Homework 3 Solutions

Problem 1

Part (a)

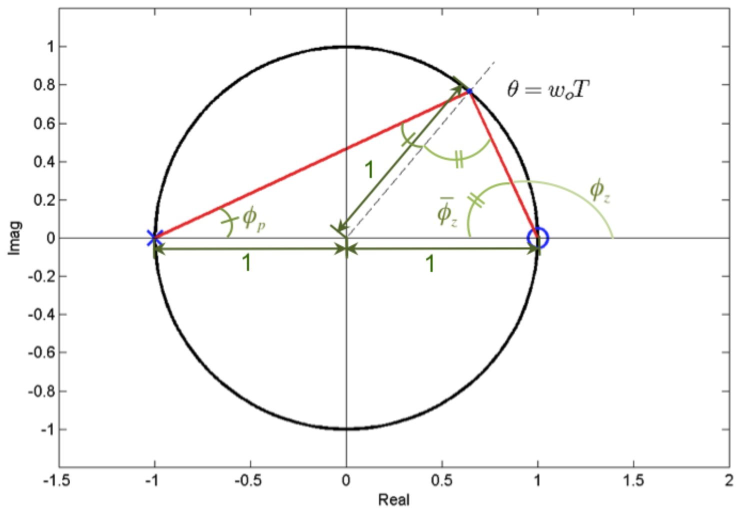

Figure 1: Pole-zero map for the transfer function $K(z-1)/(z+1)$ with the unit circle and the point $e^{j\omega_0 T}$ shown. Note that we are only concerned with the top semicircle (i.e. up to the Nyquist frequency).

The phase of a transfer function is given by the phase of its zeros minus the phase of its poles. So,

$$\angle \frac{K(z-1)}{(z+1)} = \angle(z-1) - \angle(z+1) = \phi_z - \phi_p$$

From Figure 1,

$$\begin{aligned} & 2\phi_p + 2\bar{\phi}_z = 180^\circ \\ \Rightarrow \quad & \phi_p + \bar{\phi}_z = 90^\circ \\ \Rightarrow \quad & \phi_p = 90^\circ - \bar{\phi}_z \end{aligned}$$

Because ϕ_z and $\bar{\phi}_z$ are complimentary angles,

$$\bar{\phi}_z = 180^\circ - \phi_z$$

Thus,

$$\phi_z - \phi_p = (180^\circ - \bar{\phi}_z) - (90^\circ - \bar{\phi}_z) = 90^\circ.$$

Therefore,

$$\angle \frac{K(z-1)}{(z+1)} = 90^\circ$$

Part (b)

The magnitude of a transfer function is the product of the magnitudes of its zeros and its gain divided by the magnitude of its poles. So,

$$\left|\frac{K(z-1)}{(z+1)}\right| = |K|\frac{|z-1|}{|z+1|}$$

For the point $z = e^{j\omega_0 T}$, the magnitude of the zero is the length of the segment joining the zero location and the point $z = e^{j\omega_0 T}$. Similarly, the magnitude of the pole is the length of the segment joining the pole location and the point $z = e^{j\omega_0 T}$. The angle whose vertex is at the point $e^{j\omega_0 T}$ is 90° since $\phi_p + \bar{\phi}_z = 90^\circ$. Therefore,

$$|z-1| = 2\sin(\phi_p)$$

and

$$|z+1| = 2\cos(\phi_p)$$

for $z = e^{j\omega_0 T}$.

Therefore,

$$\left|\frac{K(e^{j\omega_0 T}-1)}{(e^{j\omega_0 T}+1)}\right| = |K|\frac{2\sin(\phi_p)}{2\cos(\phi_p)} = |K|\tan(\phi_p)$$

From Figure 1,

$$\theta = 180^\circ - 2\bar{\phi}_z = 180^\circ - 2(90^\circ - \phi_p) = 2\phi_p$$

Thus

$$\left|\frac{K(e^{j\omega_0 T}-1)}{(e^{j\omega_0 T}+1)}\right| = |K|\tan\left(\frac{\theta}{2}\right) = |K|\tan\left(\frac{\omega_0 T}{2}\right)$$

Therefore, the value of K that will yield a magnitude of ω_0 when $z = e^{j\omega_0 T}$ is

$$K = \frac{\omega_0}{\tan\left(\frac{\omega_0 T}{2}\right)}$$

which matches the standard Tustin with pre-warp formula

$$s \rightarrow \alpha\frac{(z-1)}{(z+1)}, \qquad \alpha = \frac{\omega_0}{\tan\left(\frac{\omega_0 T}{2}\right)}$$

Problem 2

Part (a)

$$\frac{Y(z)}{U(z)} = \frac{0.83z}{z^2 - 1.6z + 0.81}$$

The characteristic polynomial of the above transfer function is

$$z^2 - 1.6z + 0.81 = 0$$

The poles of the system are the roots of the characteristic polynomial.

$$(z-0.8)^2 + 0.17 = 0 \Rightarrow z_{1,2} = 0.8 \pm j\sqrt{0.17} \approx 0.8 \pm j0.4123$$

The given transfer function represents the relationship between the input and the output of the system. We can find the difference equation by first separating the $Y(z)$ and $U(z)$ terms as follows.

$$Y(z)(z^2 - 1.6z + 0.81) = U(z)(0.83z)$$

We can then take the inverse Z-transform of the above equation to obtain the time-domain expression. Note that z^n represents an advance by n if $n > 0$ or a delay by n if $n < 0$. Assuming zero initial conditions,

$$y_{k+2} - 1.6y_{k+1} + 0.81y_k = 0.83u_{k+1}$$

Shifting the indices by two, we obtain the difference equation for y_k.

$$\begin{aligned} y_k - 1.6y_{k-1} + 0.81y_{k-2} &= 0.83u_{k-1} \\ y_k &= 0.83u_{k-1} + 1.6y_{k-1} - 0.81y_{k-2} \end{aligned}$$

Part (b)

(i) Using the Final Value Theorem to find the final value for a unit step input, we get

$$\begin{aligned} y_\infty &= \lim_{z\to 1}(z-1)\left[\frac{z}{(z-1)} \cdot \frac{0.83z}{(z^2 - 1.6z + 0.81)}\right] \\ &= \frac{0.83}{1 - 1.6 + 0.81} \\ y_\infty &= \frac{0.83}{0.21} \approx 3.9524 \end{aligned}$$

(ii) To find $y(n_1)$, $k = n_1 = 1$, we first need to compute $y(n_0)$. For a unit step input,

$$u(k) = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

and assuming zero initial conditions for $y(k)$ and $x(k)$,

$$y(n_0) = y(0) = 0.83u(-1) + 1.6y(-1) - 0.81y(-2) = 0$$

Now we can use the value for $y(n_0)$ to find $y(n_1)$.

$$\begin{aligned} y(n_1) &= y(1) = 0.83u(0) + 1.6y(0) - 0.81y(-1) \\ &= 0.83(1) + 1.6(0) \\ y(n_1) &= 0.83 \end{aligned}$$

(iii) We can use the value found in (ii) for $y(n_1)$ to find $y(n_2)$.

$$\begin{aligned} y(n_2) &= y(2) = 0.83u(1) + 1.6y(1) - 0.81y(0) \\ &= 0.83(1) + 1.6(0.83) \\ y(n_2) &= 2.158 \end{aligned}$$

Part (c)

We know $y(k)$ will have the form

$$y(k) = K + c_1 z_1^k + c_2 z_2^k$$

We can use the values found in Part (b) to solve for the values of K, c_1, and c_2. We can first solve for K by using the final value found in Part (b)-(i). Also recall that $\Re\{z_{1,2}\} = 0.8 < 1$ (as found in Part (a)), so z_1^k and z_2^k will converge to zero as $k \to \infty$.

$$y_\infty = \lim_{k\to\infty} y(k) = \lim_{k\to\infty} (K + c_1 z_1^k + c_2 z_2^k) = K$$

Therefore,

$$K = \frac{0.83}{0.21} \approx 3.9524$$

We can solve for c_1 and c_2 by setting up a system of equations using the values for $y(n_0)$ and $y(n_1)$ (or $y(n_1)$ and $y(n_2)$) that we found in Part (b).

$$\begin{array}{l} y(n_0) = y(0) = K + c_1 + c_2 = 0 \\ y(n_1) = y(1) = K + c_1 z_1 + c_2 z_2 = 0.83 \end{array} \Rightarrow \left\{ \begin{array}{ccccl} c_1 & + & c_2 & = & -K \\ z_1 c_1 & + & z_2 c_2 & = & 0.83 - K \end{array} \right.$$

In matrix form,

$$\begin{bmatrix} 1 & 1 \\ z_1 & z_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -K \\ 0.83 - K \end{bmatrix}$$

We can then use the MATLAB command `x = A\b` to find c_1 and c_2, where

$$x = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 1 \\ z_1 & z_2 \end{bmatrix}, \qquad \text{and} \qquad b = \begin{bmatrix} -K \\ 0.83 - K \end{bmatrix}$$

We find that

$$c_1 = -1.9762 - j0.0479$$
$$c_2 = -1.9762 + j0.0479$$

Part (d)

$$\frac{Y(z)}{U(z)} = \frac{2z^2 - z - 0.17}{z^2 - 1.6z + 0.81}$$

Following the same procedure as above

Part (d)-(a)

The poles of the system are the same as for the first transfer function since their characteristic polynomials are the same. So

$$z_{1,2} = 0.8 \pm j\sqrt{0.17} \approx 0.8 \pm j0.4123$$

Separating the terms $Y(z)$ and $U(z)$ we get

$$Y(z)(z^2 - 1.6z + 0.81) = U(z)(2z^2 - z - 0.17)$$

Thus, the difference equation is

$$y(k+2) - 1.6y(k+1) + 0.81y(k) = 2u(k+2) - u(k+1) - 0.17u(k)$$

Shifting the indices by two,

$$y(k) = 2u(k) - u(k-1) - 0.17u(k-2) + 1.6y(k-1) - 0.81y(k-2)$$

Part (d)-(b)

(i) Using the Final Value Theorem,

$$\begin{aligned} y_\infty &= \lim_{z \to 1}(z-1)\left[\frac{z}{(z-1)} \cdot \frac{2z^2 - z - 0.17}{z^2 - 1.6z + 0.81}\right] \\ &= \frac{2 - 1 - 0.17}{1 - 1.6 + 0.81} \\ y_\infty &= \frac{83}{0.21} \approx 3.9524 \end{aligned}$$

(ii) First, we compute $y(n_0)$ (using the same unit step as with the first system and assuming zero initial conditions).

$$y(n_0) = y(0) = 2u(0) - u(-1) - 0.17u(-2) + 1.6y(-1) - 0.81y(-2) = 2$$

We can now compute $y(n_1)$ using the above value for $y(n_0)$.

$$\begin{aligned} y(n_1)y(1) &= 2u(1) - u(0) - 0.17u(-1) + 1.6y(0) - 0.81y(-1) \\ &= 2 - 1 + 1.6(2) \\ y(n_1) &= 4.2 \end{aligned}$$

(iii) Using the values for $y(n_0)$ and $y(n_1)$ as found above, we can compute $y(n_2)$.

$$\begin{aligned} y(n_2)y(2) &= 2u(2) - u(1) - 0.17u(0) + 1.6y(1) - 0.81y(0) \\ &= 2 - 1 - 0.17 + 1.6(4.2) - 0.81(2) \\ y(n_2) &= 5.93 \end{aligned}$$

Part (d)-(c)

The expression for $y(k)$ will be of the form

$$y(k) = K + c_1 z_1^k + c_2 z_2^k$$

Using the values found in Part (d)-(b), the following equations are formed.

$$\begin{aligned} y(n_0) &= y(0) = K + c_1 + c_2 = 2 \\ y(n_1) &= y(1) = K + c_1 z_1 + c_2 z_2 = 4.2 \end{aligned} \Rightarrow \left\{ \begin{array}{ccccc} c_1 & + & c_2 & = 2 - K \\ z_1 c_1 & + & z_2 c_2 & = 4.2 - K \end{array} \right.$$

In matrix form,

$$\begin{bmatrix} 1 & 1 \\ z_1 & z_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2-K \\ 4.2-K \end{bmatrix}$$

We can then use the MATLAB command `x = A\b` to find c_1 and c_2, where

$$x = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 1 \\ z_1 & z_2 \end{bmatrix}, \qquad \text{and} \qquad b = \begin{bmatrix} 2-K \\ 4.2-K \end{bmatrix}$$

We find that

$$c_1 = -0.9762 - j2.1944$$
$$c_2 = -0.9762 + j2.1944$$

Problem 3

Part (a)

$$\frac{Y(z)}{U(z)} = \frac{0.83(z+1)}{z^2 - 1.6z + 0.81}, \qquad T = 0.1\text{sec}$$

The above transfer function has the following poles and zeros:

- Poles: $z_{1,2} = 0.8 \pm j\sqrt{0.17} \approx 0.8 \pm j0.4123$
- Zeros: $z_0 = -1$

To find the "matched" CT transfer function, we use the following transformation on the poles:

$$z \to e^{sT}$$

This is equivalent to

$$re^{j\theta} \to e^{sT}$$

where r is the magnitude of z and θ is the phase of z. Thus,

$$s \to \frac{1}{T}(\ln(r) + j\theta)$$

Using this transformation, we obtain the following values for the poles of the CT transfer function:

$$s_{1,2} = -1.053 \pm j4.7588$$

The zero at -1 from the DT transfer function is an addition to ensure that the denominator does not exceed the order of the numerator by more than one. Therefore, we do not take this zero into account when computing the "matched" CT transfer function. Therefore, the "matched" CT transfer function is

$$\frac{Y(s)}{U(s)} = K \cdot \frac{1}{(s + 1.0536 - j4.7588)(s + 1.5037 + j4.7588)} = \frac{K}{s^2 + 2.1072s + 23.7565}$$

To find the value of K, we need to match the DC gains of the DT and CT transfer functions. The magnitude of the DT transfer function is:

$$\begin{aligned}\left|\frac{Y(e^{j0})}{U(e^{j0})}\right| &= \left|\frac{0.83 \cdot 2}{1 - 1.6 + 0.81}\right| \\ &= \left|\frac{1.66}{0.21}\right| \\ &\approx 7.9048\end{aligned}$$

The magnitude of the CT transfer function is:

$$\left|\frac{Y(0)}{U(0)}\right| = \left|\frac{K}{23.7565}\right|$$

Therefore,

$$K = \frac{1.66}{0.21} \cdot 23.7565 \approx 187.7893$$

So, the "matched" CT transfer function is

$$\frac{Y(s)}{U(s)} = \frac{187.7893}{s^2 + 2.1072s + 23.7565}$$

Part (b)

To find the analytic expression for $y_{DT}(k)$, we can follow the same procedure as in Problem 2 (b) - (c).

We know the $y_{DK}(k)$ will be of the form

$$y_{DT}(k) = K_{DT} + c_{1DT} z_1^k + c_{2DT} z_2^k$$

Using the Final Value Theorem, we find the final value for a unit step input and, by extension, the value of K_{DT}.

$$y_{DT,\infty} = \lim_{z \to 1} (z-1) \left[\frac{z}{(z-1)} \cdot \frac{0.83(z+1)}{z^2 - 1.6z + 0.81}\right] = \frac{1.66}{0.21} \approx 7.9048 = K_{DT}$$

Solving for $y(0)$ and $y(1)$, we can set up a system of equations to solve for c_1 and c_2.

$$y_k = 0.83u_{k-1} + 0.83u_{k-2} + y_{k-1} - 0.81y_{k-2}$$

With

$$u(k) = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

and assuming initial conditions, we find

$$y(0) = 0 \qquad \text{and} \qquad y(1) = 0.83$$

Therefore,

$$\begin{aligned} y(0) &= K + c_{1DT} + c_{2DT} = 0 \\ y(1) &= K + c_{1DT} z_1 + c_{2DT} z_2 = 0.83 \end{aligned} \Rightarrow \begin{cases} c_{1DT} + c_{2DT} = -K \\ z_1 c_{1DT} + z_2 c_{2DT} = 0.83 - K \end{cases}$$

In matrix form,

$$\begin{bmatrix} 1 & 1 \\ z_1 & z_2 \end{bmatrix} \begin{bmatrix} c_{1DT} \\ c_{2DT} \end{bmatrix} = \begin{bmatrix} -K \\ 0.83 - K \end{bmatrix}$$

We can then use the MATLAB command `x = A\b` to find c_1 and c_2, where

$$x = \begin{bmatrix} c_{1DT} \\ c_{2DT} \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 1 \\ z_1 & z_2 \end{bmatrix}, \qquad \text{and} \qquad b = \begin{bmatrix} -K \\ 0.83 - K \end{bmatrix}$$

We find that

$$\begin{aligned} c_{1DT} &= -3.9524 + j0.9107 \\ c_{2DT} &= -3.9524 - j0.9107 \end{aligned}$$

So

$$y_{DT}(k) = K_{DT} + c_{1DT} z_1^k + c_{2DT} z_2^k$$

where

$$\begin{aligned} K_{DT} &= \frac{1.66}{0.21} \approx 7.9048 \\ c_{1DT} &= -3.9524 + j0.9107 \\ c_{2DT} &= -3.9524 - j0.9107 \\ z_1 &= 0.8 + j\sqrt{0.17} \approx 0.8 + j0.4123 \\ z_2 &= 0.8 - j\sqrt{0.17} \approx 0.8 - j0.4123 \end{aligned}$$

To find the analytic expression for $y_{CT}(t)$, we can use partial fraction expansion and take the inverse Laplace transform of the resulting expanded terms.

$$\frac{Y(s)}{U(s)} = \frac{187.7893}{(s - s_1)(s - s_2)}$$

where $s_{1,2} = -1.053 \pm j4.7588$.

When the input is a unit step,

$$U(s) = \frac{1}{s}$$

So

$$Y(s) = \frac{187.7893}{s(s - s_1)(s - s_2)} = 187.7893 \left(\frac{K_{CT}}{s} + \frac{c_{1CT}}{s - s_1} + \frac{c_{2CT}}{s - s_2} \right)$$

We can solve for K_{CT}, c_{1CT}, and c_{2CT} by multiplying both sides of the above equation by the product of the poles.

$$K_{CT}(s - s_1)(s - s_2) + c_{1CT} s(s - s_2) + c_{2CT} s(s - s_1) = 187.7893$$

In matrix form,

$$\begin{bmatrix} 1 & 1 & 1 \\ -(s_1 + s_2) & -s_2 & -s_1 \\ s_1 s_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} K_{CT} \\ c_{1CT} \\ c_{2CT} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 187.7893 \end{bmatrix}$$

We can then use the MATLAB command `x = A\b` to find c_1 and c_2, where

$$x = \begin{bmatrix} K_{CT} \\ c_{1CT} \\ c_{2CT} \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 1 & 1 \\ -(s_1+s_2) & -s_2 & -s_1 \\ s_1 s_2 & 0 & 0 \end{bmatrix}, \qquad \text{and} \qquad b = \begin{bmatrix} 0 \\ 0 \\ 187.7893 \end{bmatrix}$$

We find that

$$\begin{aligned} K_{CT} &= 7.9048 \\ c_{1CT} &= -3.9524 + j0.8751 \\ c_{2CT} &= -3.9524 - j0.8751 \end{aligned}$$

Taking the inverse Laplace transform of

$$Y(s) = 187.7893 \left(\frac{K_{CT}}{s} + \frac{c_{1CT}}{s-s_1} + \frac{c_{2CT}}{s-s_2} \right)$$

we find the analytic expression for $y_{CT}(t)$.

$$y_{CT}(t) = K_{CT} + c_{1CT}e^{s_1 t} + c_{2CT}e^{s_2 t}$$

where

$$\begin{aligned} K_{CT} &= 7.9048 \\ c_{1CT} &= -3.9524 + j0.8751 \\ c_{2CT} &= -3.9524 - j0.8751 \\ s_1 &= -1.053 + j4.7588 \\ s_2 &= -1.053 - j4.7588 \end{aligned}$$

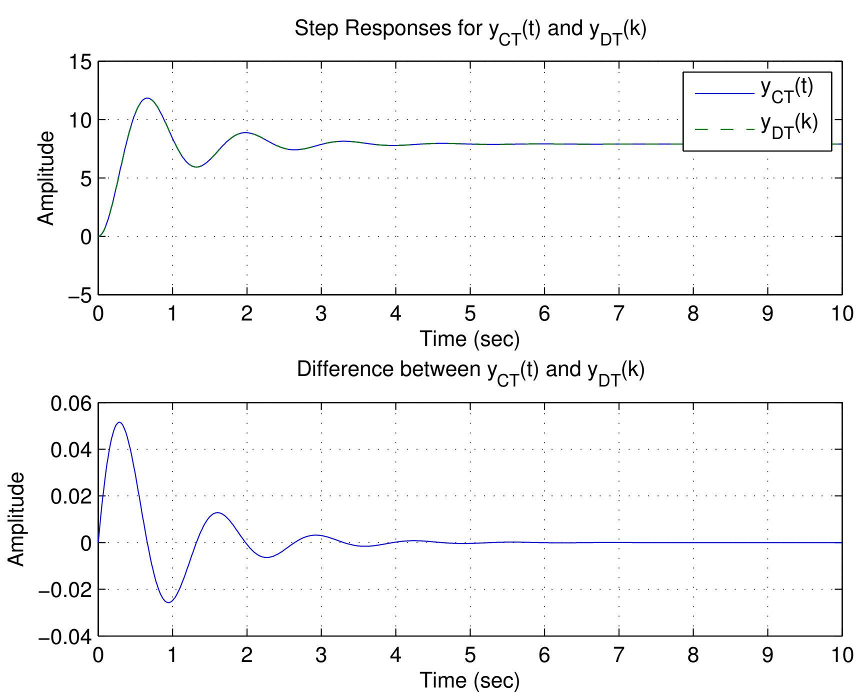

Figure 2: Plot showing the difference between the two $(y_{CT}(t) - y_{DT}(k))$ for $t = 0 : 0.01 : 10$.

Part (c)

Our new DT transfer function will still have the same poles, since we want the same basic shape, but we will adjust the zero and the gain to find a better match for the CT transfer function. Thus, our new DT transfer function will be of the form

$$\frac{Y(z)}{U(z)} = \frac{A_1 z + A_2 z}{z^2 - 1.6z + 0.81}$$

This corresponds to the difference equation

$$y_k = A_1 u_{k-1} + A_2 u_{k-2} + 1.6 y_{k-1} - 0.81 y_{k-2}$$

Since we used the "matched" method for converting between the DT and CT transfer functions, we know that

$$s \to \frac{1}{T}(\ln(r) + j\theta)$$

Therefore

$$e^{st} = e^{(\frac{1}{T}(\ln(r)+j\theta))t} = e^{\ln(r^{t/T})} e^{j\theta t/T} = r^{t/T} e^{j\theta t/T} = (re^{j\theta})^{t/T} = z^{t/T}$$

Therefore, the analytic expression we found for $y_{CT}(t)$ in the Part (b) is equivalent to

$$y_{CT}(t) = K_{CT} + c_{1CT} z_1^{t/T} + c_{2CT} z_2^{t/T}$$

At the points where $t = kT$, this expression becomes

$$y_{CT}(t) = K_{CT} + c_{1CT} z_1^k + c_{2CT} z_2^k$$

So if we want to find a DT transfer function whose step response will exactly match that of the CT transfer function, we simply need to set the coefficients equal to each other. So

$$c_{1DT} = c_{1CT} \qquad \text{and} \qquad c_{2DT} = c_{2CT}$$

Note that K_{DT} is already equal to K_{CT}.

At time $k = 0$,

$$y_{DT}(0) = K_{DT} + c_{1CT} + c_{2CT} = 0$$

In terms of the difference equation, this means that

$$y_0 - 1.6 y_{-1} + 0.81 y_{-2} = 0$$

Therefore, the input must be zero at time $k = 0$. This also shows that there is no u_k term in the difference equation and, by extension, no z^2 in the numerator of the DT transfer function.

At time $k = 1$,

$$y_{DT}(1) = K_{DT} + c_{1CT} z_1 + c_{2CT} z_2$$

In terms of the difference equation, this means that

$$y_1 = A_1 u_0 + A_2 u_{-1} + 1.6 y_0 - 0.81 y_{-1} = A_1 = K_{DT} + c_{1CT} z_1 + c_{2CT} z_2 \approx 0.8594$$

since $u_{-1} = y_0 = y_{-1} = 0$ and $u_0 = 1$ ($u(k)$ is a unit step input). So

$$A_1 \approx 0.8594$$

and

$$\frac{Y(z)}{U(z)} = \frac{0.8594z + A_2 z}{z^2 - 1.6z + 0.81}$$

We can now use the Final Value theorem to find A_2.

$$y_\infty = \lim_{z\to 1}(z-1)\left[\frac{z}{(z-1)} \cdot \frac{A_1 z + A_2 z}{z^2 - 1.6z + 0.81}\right] = \frac{A_1 + A_2}{0.21} \approx 7.9048 = K_{DT}$$

So

$$A_2 = 0.21K_{Dt} - A_1 \approx 0.21 \cdot 7.9048 - 0.8594 \approx 0.8006$$

Therefore, the new DT transfer function

$$\frac{Y(z)}{U(z)} = \frac{0.8594z + 0.8006z}{z^2 - 1.6z + 0.81}$$

will match our CT transfer function exactly.

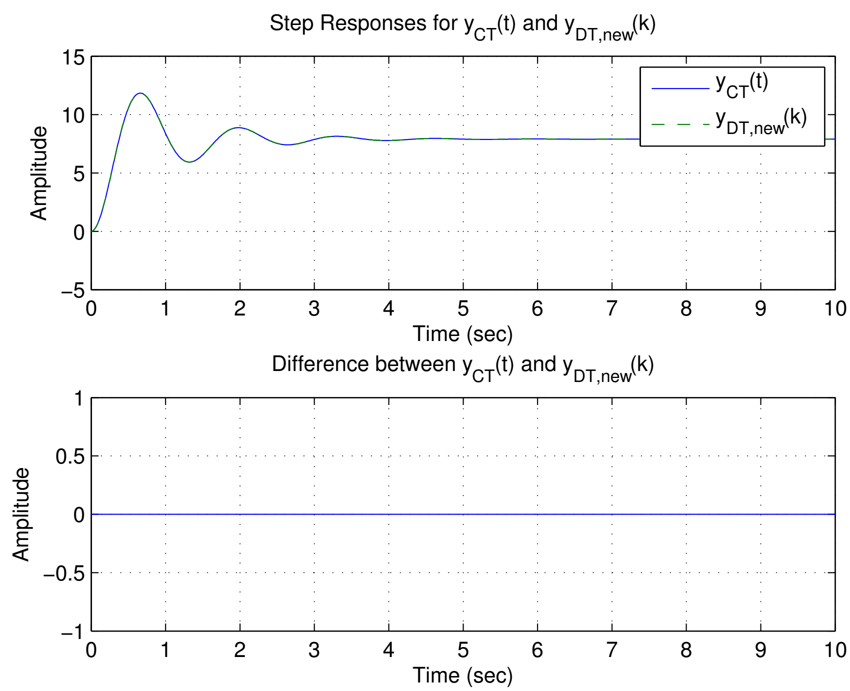

Figure 3: Plot showing the difference between the step responses of the CT transfer function and the new DT transfer function designed to exactly match the CT transfer function using the analytic expressions for the step responses. As seen in the bottom figure, the difference is zero for all t.

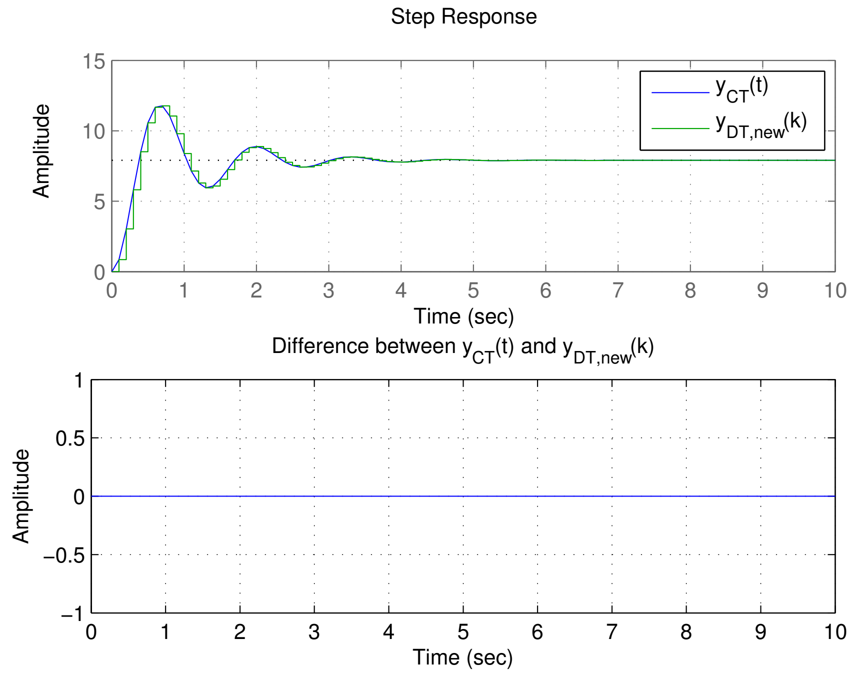

Figure 4: Plot showing the difference between the step responses of the CT transfer function and the new DT transfer function using the `step` command in MATLAB. This problem shows that adding a zero at -1 to make the difference of degrees between the numerator and denominator of a matched DT transfer function at most one is a good approximation, but not exact. The approximation may or may not be acceptable depending on the application.

Problem 4

Part (a)

$$G(z) = \frac{(z - 0.625)}{(z - 1.25)(z - 1.25)}$$

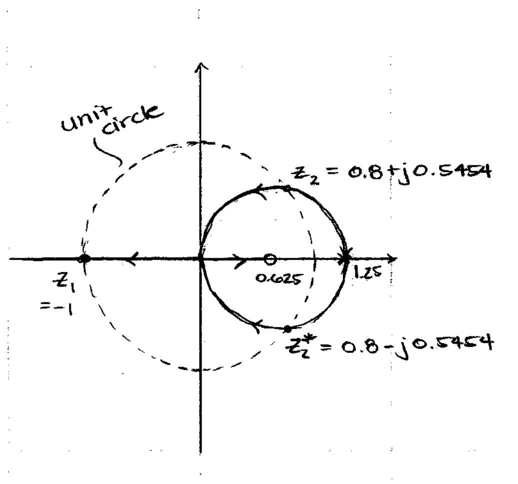

Figure 5: Root locus plot for $G(z)$.

The characteristic equation for this system is:

$$(z - 1.25)^2 + K(z - 0.625) = z^2 + (K - 2.5)z + (1.25^2 - 0.625K) = 0$$

The two poles at 1.25 will branch out as K increases and will rejoin on the real line at some point to the left of the zero 0.625. One will then head to the zero while the other will go off to negative infinity. The point at which the rejoin on the real can be found by using the following equation whose roots are the locations of multiple roots (of the same value):

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(-\frac{1}{G(z)}\right) = 0$$

For the given transfer function,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z}\left(-\frac{1}{G(z)}\right) &= \frac{\mathrm{d}}{\mathrm{d}z}\left(-(z - 0.625)^{-1}(z - 1.25)^2\right) \\
&= (z - 0.625)^{-2}(z - 1.25)^2 - 2(z - 0.625)^{-1}(z - 1.25) \\
&= (z - 0.625)^{-2}(z - 1.25)\left[z - 1.25 - 2(z - 0.625)\right] \\
&= (z - 0.625)^{-2}(z - 1.25)(-z)
\end{aligned}$$

which is zero when

$$z = 0 \qquad \text{or} \qquad z = 1.25$$

The two poles start at $z = 1.25$, and now we know that they will rejoin on the real axis at the origin.

The closed-loop system will be stable when the closed-loop poles are inside the unit circle. Thus, the values of K for which the root locus crosses the unit circle will give us the bounds on K for stability.

Clearly, the root locus will cross the unit circle on the real line at $z_1 = -1$. The value of K at this point can be found by plugging $z = -1$ into the characteristic equation.

$$(-1 - 1.25)^2 + K(-1 - 0.625) = 5.0625 - 1.625K = 0 \Rightarrow K \approx 3.1154$$

To find the locations of the two other poles on the root locus that cross the unit circle, we can take advantage of the fact that all points on the unit circle have magnitude one.

$$|z_2| = 1 \Rightarrow |z_2|^2 = 1$$

From the characteristic equation, we can express the poles as a function of K.

$$z = \frac{-(K - 2.5) + \sqrt{(K - 2.5)^2 - 4(1.25^2 - 0.625K)}}{2}$$

We know that z_2 is complex, so the term in the square root will be negative, thus resulting in an imaginary component. Therefore, we can express z as follows:

$$z = \frac{-(K - 2.5) + j\sqrt{4(1.25^2 - 0.625K) - (K - 2.5)^2}}{2}$$

Therefore,

$$\begin{aligned}
|z_2|^2 &= \left| \frac{-(K - 2.5) + j\sqrt{4(1.25^2 - 0.625K) - (K - 2.5)^2}}{2} \right| \\
&= \frac{1}{4}\left[(K - 2.5)^2 + \left(4(1.25^2 - 0.625K) - (K - 2.5)^2\right)\right] \\
&= \frac{1}{4}\left[4(1.25^2 - 0.625K)\right] \\
|z_2|^2 &= 1.25^2 - 0.625K
\end{aligned}$$

Since $|z_2|^2 = 1$, we can now solve for K.

$$1.25^2 - 0.625K = 1 \Rightarrow K = 0.9$$

We can now plug this value of K back into our general equation for z as a function of K to find z_2.

$$z_2 = \frac{-(0.9 - 2.5) + \sqrt{(0.9 - 2.5)^2 - 4(1.25^2 - 0.625 \cdot 0.9)}}{2} \approx 0.8 + j0.5454$$

Therefore, the values of the z-plane poles when the root locus crosses the unit circle are:

$$z_1 = -1\ , \qquad z_2 = 0.8 + j0.5454\ , \qquad \text{and} \qquad z_2^* = 0.8 - j0.5454$$

The range of K for stability is

$$0.9 < K < 0.31154$$

Part (b)

$$H(z) = \frac{1}{(z-0.6)(z-0.6)(z-0.6)}\ , \qquad T = 0.02\text{s}$$

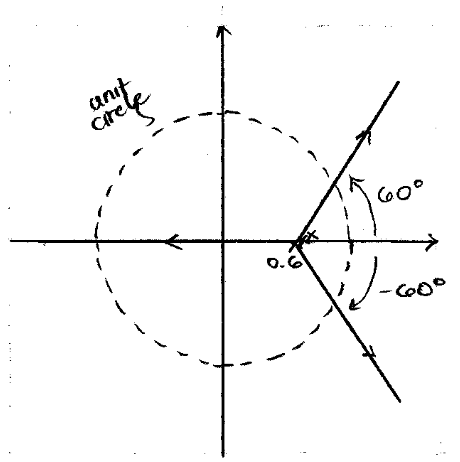

Figure 6: Root locus plot for $H(z)$.

The characteristic equation for this system is:

$$(z-0.6)^3 + K = 0$$

To draw the root locus, we need to calculate the number of asymptotes and the angle of the asymptotes. The number of asymptotes is the difference between the number of poles and the number of zeros. Therefore, there will be three asymptotes. The angles of the asymptotes are:

$$\phi_k = \frac{(2k+1)180^\circ}{N-M}\ , \qquad k \in \{0, 1, ..., N-M-1\}$$

where N is the number of open-loop poles and M is the number of open-loop zeros. Therefore,

$$\phi_k = \frac{(2k+1)180^\circ}{3}\ , \qquad k \in \{0, 1, 2\}$$

and

$$\phi_0 = 60^\circ\ , \qquad \phi_1 = 180^\circ\ , \qquad \text{and} \qquad \phi_2 = 300^\circ = -60^\circ$$

The centroid from which the asymptotes radiate is given by

$$\alpha = \frac{\sum_{n=1}^{N} p_n - \sum_{m=1}^{M} z_m}{N-M}$$

where p_n is the nth pole, $1 \leq n \leq N$, and z_m is the mth zero, $1 \leq m \leq M$. Therefore,

$$\alpha = \frac{3 \cdot 0.6 - 0}{3} = 0.6$$

The characteristic equation is

$$1 + KH(z) = 0 \Rightarrow KH(z) = -1 \Rightarrow |KH(z)| = 1$$

For $z = 0$,

$$|KH(0)| = |K| \cdot \left| \frac{1}{(-0.6)^3} \right| = |K|(0.216)^{-1} = 1$$
$$\Rightarrow K = 0.216$$

To find the locations of the other two poles for this value of K, we can plug the value of K found above into the characteristic equation.

$$\begin{aligned}(z-0.6)^3 + 0.216 &= z^3 - 1.8z^2 + 1.08z \\ &= z(z^2 - 1.8z + 1.08) \\ &= z\left((z-0.9)^2 + 0.27\right) = 0 \\ &\Rightarrow z = 0\ ,\ 0.9 \pm j\sqrt{0.27}\end{aligned}$$

So the locations of the poles when $K = 0.216$ are

$$z_0 = 0\ ,\ z_{1,2} = 0.9 \pm j\sqrt{0.27}$$

The corresponding underdamped s-plane poles are found using the "matched" transformation

$$s \to \frac{1}{T}(\ln(r) + j\theta)$$

where $z = re^{j\theta}$. For the underdamped z-plane poles,

$$r = \sqrt{0.81 + 0.27} = \sqrt{1.08}\ , \qquad \text{and} \qquad \theta_{1,2} = \tan^{-1}\left(\pm\frac{\sqrt{0.27}}{0.9}\right) \approx \pm 0.5236$$

Therefore,

$$s_{1,2} = \frac{1}{0.02}(\ln(\sqrt{1.08} \pm j0.5236) \approx 1.9240 \pm j26.1799$$

A second-order characteristic equation can expressed in the form

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

The following manipulations show how to express s in terms of σ and ω_d.

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = (s + \zeta\omega_n)^2 - \zeta^2\omega_n^2 + \omega_n^2 = (s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2) = 0$$

$$\Rightarrow \qquad \begin{aligned} s &= -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \\ s &= -\sigma \pm j\omega_d \end{aligned}$$

where $\sigma = \zeta\omega_n$ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$.

So, for the corresponding underdamped s-plane poles,

$$\sigma = -1.9240\ , \qquad \text{and} \qquad \omega_d = 26.1799$$

Part (c)

Using the transformation

$$z \to e^{sT}$$

we can find the corresponding parameter specifications in the z-plane.

- $\sigma = 5$ rad/s.

  $$s = -\sigma + j\omega_d \leftrightarrow z = e^{sT} = e^{-\sigma T} e^{j\omega_d T}$$

  So, for $\sigma = 5$ rad/s and $T = 0.02$ s,

  $$z = e^{-5T} e^{j\omega_d T}$$

  which corresponds to a circle of radius $r = e^{-5 \cdot 0.02} = e^{-0.1} \approx 0.9048$.

  The case where $s = -\sigma - j\omega_d$ yields essentially the same result.

- $\omega_d = 25$ rad/s

  $$z = e^{-\sigma T} e^{j25T} = e^{-\sigma T}\cos(25T) + je^{-\sigma T}\sin(25T)$$

  which corresponds to a line at an angle of $\theta = 25T = 0.5$ rad, $T = 0.02$ s.

  For the case where $s = -\sigma - j\omega_d$,

  $$z = e^{-\sigma T} e^{-j25T} = e^{-\sigma T}\cos(25T) - je^{-\sigma T}\sin(25T)$$

  which corresponds to a line at an angle of $\theta = -25T = -0.5$ rad, $T = 0.02$ s.

- $\zeta = 0.7$
  As noted in Part (b),

  $$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

  which implies that

  $$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}}$$

  Therefore,

  $$s = -\sigma + j\omega_d = -\zeta\omega_n + j\omega_d = -\frac{\zeta\omega_d}{\sqrt{1-\zeta^2}} + j\omega_d$$

  which, in the z-plane, corresponds to

  $$z = \exp\left(-\frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\right) \cdot \exp(j\omega_d)$$

  For $\zeta = 0.7$,

  $$z = \exp\left(-\frac{0.7\omega_d}{\sqrt{1-0.7^2}}\right) \cdot \exp(j\omega_d) \approx \exp(-0.9802\omega_d) \cdot \exp(j\omega_d)$$

  which corresponds to a logarithmic spiral.

  The case where $s = -\sigma - j\omega_d$ yields essentially the same result.

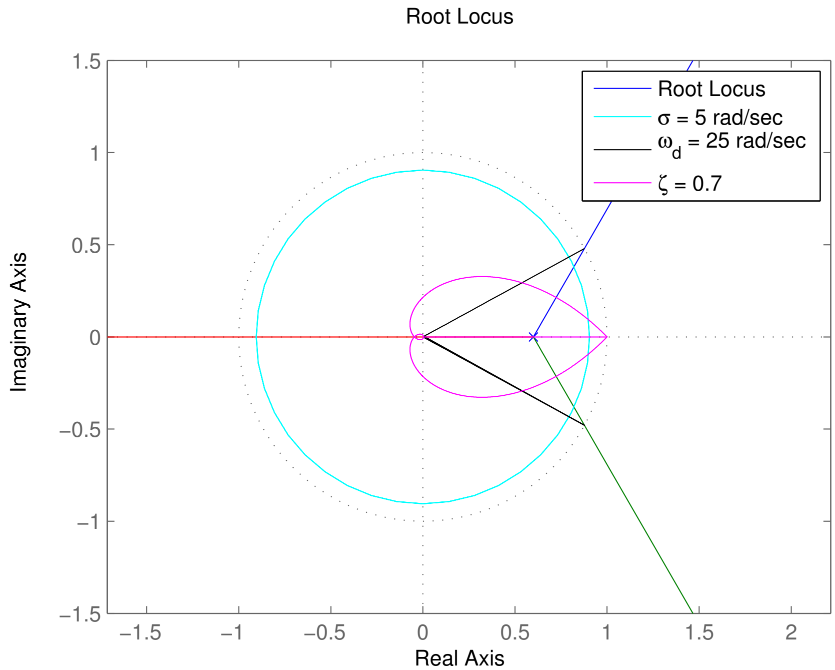

Figure 7: Root locus plot of $H(z)$ with mapped construction lines for $\sigma = 5$ rad/s, $\zeta = 0.7$, and $\omega_d = 25$ rad/s. The intersections of the construction lines with the root locus clearly show where the complex conjugate pair poles will correspond to the three parameter specifications.

Problem 5

Part (a)

$$G(z) = \frac{1}{z - 0.8}\ , \qquad T = 0.1\text{s}$$

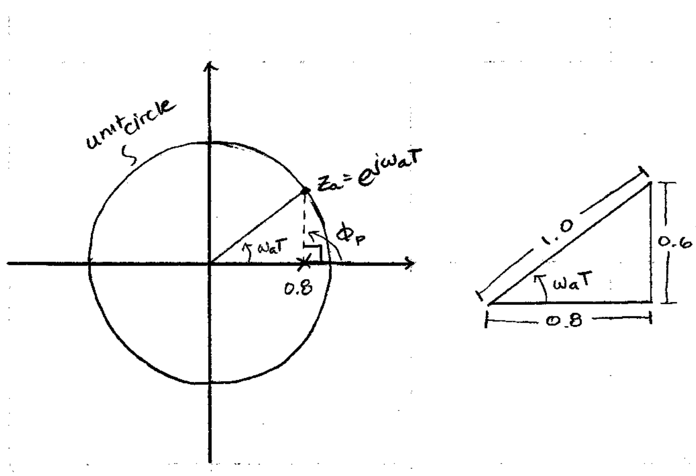

Figure 8: Pole-zero plot for $G(z)$.

The pause of a transfer function is given by the phase of its zeros minus the phase of its poles. So,

$$\underline{/G(z_a)} = -\phi_p = -90^\circ$$

Thus,

$$\omega_a T = \cos^{-1}\left(\frac{0.8}{1.0}\right) \approx 0.6435\text{rad}$$

Therefore,

$$\omega_a = 6.4350\text{rad/s}$$

and

$$z_a = e^{j0.6435} = 0.8 + j0.6$$

Part (b)

$$G(z) = \frac{1}{(z-0.8)(z-0.8)}$$
$$C(z) = K(z+a)$$

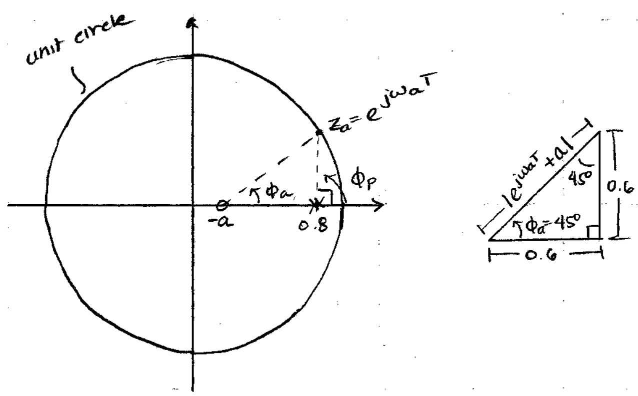

Figure 9: Pole-zero plot for $G(z)$.

There are now two poles at $z = 0.8$, so the phase is twice that of the previous transfer function (from Part (a)). Therefore,

$$\underline{/G(z_1)} = 2 \cdot (-90^\circ) = -180^circ$$

In order to add 45° of phase to the system, we need to add a zero that has a phase of 45°. Therefore,

$$-a = 0.8 - 0.6 = 0.2$$
$$\Rightarrow a = -0.2$$

At the crossover frequency, the magnitude is 0 dB = 1. Therefore,

$$\begin{aligned}
&|C(e^{j\omega_a T})G(e^{j\omega_a T)}| = 1 \\
\Rightarrow \quad &|K(e^{j\omega_a T} + a)||G(e^{j\omega_a T})| = 1 \\
&|K| = \frac{1}{|(e^{j\omega_a T} + a)||G(e^{j\omega_a T})|} \\
&\quad = \frac{|e^{j\omega_a T} - 0.8|^2}{|e^{j\omega_a T} - 2|} \\
&\quad = \frac{|j0.6|^2}{|0.6 + j0.6|} \\
&|K| = \frac{0.6^2}{0.6\sqrt{2}} \\
&K = \frac{0.6}{\sqrt{2}} \approx 0.4243
\end{aligned}$$

Therefore,

$$C(z) = \frac{0.6}{\sqrt{2}}(z - 0.2) \approx 0.4243(z - 0.2)$$

Problem 6

Part (a)

$$P(z) = \frac{z}{z^2 - 1.2z + 1.9}$$
$$C(z) = K(z + a)$$

Desired closed-loop poles: $z_1 = 0.95$, $z_2 = 0.4$

The characteristic equation is

$$1 + CP = z^2 - 1.2z + 1.9 + Kz(z + 1) = (K + 1)z^2 + (aK - 1.2)z + 1.9 = 0$$

We can now plug in our two desired closed-loop poles and obtain a system of (two) equations to solve for K and a.

For $z = 0.95$,

$$(K+1)z^2+(aK-1.2)z+1.9 = (K+1)0.9025+(aK-1.2)0.95+1.9 = 0.9025K+0.95aK+1.6625 = 0$$

For $z = 0.5$,

$$(K + 1)z^2 + (aK - 1.2)z + 1.9 = (K + 1)0.16 + (aK - 1.2)0.4 + 1.9 = 0.16K + 0.4aK + 1.58 = 0$$

In matrix form,

$$\begin{bmatrix} 0.9025 & 0.95 \\ 0.16 & 0.4 \end{bmatrix} \begin{bmatrix} K \\ \alpha K \end{bmatrix} = \begin{bmatrix} -1.6625 \\ -1.58 \end{bmatrix}$$

We can then use the MATLAB command `x = A\b` to find K and αK, where

$$x = \begin{bmatrix} K \\ \alpha K \end{bmatrix}, \qquad A = \begin{bmatrix} 0.9025 & 0.95 \\ 0.16 & 0.4 \end{bmatrix}, \qquad \text{and} \qquad b = \begin{bmatrix} -1.6625 \\ -1.58 \end{bmatrix}$$

We find that

$$\begin{aligned} K &= 4 \\ aK &= -5.55 \Rightarrow a = -1.3875 \end{aligned}$$

So

$$C(z) = 4(z - 1.3875)$$

Part (b)

$$\begin{aligned} P(z) &= \frac{z}{z^2 - 1.2z + 1.9} \\ C(z) &= K(z + a) \end{aligned}$$

Desired closed-loop poles: $z_1 = 0.95$, $z_2 = 0.01$

The characteristic equation is

$$1 + CP = z^2 - 1.2z + 1.9 + Kz(z+1) = (K+1)z^2 + (aK - 1.2)z + 1.9 = 0$$

We can now plug in our two desired closed-loop poles and obtain a system of (two) equations to solve for K and a.

For $z = 0.95$,

$$(K+1)z^2 + (aK-1.2)z + 1.9 = (K+1)0.9025 + (aK-1.2)0.95 + 1.9 = 0.9025K + 0.95aK + 1.6625 = 0$$

For $z = 0.01$,

$$(K+1)z^2 + (aK-1.2)z + 1.9 = (K+1)0.0001 + (aK-1.2)0.01 + 1.9 = 0.0001K + 0.01aK + 1.8881 = 0$$

In matrix form,

$$\begin{bmatrix} 0.9025 & 0.95 \\ 0.0001 & 0.01 \end{bmatrix} \begin{bmatrix} K \\ \alpha K \end{bmatrix} = \begin{bmatrix} -1.6625 \\ -1.881 \end{bmatrix}$$

We can then use the MATLAB command `x = A\b` to find K and αK, where

$$x = \begin{bmatrix} K \\ \alpha K \end{bmatrix}, \qquad A = \begin{bmatrix} 0.9025 & 0.95 \\ 0.0001 & 0.01 \end{bmatrix}, \qquad \text{and} \qquad b = \begin{bmatrix} -1.6625 \\ -1.881 \end{bmatrix}$$

We find that

$$\begin{aligned} K &= 199 \\ aK &= -190.8 \Rightarrow a = -0.9588 \end{aligned}$$

So

$$C(z) = 199(z - 0.9588)$$

Part (c)

If one of the desired poles is at the origin, the desired characteristic equation would be

$$z(z-p) = z^2 - pz = 0$$

where p is the other desired pole location. In order for our actual characteristic equation

$$\begin{aligned} &(K+1)z^2 + (aK-1.2)z + 1.9 = 0 \\ \text{or} \qquad &z^2 + \left(\frac{aK-1.2}{K+1}\right) + \left(\frac{1.9}{K+1}\right) = 0 \end{aligned}$$

to equal the desired characteristic equation, we would have to have

$$\frac{1.9}{K+1} = 0$$

which would only be possible if $K \to \infty$, which is not physically feasible.

Part (d)

The desired characteristic equation for a deadbeat controller is

$$z^n = 0$$

where n is the number of poles of the closed-loop system. With a controller, $C_{dbc}(z)$, our actual characteristic equation is

$$z^2 - 1.2z + 1.9 + zC_{dbc}(z) = 0$$

In order for this equation to have the desired form, $C_{dbc}(z)$ will have to be designed to cancel out the terms of order less than two. We can do this by having a pole at the origin to cancel out the zero from the plant and having a numerator of $(1.2z - 1.9)$ to cancel out the terms of order less than two from the denominator of the plant transfer function. Therefore,

$$C_{dbc}(z) = \frac{1.2z - 1.9}{z}$$

Problem 7

Part (a)

$$P(z) = \frac{1}{z^3 + 0.8}$$

The poles of this system are

$$\begin{aligned} z_1 &= -0.9283 \\ z_2 &= 0.4642 + j0.8039 \\ z_3 &= 0.4642 - j0.8039 \end{aligned}$$

To find the analytic expression for $y(k)$, we can follow the same procedure as in Problem 2 (a) - (c).

Separating the $Y(z)$ and $U(z)$ terms:

$$Y(z)(z^3 + 0.8) = U(z)$$

Taking the inverse Z-transform,

$$y_{k+3} + 0.8y_k = u_k$$

Shifting the indices by three,

$$y_k = u_{k-3} - 0.8y_{k-3}$$

For a unit step input, we know the $y_(k)$ will be of the form

$$y(k) = K + c_1 z_1^k + c_2 z_2^k + c_3 z_3^k$$

Using the Final Value Theorem, we find the final value for a unit step input and, by extension, the value of K.

$$K = y_\infty = \lim_{z\to 1}(z-1)\left[\frac{z}{(z-1)} \cdot \frac{1}{z^3 + 0.8}\right] = \frac{1}{0.1.8}\frac{5}{9} \approx 0.556$$

Solving for $y(0)$ and $y(1)$, we can set up a system of equations to solve for c_1 and c_2.

$$y_k = u_{k-3} - 0.8y_{k-3}$$

With

$$u(k) = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

and assuming initial conditions, we find

$$y(0) = 0\ , \qquad y(1) = 0\ , \qquad \text{and} \qquad y(2) = 0$$

Therefore,

$$\begin{matrix} y(0) = K + c_1 + c_2 + c_3 = 0 \\ y(1) = K + c_1 z_1 + c_2 z_2 + c_3 z_3 = 0 \\ y(2) = K + c_1 z_1^2 + c_2 z_2^2 + c_3 z_3^2 = 0 \end{matrix} \Rightarrow \begin{cases} c_1 + c_2 + c_3 = -K \\ z_1 c_1 + z_2 c_2 + z_3 c_3 = -K \\ z_1^2 c_1 + z_2^2 c_2 + z_3^2 c_3 = -K \end{cases}$$

In matrix form,

$$\begin{bmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_1^2 & z_2^2 & z_3^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -K \\ -K \\ -K \end{bmatrix}$$

We can then use the MATLAB command `x = A\b` to find c_1 and c_2, where

$$x = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}\ , \qquad A = \begin{bmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_1^2 & z_2^2 & z_3^2 \end{bmatrix}\ , \qquad \text{and} \qquad b = \begin{bmatrix} -K \\ -K \\ -K \end{bmatrix}$$

We find that

$$c_1 = -2.006$$
$$c_2 = -0.1775 + j0.3589$$
$$c_3 = -0.1775 - j0.3589$$

So

$$y_(k) = K + c_1 z_1^k + c_2 z_2^k + c_3 z_3^k$$

where

$$\begin{aligned}
K &= \frac{5}{9} \approx 0.556 \\
c_1 &= -2.006 \\
c_2 &= -0.1775 + j0.3589 \\
c_3 &= -0.1775 - j0.3589 \\
z_1 &= -0.9283 \\
z_2 &= 0.4642 + j0.8039 \\
z_3 &= 0.4642 - j0.8039
\end{aligned}$$

Part (b)

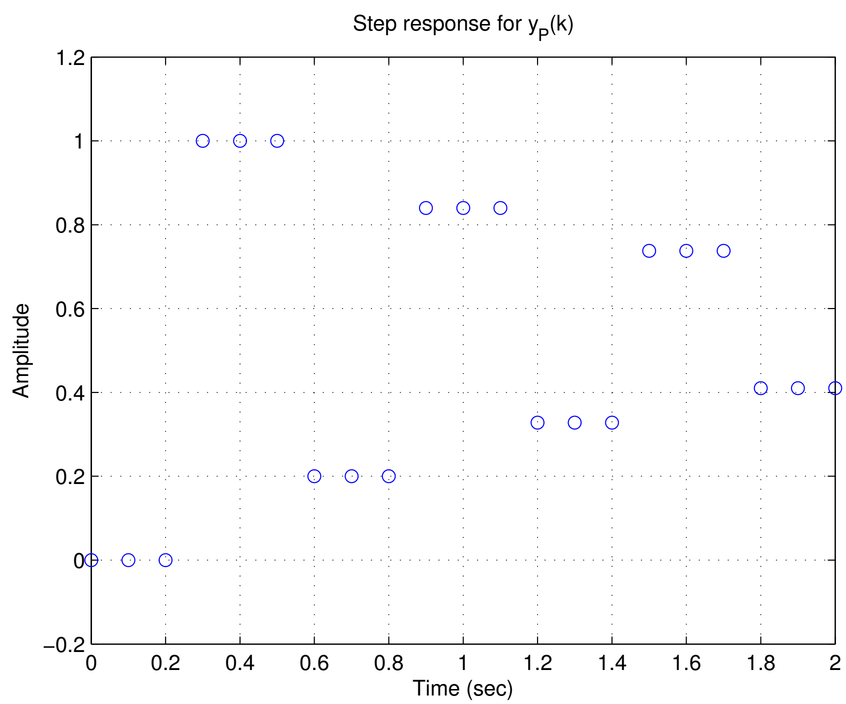

Figure 10: Step response for the DT system, $P(z)$, with $T = 0.1$ seconds.

Part (c)

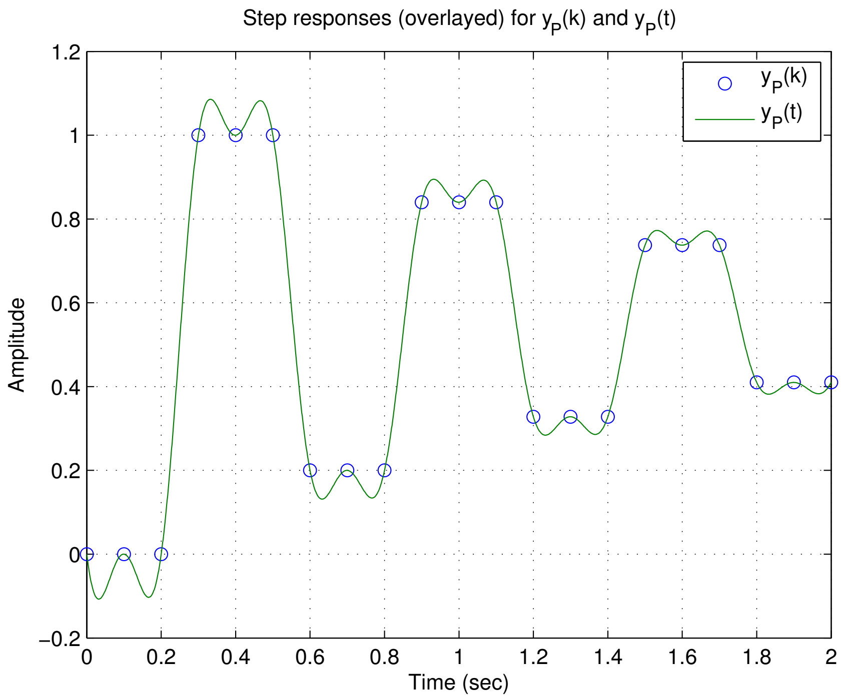

Figure 11: Overlay of "continuous" step response onto discrete step response. Note the ripples between samples. This shows that we must be careful when designing discrete controllers because a seemingly well-behaved sampled response may, in reality, stem from a poorly-behaved system.