Name SOLUTIONS

ECE 147B

Midterm Exam

Feb. 14, 2012

- You are allowed one, single-side 8 ½ x 11 sheet of self-prepared notes.
- Submit this "Cheat Sheet" with your exam. (Worth 2 pts.) It will be returned with your graded exam, as is.
- **No calculators, laptops, or other computing devices are allowed.**

____/2	**Cheat Sheet:**	**Turn in Cheat Sheet with your exam!**
____/25	**Problem 1:**	**Step vs Impulse response**
____/18	**Problem 2:**	**C(z) frequency response**
____/20	**Problem 3:**	**Estimating P(z) from a time response**
____/20	**Problem 4:**	**Tustin with pre-warp**
____/15	**Problem 5:**	**State space (discrete-time) equations**
____/100	**TOTAL**	

Midterm ID number.

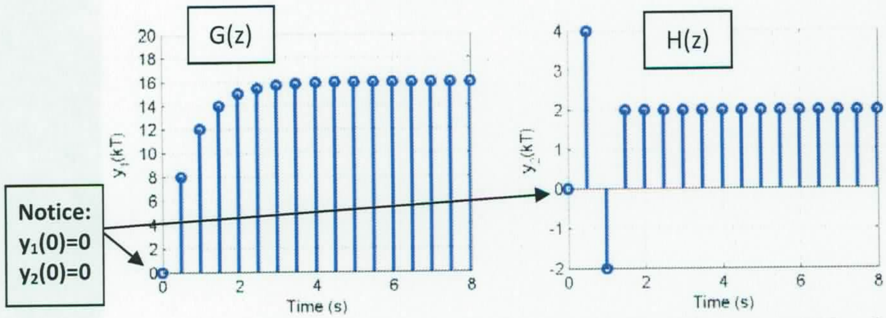
Below are two matched random numbers. Tear off the non-taped number and keep it.

We will post exam grades and statistics using these numbers instead of student names, to hide identities.

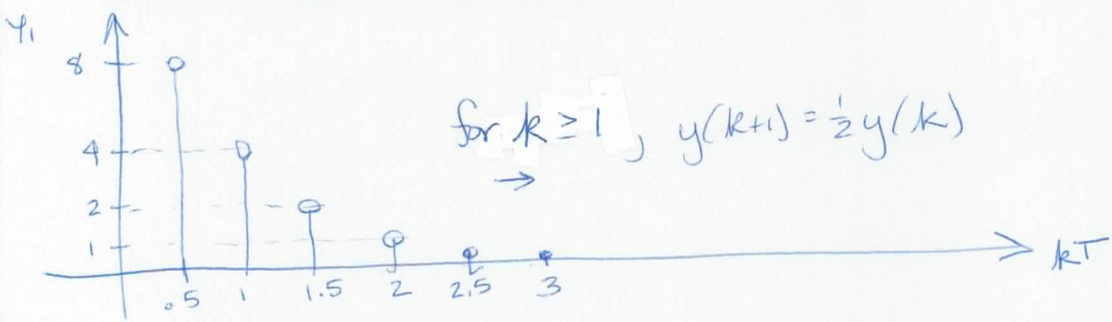
(Taped side)	**TEAR AND KEEP TAB BELOW**

1) At top below is a STEP response, $y_{tot}(kT)$, for a discrete-time system P(z) with sampling time T=0.5 seconds. A colleague has noticed that after the first few samples, the response seems to be decaying exponentially to a final value of 18. She suggests separating the response into two components, corresponding to DT TF's G(z) and H(z), which, when added together, give the total step response. These two components, $y_1(kT)$ and $y_2(kT)$, are shown in the lower two plots below, respectively.

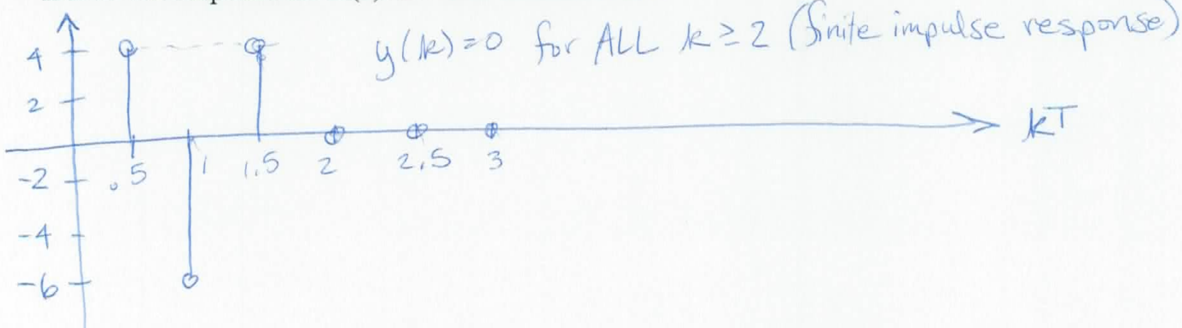
kT	$y_{tot}=y_1+y_2$	18-y_{tot}	y_2
0	0	18.0	0
0.5	12.0	6.0	4
1.0	10.0	8.0	-2
1.5	16.0	2.0	2
2.0	17.0	1.0	2
2.5	17.5	0.5	2
3.0	17.75	0.25	2
3.5	17.875	0.125	2
4.0	17.9375	0.0625	2
4.5	17.96875	0.03125	2
5.0	17.984375	0.015625	2
5.5	17.9921875	0.0078125	2
6.0	17.99609375	0.00390625	2
…	…	…	…
kT	$18-(1/2)^{(k-4)}$	$(1/2)^{(k-4)}$	2



5 points a) Given the unit step response for G(z) [lower LEFT subplot], sketch the corresponding unit IMPULSE response for G(z). LABEL CAREFULLY.



5 points b) Given the unit step response for H(z) [lower RIGHT], sketch the corresponding unit IMPULSE response for H(z). LABEL CAREFULLY.



15 points c) Using your answers to a) and b), calculate expressions for all three (3) DT TF's: G(z), H(z), and the complete system, P(z)=G(z)+H(z).

$$\boxed{G(z) = \frac{8}{z-.5}}, \text{ pole @ } z=.5, \text{ F.V. @ } \frac{8}{.5}=16$$

$$H(z) = \frac{4}{z} - \frac{6}{z^2} + \frac{4}{z^3}$$

$$\boxed{H(z) = \frac{4z^2-6z+4}{z^3}}$$

$$P(z) = \frac{[8z^3] + [(z-.5)(4z^2-6z+4)]}{z^3(z-.5)}$$

$$= \frac{8z^3 + (4z^3-6z^2+4z) + (-2z^2+3z-2)}{z^3(z-.5)}$$

$$\boxed{P(z) = \frac{12z^3 - 8z^2 + 7z - 2}{z^3(z-.5)}}$$

2) Given $C_1(z)$, below, with sampling time T=.001 seconds:

$$C_1(z) = 50\frac{(z-.98)}{(z-.95)}$$

a) What is the DC gain of $C_1(z)$?

$\lim z \to 1,\ C_1(z{=}1) = 50\frac{(1-.98)}{(1-.95)} = 50\left(\frac{.02}{.05}\right) = \boxed{20}$ ← DC gain

6 points b) Use the forward Euler approximation to estimate a corresponding CT TF, C(s). What is the high-frequency gain for C(s)?

$z \to 1+sT,\quad C(s) \approx 50\frac{(1-s(.001)-.98)}{(1-s(.001)-.95)} = \boxed{50\frac{(s+20)}{(s+50)}}$ ← C(s)

@ high freq., $s \to \infty$, $|C(s{=}\infty)| = \boxed{50}$ ← @ high freq.

6 points c) Now, assume our control loop is slower, with T=.005 seconds. Again using forward Euler, solve for a new controller, $C_2(z)$, and verify that it provides the same DC gain.

$s \to \frac{(z-1)}{T},\quad C_2(z) = 50\frac{(z-1+20T)}{(z-1+50T)} = 50\frac{(z-1+.1)}{(z-1+.25)} = \boxed{50\frac{(z-.9)}{(z-.75)}}$

@ $z \to 1$, $50\frac{(1-.9)}{(1-.75)} = 50\frac{.1}{.25} = \boxed{20}$ matches for DC gain!

2 points d) For $C_2(z)$, (T=.005 seconds) what is the Nyquist frequency, ω_{NyC2}?

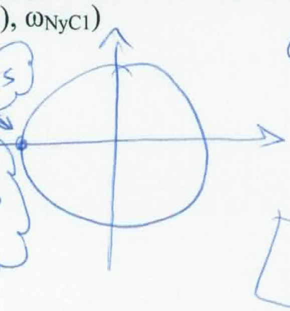
$\omega_{Nyquist} = \frac{\pi}{T} \approx \frac{3.14}{.005} = \frac{314}{.5} = 2 * 314 = \boxed{628\ \frac{rad}{s}}$

4 points e) At ω_{NyC2} for $C_2(z)$, what is the gain and phase of $C_2(z)$? (If you could not solve for $C_2(z)$, perform this calculation for $C_1(z)$ instead, with respect to the Nyquist frequency for $C_1(z)$, ω_{NyC1})

$C_2(z{=}-1) = 50\frac{(-1-.9)}{(-1-.75)} = 50\frac{1.9}{1.75} = \ldots$

$= 50\left(\frac{38}{35}\right) = \left(\frac{10}{7}\right)\cdot 38 = \frac{380}{7}$ ← gain.

$\omega_{Nyquist}$ is ALWAYS @ z=-1!

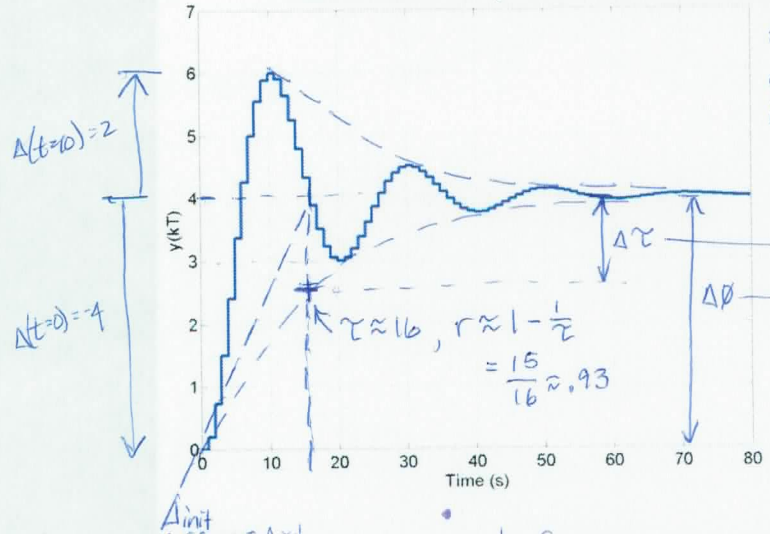


$\boxed{\text{Phase is } 0^\circ\text{, b/c } C_2(z{=}-1) \text{ is real valued and positive.}}$ ← phase.

3) In both parts a) and b) below, assume zero initial conditions, so y(kT) is zero for all k<0.)

10 points

a) Below is a unit step response for some 2^{nd}-order DT transfer function, P(z), with T=1 second. **Sketch the pole locations for P(z) the z-plane** (showing unit circle). Clearly label r and θ. (Write a math expression for r, but do not waste time estimating its exact value. You pole locations should look "close", however.)

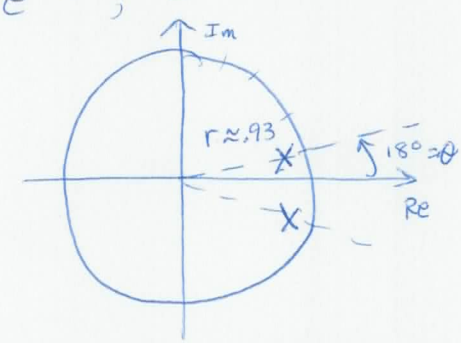


τ Slope $= A \cdot \frac{1}{\tau}$, so @ k=1, we go $\frac{1}{\tau}$ of the way to F.V. (approx).

After 10 sample, envelope has decayed to $\frac{1}{2}$ value, & oscillation is π later in phase, ∴

$r^{10} = \frac{1}{2}$, $\boxed{r = \left(\frac{1}{2}\right)^{1/10}}$, $\boxed{\theta = \frac{\pi}{10} = 18^\circ}$

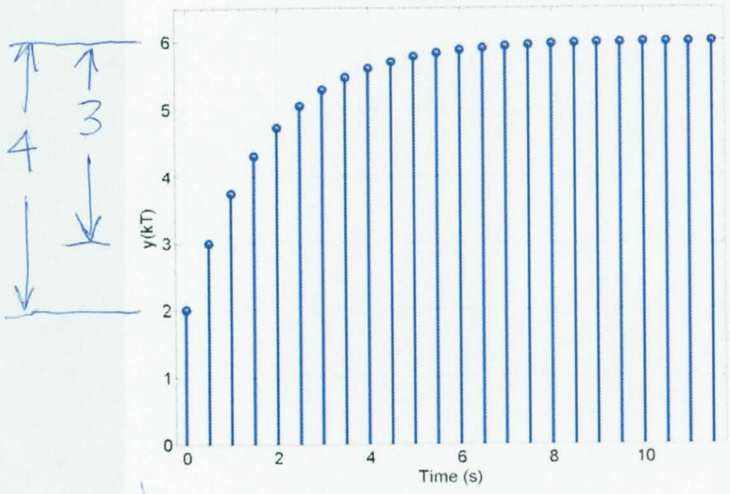
$\frac{\Delta\tau}{\Delta\theta} = \frac{1}{e} = e^{-1}$, $\tau \approx 16$ sec.



z-plane

10 points

b) Below is a unit step response for a first-order DT TF, G(z), with T=0.5 seconds. Assume the system has the form $G(z) = \frac{az+b}{z+c}$, and **solve for a, b, and c**.



a) @ k=0, y(0)=2, ∴ $\boxed{a=2}$

c) For k>0, $\frac{(y(k+1)-y(\infty))}{(y(k)-y(\infty))} = \frac{3}{4} = .75$,

∴ $\boxed{c = -.75}$

$(y(k+1) = .75(y(k)) +$ "stuff"$)$,

$[y(k+1) - .75 \cdot y(k)] =$ "stuff"...

b) F.V.T.: $\frac{2z+b}{z-.75} = 6$, for z=1,

$2+b = 6(1-.75)$

$b = 6 * .25 - 2$

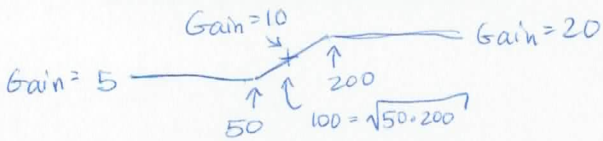
$= 1.5 - 2$

$\boxed{b = -0.5}$

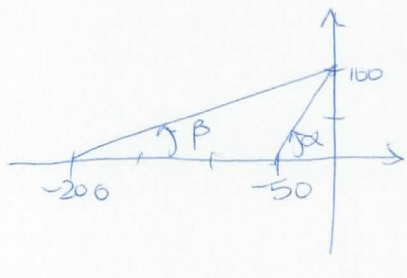
$\boxed{\therefore G(z) = \frac{2z-0.5}{z-0.75}}$

4) In this problem, you are asked to use emulation to design a DT controller.

6 points a) For a continuous-time controller, $C(s) = 20\frac{(s+50)}{(s+200)}$, **estimate the phase and gain** contributed by C(s) at a desired crossover frequency of 100 rad/s. (**See Appendix** for plot and table of tangent function.)



$\leftarrow$ Gain @ $\omega = 100\frac{rad}{s} = \boxed{10}$ (or "6 dB")



phase @ $100\frac{rad}{s}$ is: $\alpha - \beta$

$\tan\alpha = \frac{100}{50} = 2 \rightarrow \alpha \cong 63°$

$\tan\beta = \frac{100}{200} = \frac{1}{2} \rightarrow \beta \cong 27°$

phase @ $100\frac{rad}{s} \cong 63° - 27° = \boxed{36°}$

7 points b) Leave α (or any similar variable that is a function of pre-warp frequency) as a variable and use **Tustin with pre-warp** to convert C(s) to a DT TF C(z). Use a sampling time of **T=0.02 seconds**.

$s \rightarrow \alpha\left(\frac{z-1}{z+1}\right)$, $C(z) = 20\left(\frac{\alpha z - \alpha + 50z + 50}{\alpha z - \alpha + 200z + 200}\right)$

$$\boxed{= 20\left(\frac{(50+\alpha)z + (50-\alpha)}{(200+\alpha)z + (200-\alpha)}\right)}$$

7 points c) Now, solve numerically for C(z), using a **pre-warp frequency of 100 rad/s**.

$$\alpha = \frac{\omega}{\tan\left(\frac{\omega T}{2}\right)} = \frac{100}{\tan\left(\frac{100 \times 0.02}{2}\right)} = \frac{100}{\tan(1)} \approx \frac{100}{1.6} \approx \frac{5}{8} * 100 = 62.5$$

$$C(z) \cong 20\left(\frac{(50+63)z + (50-63)}{(200+63)z + (200-63)}\right) = \boxed{20\left(\frac{113z - 13}{263z + 137}\right)}$$

$$\text{or, } \cong \left(\frac{113}{263}\right) \times 20 \times \left[\frac{z - \frac{13}{113}}{z + \frac{137}{263}}\right]$$

5) Recall the discrete-time state space representation of a dynamic system, given below:

$$x(k+1) = A_d x(k) + B_d u(k)$$
$$y(k) = C_d x(k) + D_d u(k)$$

In this problem, $\boldsymbol{A_d}$ and $\boldsymbol{B_d}$ are defined as:

$$A_d = \begin{bmatrix} 0 & 0 & 1 \\ 13 & 0 & 147 \\ -1 & 0 & 1.6 \end{bmatrix} \qquad B_d = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

3 points a) Write out the matrix "zI-A_d". ("z times identity matrix, minus A_d.")

$$\begin{bmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{bmatrix} - A_d = \begin{bmatrix} z & 0 & -1 \\ -13 & z & -147 \\ 1 & 0 & (z-1.6) \end{bmatrix}$$

5 points b) Write the characteristic equation of the open-loop system, via "$det\{zI - A_d\} = 0$".

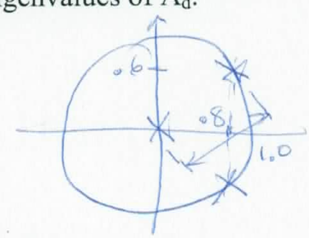
$$z \cdot z \cdot (z-1.6) + 0 + 0 - (-z + 0 + 0) = 0$$
$$\boxed{z^3 - 1.6z^2 + z = 0}$$

7 points c) Solve for the poles of the system. These are the solutions to the characteristic equation and are also the eigenvalues of A_d.

$$z(z^2 - 1.6z + 1) = 0$$

$-\sigma = .8$ $\quad \omega_n^2 = 1$

$\boxed{z_1 = 0}$



$$\boxed{\begin{aligned} z_2 &= .8 + .6j \\ z_3 &= .8 - .6j \end{aligned}}$$

Extra Credit d) Solve for vector K for a feedback regulator of the form

$u = -Kx$

such that the poles (eigenvalues) of "A_d-B_dK" are all at z=0 (i.e., deadbeat control).

$$A_d - B_d K = \begin{bmatrix} 0 & 0 & 1 \\ 13 & 0 & 147 \\ -1-K_1 & -K_2 & 1.6-K_3 \end{bmatrix}, \quad det\{zI - (A_d - B_d K)\} = 0$$

$$z \cdot z \cdot (z + K_3 - 1.6) + \ldots$$
$$13K_2 - [147K_2 z + (K_1 + 1)z] = 0$$

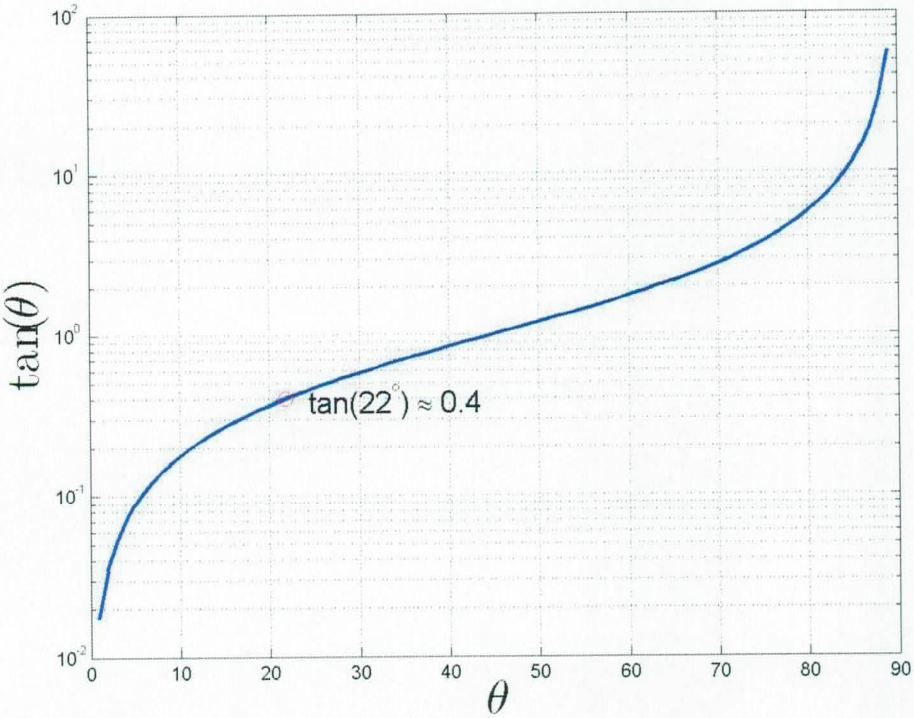
$$z^3 + \underbrace{(K_3 - 1.6)}_{\boxed{K_3 = 1.6}} z^2 - \underbrace{(147K_2 + K_1 + 1)}_{\text{since } K_2 = 0,\ \boxed{K_1 = -1}} z + \underbrace{13K_2}_{\boxed{K_2 = 0}} = 0 \Rightarrow \boxed{K = [-1, 0, 1.6]}$$

Appendix: The tangent function.

Note: This Appendix is likely to appear in the actual Midterm, too, so ***learn to use it now*** *for problems involving phase calculations for both CT and DT transfer functions.*

Table of tan fn.

θ	θ	tanθ
(deg)	(rad)	-
0	0	0
5.7	.1	.1
11.5	.2	.2
17.2	.3	.3
22.9	.4	.4
28.6	.5	.5
34.4	.6	.7
40.1	.7	.8
45.8	.8	1.0
51.6	.9	1.3
57.3	1.0	1.6
63.0	1.1	2.0
68.8	1.2	2.6
74.5	1.3	3.6
80.2	1.4	5.8
85.9	1.5	14.1



The examples and figure below illustrate how to use semilog plot above:

- The complex number 5+2j has phase: $\text{atan}\left(\frac{2}{5}\right) \approx 22^\circ$.
- The complex number 5-2j has phase: $\text{atan}\left(\frac{-2}{5}\right) \approx -22^\circ$.
- The complex number -5-2j has phase: $\text{atan}\left(\frac{-2}{-5}\right) \approx -180^\circ + 22^\circ = -158^\circ$.