Homework 8 Solutions

Problem 1

$$A_d = \begin{bmatrix} 0.5 & -1.2 \\ 1 & 0 \end{bmatrix}, \qquad B_d = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad C_d = \begin{bmatrix} 0 & 1 \end{bmatrix}, \qquad D_d = \begin{bmatrix} 0 \end{bmatrix}, \qquad T = 0.1 \text{ seconds}$$

Part (a)

The open-loop poles are given by the eigenvalues of A_d. The poles/eigenvalues of A_d are found by setting the determinant of $|zI - A_d|$ to zero and solving for z.

$$|zI - A_d| = \begin{vmatrix} z - 0.5 & 1.2 \\ -1 & z \end{vmatrix} = z(z-0.5) + 1.2 = z^2 - 0.5z + 1.2 = 0$$

when

$$z = \frac{1}{2}\left(0.5 \pm \sqrt{0.5 - 4.8}\right) = \frac{1}{2}\left(0.5 \pm j\sqrt{4.55}\right) \approx 0.25 \pm j1.0665$$

Thus, the open-loop poles are

$$z_1 = 0.25 + j1.0665, \qquad z_2 = 0.25 - j1.0665$$

Since the imaginary part of the poles is greater than unity, the magnitude will be greater than unity, thus the poles will be outside the unit circle. Therefore, the system is unstable.

Part (b)

$$T_p = 1 \text{ second}, \qquad \%OS \approx 5\%$$

We can use the above information to find the s-plane poles for the continuous-time system, then use the matched transformation to find the equivalent z-plane poles. The s-plane poles will be of the form

$$s = -\sigma \pm j\omega_d$$

where $\sigma = \zeta\omega_n$ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the damped frequency. We can find the damped frequency using the peak time, T_p.

$$T_p = \frac{\pi}{\omega_d} \implies \omega_d = \frac{\pi}{T_p} = \pi$$

A $\%OS$ of approximately 5% corresponds to a damping ratio, ζ, of around 0.7. Using this value for ζ and the value of ω_d found above, we can determine the value of $\sigma = \zeta\omega_n$.

$$\sigma = \zeta\omega_n = \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}} = \frac{0.7\pi}{\sqrt{1-0.7^2}} \approx 3.0794$$

Therefore, the s-plane poles are

$$s_{1,2} = -3.0794 \pm j\pi$$

Using the matched transformation

$$z = e^{sT}$$

the equivalent z-plane poles are

$$z_{1,2} = 0.6990 \pm j0.2271$$

Part (c)

Because the state-space matrices are given in control canonical form, we can easily find the characteristic equation of the system closed-loop system. The closed-loop characteristic equation is found by taking the determinant of the matrix $[A_d - B_d K]$, $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$, and setting it equal to zero.

$$\begin{bmatrix} A_d - B_d K \end{bmatrix} = \begin{bmatrix} 0.5 - K_1 & -1.2 - K_2 \\ 1 & 0 \end{bmatrix}$$

Thus, the characteristic equation is

$$z^2 - (0.5 - K_1)z - (-1.2 - K_2) = 0$$

We would like the characteristic equation to be of the form

$$(z - 0.6990 + j0.2271)(z - 0.6990 - j0.2271) = z^2 - 1.3980z + 0.5402 = 0$$

in order to place the poles at the pole locations found in Part (b). Thus,

$$\begin{aligned} 0.5 - K_1 = 1.3980 &\implies K_1 = -0.8980 \\ 1.2 + K_2 = 0.5402 &\implies K_2 = -0.6598 \end{aligned}$$

Therefore,

$$K = \begin{bmatrix} -0.8980 & -0.6598 \end{bmatrix}$$

Part (d)

We want our peak time to be half that of Part (b), although our ζ will remain the same. Therefore,

$$\overline{\omega}_d = \frac{\pi}{0.5T_p} = \frac{2\pi}{T_p} = 2\omega_d$$

and

$$\overline{\sigma} = \zeta\omega_n = \frac{\zeta(2\omega_d)}{\sqrt{1-\zeta^2}} = 2\frac{\zeta\omega_d}{\sqrt{1-\zeta^2}} = 2\sigma$$

Thus,

$$\overline{s} = 2s$$

and

$$\overline{z} = e^{\overline{s}T} = e^{2sT} = \left(e^{sT}\right)^2 = z^2$$

Therefore, the desired estimator poles are

$$\overline{z}_{1,2} = z_{1,2}^2 = 0.4370 \pm j0.3175$$

Part (e)

The matrix $[A_d - L_p C_d]$, $L_p = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$, is given by

$$[A_d - L_p C_d] = \begin{bmatrix} 0.5 & -1.2 - L_1 \\ 1 & -L_2 \end{bmatrix}$$

The characteristic equation is given by $|zI - (A_d - L_pC_d)| = 0$. Therefore, the characteristic equation for the estimator is

$$\left|zI - (A_d - L_pC_d)\right| = \begin{vmatrix} z - 0.5 & 1.2 + L_1 \\ -1 & z + L_2 \end{vmatrix} = (z-0.5)(z+L_2)-(-1.2-L_1)(1) = z^2+(L_2-0.5)z+(L_1-0.5L_2+1.2)$$

The desired characteristic equation for the estimator is

$$(z - 0.4370 + j0.3175)(z - 0.4370 - j0.3175) = z^2 - 0.8740 + 0.2918 = 0$$

Thus,

$$L_2 - 0.5 = -0.8740 \implies L_2 = -0.3740$$
$$L_1 - 0.5L_2 + 1.2 = 0.2918 \implies L_1 = -1.0952$$

Therefore,

$$L_p = \begin{bmatrix} -1.0952 \\ -0.3740 \end{bmatrix}$$

Problem 2

Part (a)

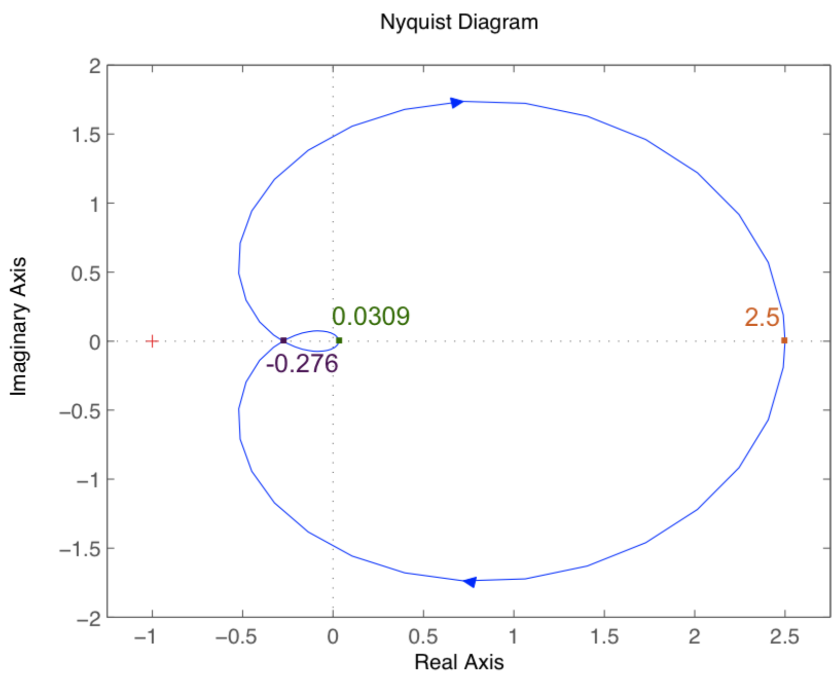


Figure 1: Nyquist plot for this transfer function $G(z) = \frac{0.1}{(z-0.8)^2}$.

Part (b)

To determine the number of unstable closed-loop poles, we can use the following equation

$$Z = N + P$$

where Z is the number of unstable closed-loop poles, N is the number of clockwise encirclements of -1, and P is the number of unstable open-loop poles. In this case, $P = 0$ and we want no unstable closed-loop poles. Thus, we want zero encirclements of -1. Therefore, the range of K for stability, considering both positive and negative values of K, is

$$\frac{1}{-2.5} < K < \frac{1}{0.276}$$

which is equivalent to

$$-0.4 < K < 3.6232$$

Part (c)

In order to have two unstable closed-loop poles, we want $Z = 2$, therefore we want two clockwise encirclements of -1. Therefore, the range of K for two unstable closed-loop poles is

$$K > \frac{1}{0.276} \qquad \text{or} \qquad K < \frac{1}{-0.0309}$$

which is equivalent to

$$K > 3.6232 \qquad \text{or} \qquad K < -32.3625$$

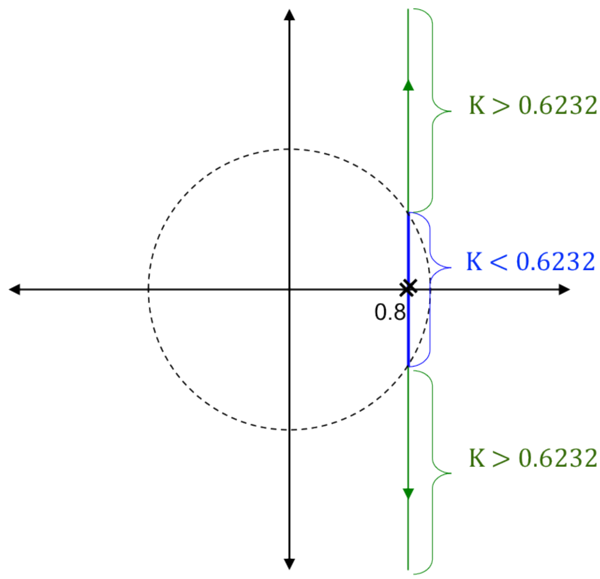
In order to have one unstable closed-loop pole, we want $Z = 1$, therefore we want one clockwise encirclement of -1. Therefore, the range of K for one unstable closed-loop pole is

$$\frac{1}{-0.0309} < K < \frac{1}{2.5}$$

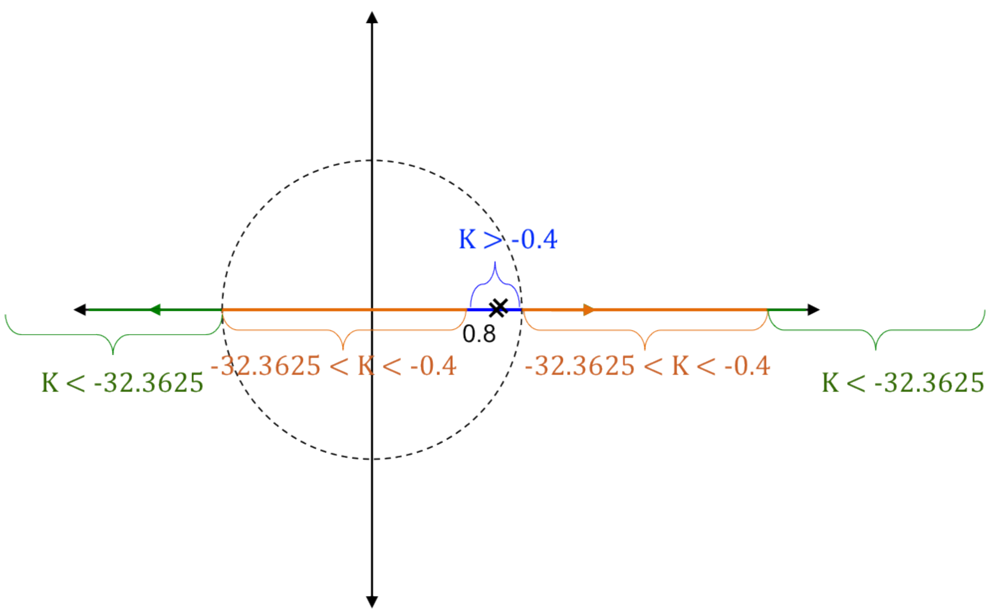
which is equivalent to

$$-32.3625 < K < -0.4$$

Part (d)



(a) Root locus plot for the system $G(z)$ for positive values of K.



(b) Root locus plot for the system $G(z)$ for negative values of K.

Figure 2: Root locus plots for $G(z)$ considering both positive and negative values of K. The regions of K found in Parts (b) and (c) are labeled.

Problem 3

$$A_d = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

To determine controllability, we test the rank of the controllability matrix. The system is controllable if the controllability matrix is full rank. The controllability matrix for the above system is given by

$$\mathcal{C} = \begin{bmatrix} B_d & A_d B_d \end{bmatrix}$$

Part (a)

$$B_d = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The controllability matrix is thus,

$$\mathcal{C} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$

The controllability matrix has two linearly independent columns, therefore the matrix has full rank and the system is controllable.

Part (b)

$$B_d = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The controllability matrix is thus,

$$\mathcal{C} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

The controllability matrix has only one linearly independent column, therefore the matrix does not have full rank and the system is not controllable.

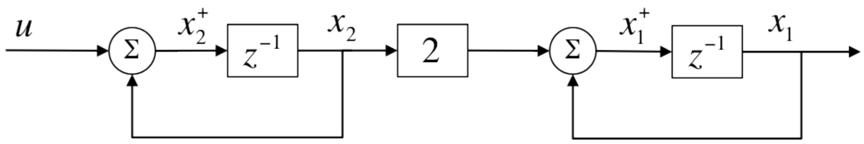
Part (c)

The state equations for the system in Part (a) are:

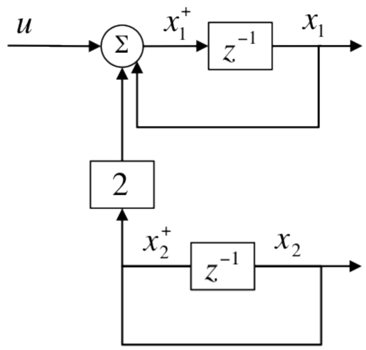
$$\begin{aligned} x_1^+ &= x_1 + 2x_2 \\ x_2^+ &= x_2 + u \end{aligned}$$

The state equations for the system in Part (b) are:

$$\begin{aligned} x_1^+ &= x_1 + 2x_2 + u \\ x_2^+ &= x_2 \end{aligned}$$



(a) Block diagram for $B_d = [0, 1]^T$.



(b) Block diagram for $B_d = [1, 0]^T$.

Figure 3: Block diagrams for the given A_d and B_d matrices. Clearly, the input u affects both x_2 (directly) and x_1 (indirectly, through x_2) in (a), meaning that we can "control" the states using the input. On the other hand, the input u only affects x_1 in (b). There is not connection between u and x_2, directly or indirectly. Thus, we would only be able to "control" the state x_1, and x_2 would be uncontrollable.

Problem 4

$$P(z) = \frac{z + 0.5}{z^3 + 1.2z^2 + 0.5z - 1}, \qquad T = 0.1 \text{ seconds}$$

Part (a)

$$A_d = \begin{bmatrix} -1.2 & -0.5 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad B_d = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad C_d = \begin{bmatrix} 0 & 1 & 0.5 \end{bmatrix}, \qquad D_d = \begin{bmatrix} 0 \end{bmatrix}$$

Part (b)

The closed-loop characteristic equation is found by taking the determinant of the matrix $[A_d - B_d K]$, $K = \begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix}$, and setting it equal to zero.

$$\begin{bmatrix} A_d - B_d K \end{bmatrix} = \begin{bmatrix} -1.2 - K_1 & -0.5 - K_2 & 1 - K_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Thus, the characteristic equation is

$$z^3 - (-1.2 - K_1)z^2 - (-0.5 - K_2)z - (1 - K_3) = 0$$

For deadbeat control, we want all closed-loop poles at $z = 0$. Therefore, the desired closed-loop characteristic equation is

$$z^3 = 0$$

Thus,

$$\begin{aligned} -1.2 - K_1 = 0 &\implies K_1 = -1.2 \\ -0.5 - K_2 = 0 &\implies K_2 = -0.5 \\ 1 - K_3 = 0 &\implies K_3 = 1 \end{aligned}$$

Therefore,

$$K = \begin{bmatrix} -1.2 & -0.5 & 1 \end{bmatrix}$$

Part (c)

In this case, the desired characteristic equation is

$$(z + 0.5)(z - 0.8)^2 = z^3 - 1.1z^2 - 0.16z + 0.32 = 0$$

Thus,

$$\begin{aligned} -1.2 - K_1 = 1.1 &\implies K_1 = -2.3 \\ -0.5 - K_2 = 0.16 &\implies K_2 = -0.66 \\ 1 - K_3 = -0.32 &\implies K_3 = 1.32 \end{aligned}$$

Therefore,

$$K = \begin{bmatrix} -2.3 & -0.66 & 1.32 \end{bmatrix}$$

Part (d)

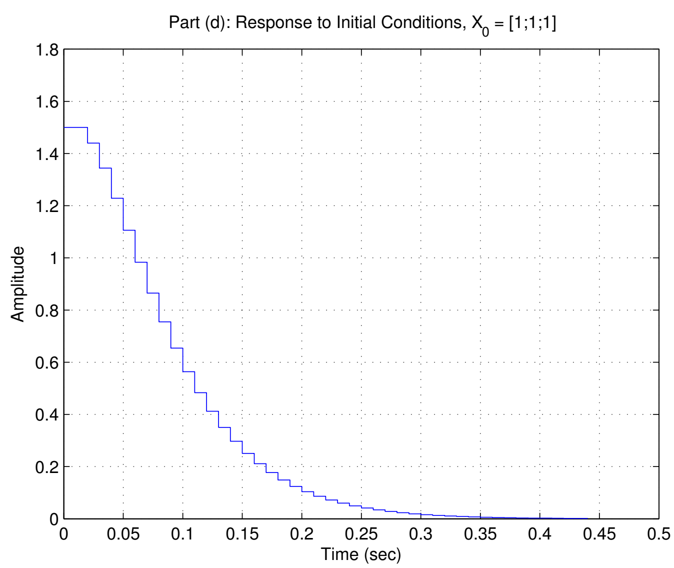


Figure 4: Initial condition response for the closed-loop system of Part (c) for $X_0 = [1; 1; 1]$.

Part (e)

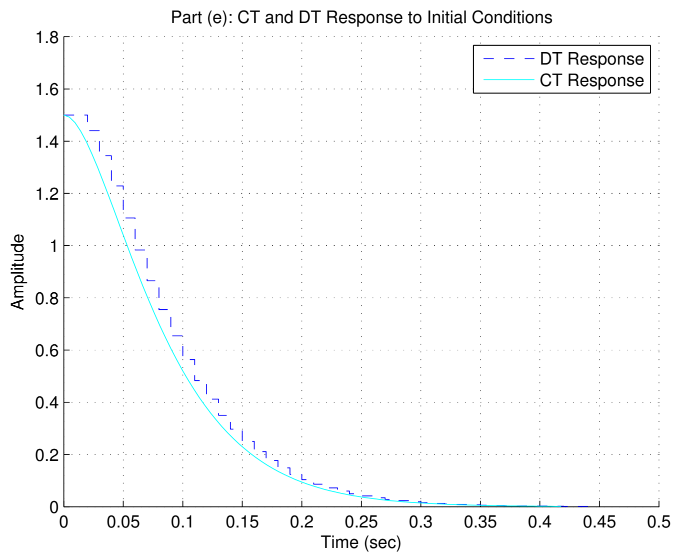


Figure 5: Comparison of the initial condition response for the closed-loop DT system of Part (c) with $X_0 = [1; 1; 1]$ and the initial condition response for CT system with two s-plane poles corresponding to $z = 0.8, T = 0.01$ seconds. The initial conditions for the CT system were $X_{C0} = [0; 1.5]$ to match the initial conditions of the DT system. The CT response is very closed to the closed-loop DT response.

Part (f)

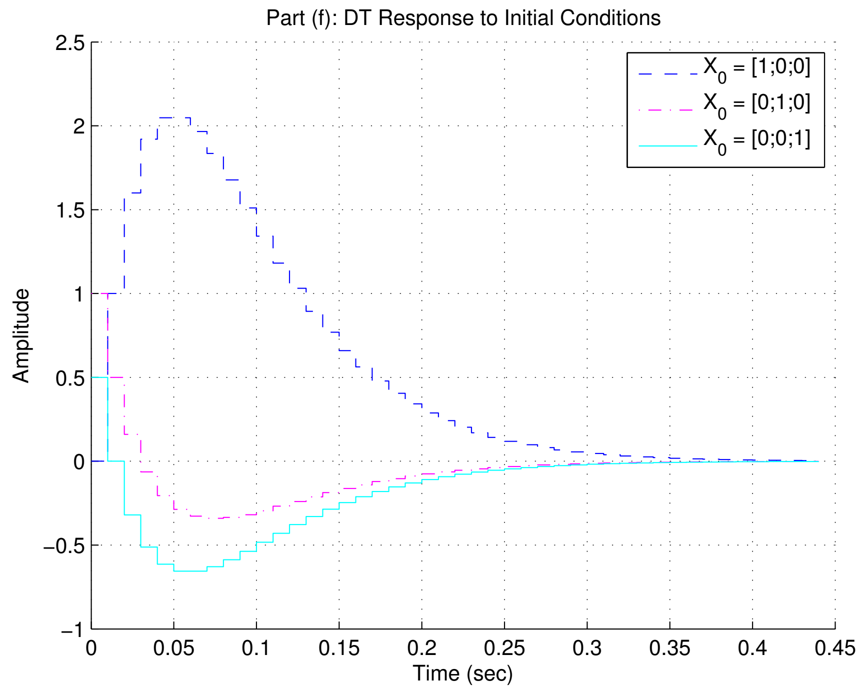


Figure 6: Initial condition response for the closed-loop system of Part (c) for $X_0 = [1;0;0]$, $X_0 = [0;1;0]$, $X_0 = [0;0;1]$.

Part (g)

The state equation for the system is given by

$$x^+ = (A_d - B_d K)x$$

where x and x^+ are shorthand notation for $x(k)$ and $x(k+1)$, respectively. Thus,

$$x(k=1) = (A_d - B_d K)x(k=0) = (A_d - B_d K)x_0$$

where $x_0 = \begin{bmatrix} x_{01} & x_{02} & x_{03} \end{bmatrix}^T$. So,

$$x(k=1) = \begin{bmatrix} -1.1 & -0.16 & 0.32 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix} = \begin{bmatrix} -1.1x_{01} - 0.16x_{02} + 0.32x_{03} \\ x_{01} \\ x_{02} \end{bmatrix}$$

Part (h)

The output equation for the system is given by

$$y = C_d x$$

Therefore,

$$y(k=0) = C_d x_0 = \begin{bmatrix} 0 & 1 & 0.5 \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix} = x_{02} + 0.5x_{03}$$

and

$$y(k=1) = C_d x(k=1) = \begin{bmatrix} 0 & 1 & 0.5 \end{bmatrix} \begin{bmatrix} -1.1x_{01} - 0.16x_{02} + 0.32x_{03} \\ x_{01} \\ x_{02} \end{bmatrix} = x_{01} + 0.5x_{02}$$

The values for $y(k=0)$ and $y(k=1)$ for the various initial conditions are summarized in the following table.

x_0	$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$	$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$	$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$	$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$
$y(k=0)$	0	1	0.5	1.5
$y(k=1)$	1	0.5	0	1.5

The values in the above table match the values of y at $k=0$ and $k=1$ in the plots from parts (d) and (f).

Problem 5

$$P(s) = \frac{10}{s^2 + 2s + 30}, \qquad T = 0.1\text{seconds}$$

Part (a)

The state-space model in control canonical form for the continuous-time system, $P(s)$, is

$$A = \begin{bmatrix} -2 & -30 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 10 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 \end{bmatrix},$$

Using the ZOH transformation, the A_d and B_d matrices for the discrete-time state-space model are found as follows

$$\left[\begin{array}{c|c} A_d & B_d \\ \hline 0_{(1\times 2)} & 1 \end{array}\right] = e^{\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} T} = \left[\begin{array}{cc|c} 0.6906 & -2.5852 & 0.0862 \\ 0.0862 & 0.8629 & 0.0046 \\ \hline 0 & 0 & 1 \end{array}\right]$$

Thus,

$$A_d = \begin{bmatrix} 0.6906 & -2.5852 \\ 0.0862 & 0.8629 \end{bmatrix}, \qquad B_d = \begin{bmatrix} 0.0862 \\ 0.0046 \end{bmatrix}$$

Using the ZOH transformation, the C_d and D_d matrices are the same as the C and D matrices, respectively, for the continuous-time system.

Part (b)

The N_x and N_u matrices are found as follows

$$\begin{bmatrix} A_d - I & B_d \\ C_d & 0 \end{bmatrix} \begin{bmatrix} N_x \\ N_u \end{bmatrix} = \begin{bmatrix} 0_{(2\times1)} \\ 1 \end{bmatrix}$$

where N_x is (2×2) and N_u is (1×1).

$$\begin{bmatrix} A_d - I & B_d \\ C_d & 0 \end{bmatrix} = \begin{bmatrix} -0.3094 & -2.5852 & 0.0862 \\ 0.0862 & -0.1371 & 0.0046 \\ 0 & 10 & 0 \end{bmatrix}$$

and is invertible, therefore,

$$\begin{bmatrix} N_x \\ N_u \end{bmatrix} = \begin{bmatrix} A_d - I & B_d \\ C_d & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Thus,

$$N_x = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \qquad N_u = 3$$

Part (c)

We would like a critically damped response, thus $zeta = 1$. Therefore, for $\omega_n = 30$ rad/sec, the desired characteristic equation for the continuous-time system is

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 60s + 900 = 0$$

which corresponds to two poles at $s = -30$. Using the matched transformation, the equivalent z-plane poles are

$$z = e^{sT} = e^{-30\cdot0.1} \approx 0.0498$$

Using the command `K = acker(A_d,B_d,p_d)` in MATLAB, where `p_d = [0.0498 0.0498]`, we obtain the following feedback gain matrix, K:

$$K = \begin{bmatrix} 13.0480 & 72.1469 \end{bmatrix}$$

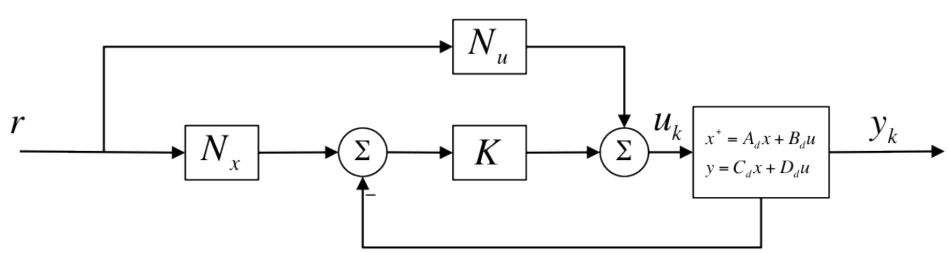


Figure 7: Block diagram for reference trajectory following control.

Part (d)

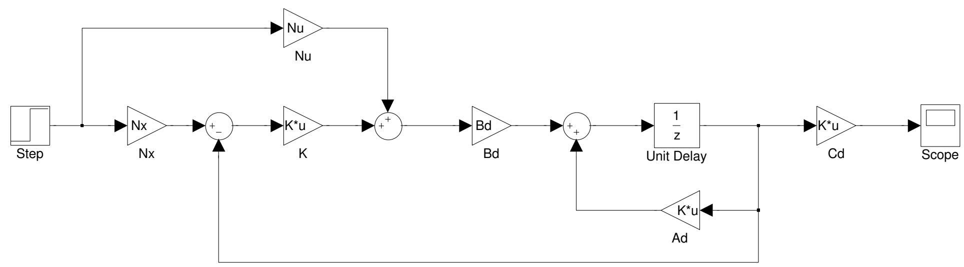


Figure 8: Simulink diagram used to simulate system.

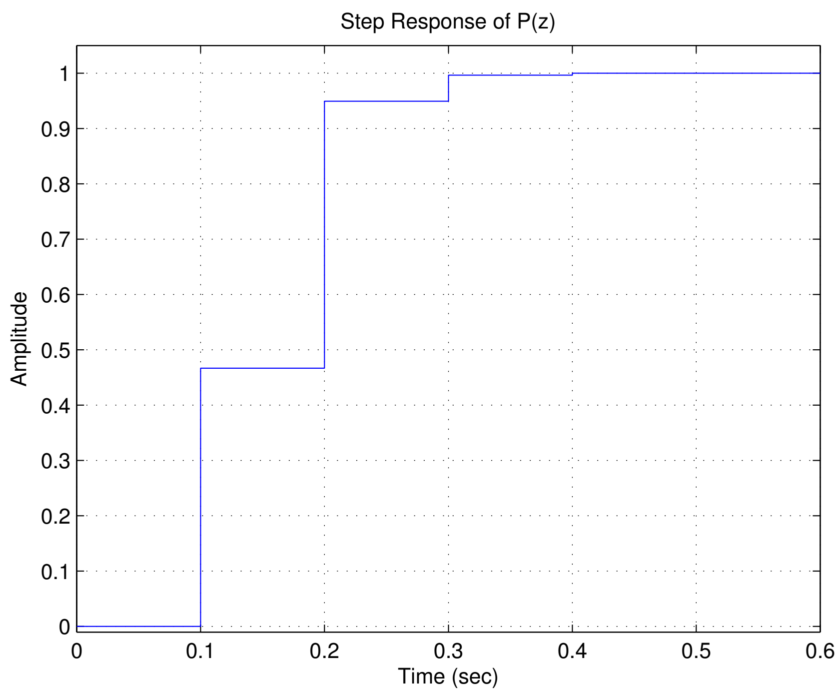


Figure 9: Unit step response of the DT system of Part (a) designed for reference tracking with N_x, N_u, and K from parts (b) and (c). As evident from the plot, there is no steady-state error, despite this being a Type 0 system with no integral control.