$$GM \approx 8 \text{ dB} \approx 2.5$$

and

$$PM \approx 30°$$

The crossover frequency, ω_G, is the frequency at which the magnitude is 0 dB. Therefore,

$$\omega_G \approx 10^{1.9} \approx 80 \text{ rad/sec}$$

The crossover frequency, ω_P, is the frequency at which the phase is $-180°$. Therefore,

$$\omega_P \approx 10^{2.02} \approx 105 \text{ rad/sec}$$

(b)

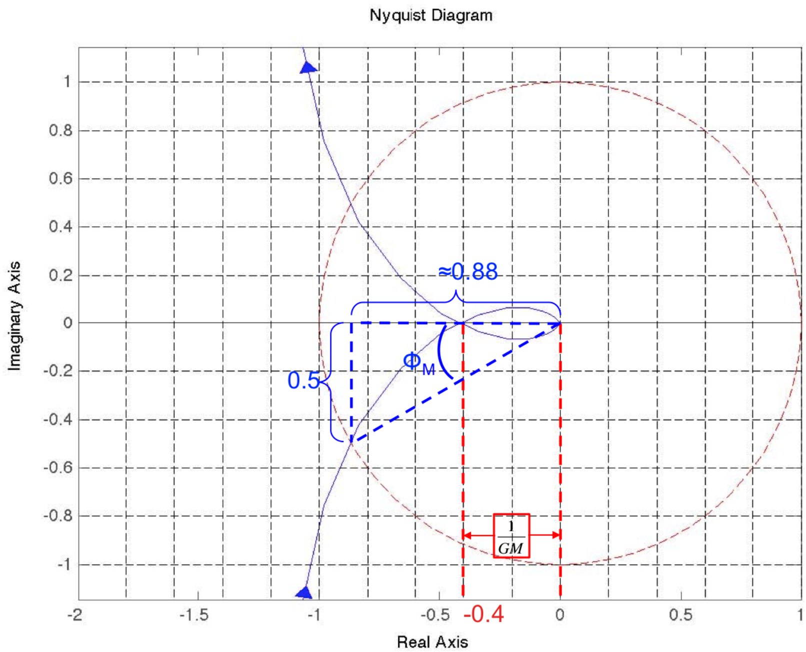


Figure 13: Nyquist plot for the given system with the margins labeled.

Assuming there are no unstable poles, $P = 0$. There are currently no encirclements of -1. If we increase the gain by too much, however, there will be encirclements of -1, so there will be an upper limit on the gain. For negative values of gain, we can consider the Nyquist plot flipped about the imaginary-axis (which would be the Nyquist for $-G(s)$, where $G(s)$ is the transfer function of the system). There will an encirclement of -1 for all gain values, therefore the gain must be positive.

From Figure 13,

$$GM = \frac{1}{0.4} = 2.5$$

and

$$PM \approx \tan^{-1}\left(\frac{0.5}{0.88}\right) \approx 29.6^\circ$$