Name SOLUTIONS

ECE 147B

Final Exam

Mar. 22, 2011

- Extra blank space provide on page 12.
- You are allowed two, single-side 8 ½ x 11 sheets of self-prepared notes.
- Use **must** submit these "Cheat Sheets" with your exam; they will later be returned, unmarked. **PUT YOUR NAME ON YOUR CHEAT SHEET!**
- **No calculators, laptops, or other computing devices are allowed.**

____ / 20 Problem 1: P(z) time response.

____ / 20 Problem 2: Tustin transformation.

____ / 20 Problem 3: Time response for z-plane poles.

____ / 20 Problem 4: Freq. response (Bode) for DT TF.

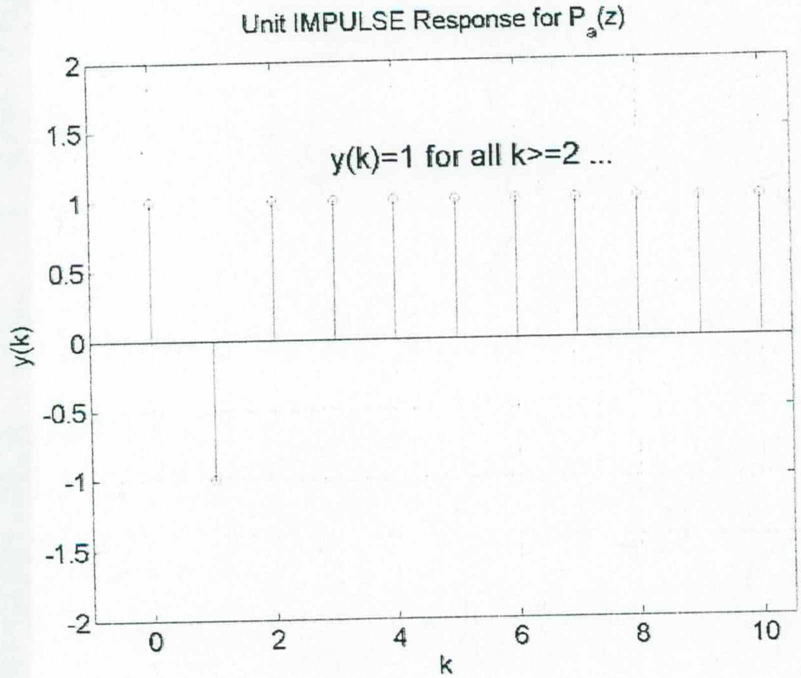
____ / 20 Problem 5: State space: discrete-time conversion / control.

____ / 10 Extra Credit: Optimal Estimation.

100 total points possible on the Final.
(Extra credit can bring your grade to up to *only 100 pts, max.*)

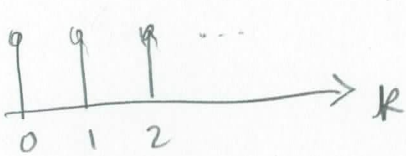
1) Estimating a discrete-time transfer function from a time response.
Please note whether each shows an **IMPULSE** ***(part a)*** *or* **STEP (part b)** *response!!*

a) For the **unit impulse** response shown below, estimate the corresponding plant, $P_a(z)$. ($y(k)=0$ for all $k<0$ here.)



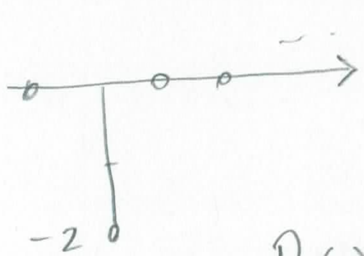
Break into 2 parts:

1)



is $P_1(z) = \frac{z}{z-1}$

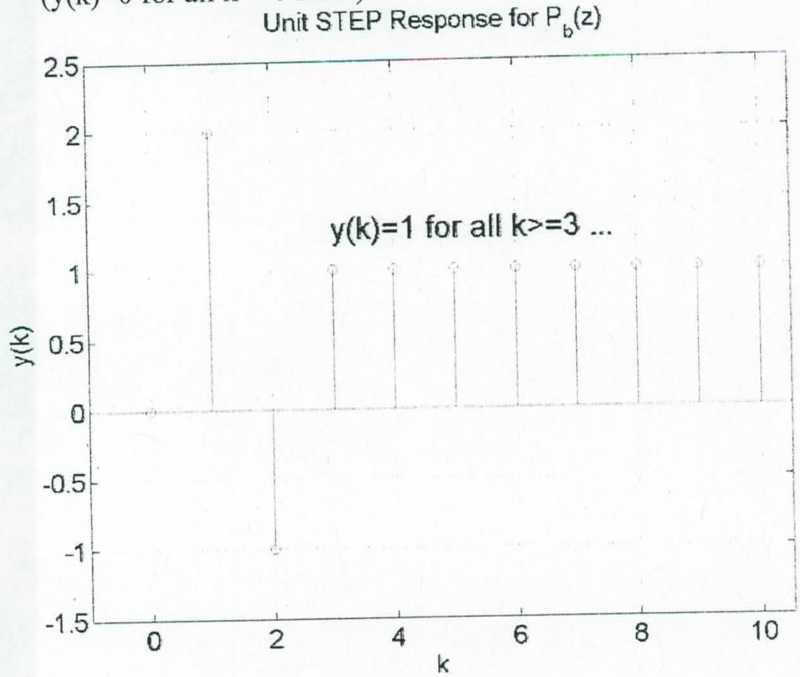
2)



is $P_2(z) = \frac{-2}{z}$

$$P_a(z) = \frac{(z)}{(z-1)} \cdot \frac{(z)}{(z)} + \frac{(-2)(z-1)}{(z)(z-1)} = \boxed{\frac{z^2-2z+2}{z^2-z}}$$

b) For the **unit step** response shown below, estimate the corresponding plant, $P_b(z)$. ($y(k)=0$ for all $k<=0$ here.)



First, calculate impulse response:

	$k=0$	1	2	3	4		
this: $P_b(k)=$	0	2	-1	1	1		...
minus a shifted copy: $-P_b(k-1)=$	0	0	-2	+1	-1		...
	0	2	-3	2	0	0	...

$$\therefore P_b(z) = 0z^0 + 2z^{-1} - 3z^{-2} + 2z^{-3}$$

$$\boxed{P_b(z) = \frac{2z^2 - 3z + 2}{z^3}}$$

2) Below is the transfer function for a continuous-time controller:

$$C(s) = \frac{s+1}{s+4}$$

a) Convert *C(s)* to a discrete-time transfer function, *C(z)*, using the Tustin transformation (no "pre-warp"). Use a sampling time of T=0.01 seconds.

Tustin: $s \leftrightarrow \left(\frac{2}{T}\right)\frac{(z-1)}{(z+1)}$

$$C(z) = \frac{2(z-1) + 1\cdot T(z+1)}{2(z-1) + 4\cdot T(z+1)}$$

$$= \frac{(2+T)z - (2-T)}{(2+4T)z - (2-4T)}$$

$$= \left(\frac{2+T}{2+4T}\right)\cdot \frac{z - \left(\frac{2-T}{2+T}\right)}{z - \left(\frac{2-4T}{2+4T}\right)}$$

$$\boxed{C(z) = \left(\frac{2.01}{2.04}\right)\cdot\left(\frac{z - \left(\frac{1.99}{2.01}\right)}{z - \left(\frac{1.96}{2.04}\right)}\right)}$$

b) *Rapidly* estimate the gain and phase of C(z) at 2 rad/sec. (Hint: evaluate C(s), using appendix to estimate phase of C(s) quickly! C(s) and C(z) should be close here.)

Using Appendix "B", phase for $\alpha=2$ is $\boxed{+37°}$.
(Adds phase.)

DC gain is $\frac{1}{4}$ $(s=0)$, & high freq. gain is 1 $(s=\infty)$.

At $s=2j$ rad/s, gain is the geometric mean of these limits

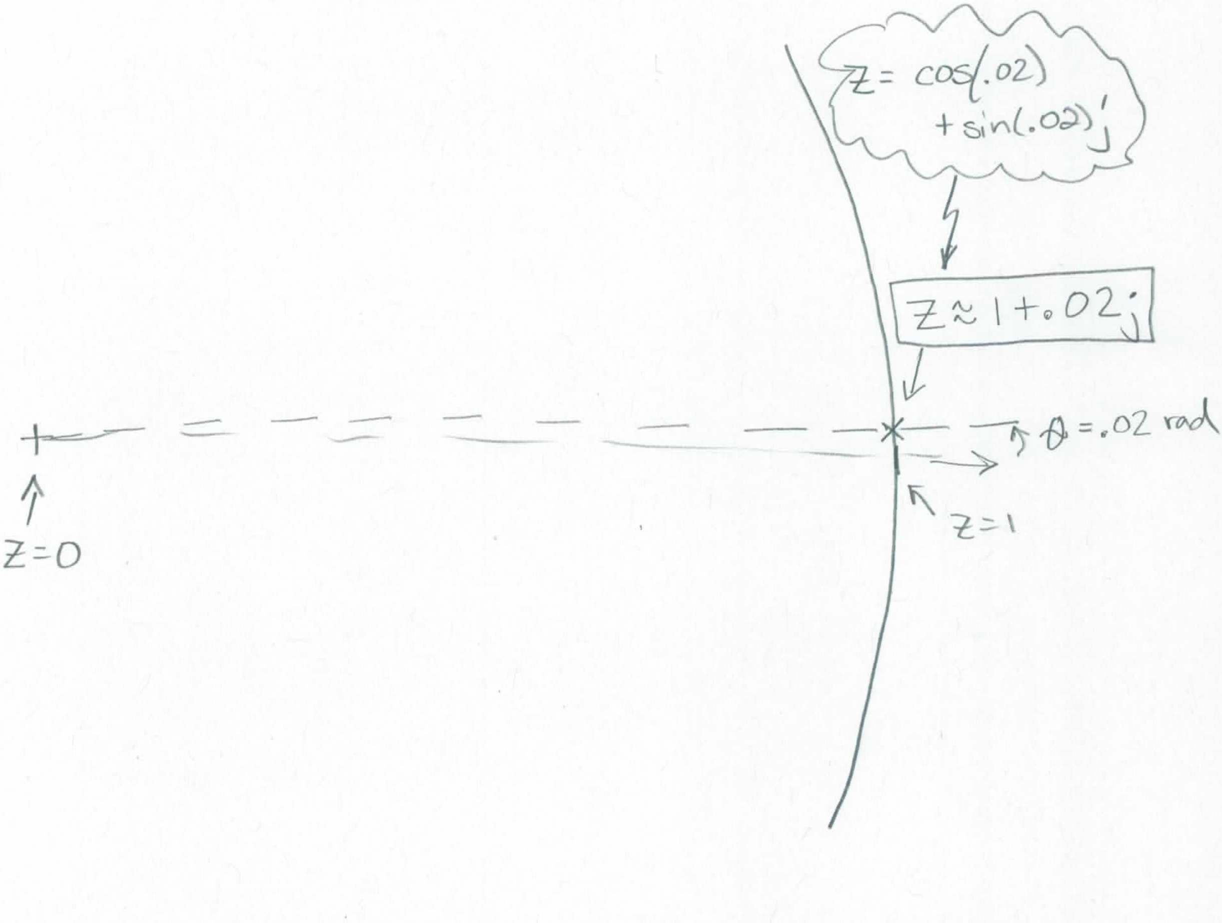
$|C(z)| \approx \sqrt{\frac{1}{4}\cdot 1} = \boxed{\frac{1}{2}}$ ← or about -6 dB

c) To evaluate the *exact* frequency response of C(z) at 2 rad/sec requires plugging in some value for z. Estimate numerically what this value of "z" is, and SKETCH this value of z on the z-plane. (Show all or a blown-up subsection of the unit circle in this plot. Recall, T=0.01 sec.)

$$z = e^{sT}, \quad s = j\omega$$

$$\therefore \quad z = e^{2j * .01}$$

$$= e^{.02j}$$

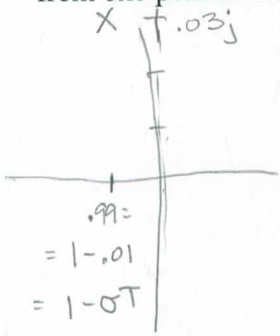


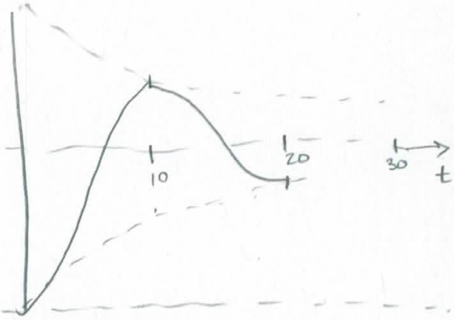
3) Discrete-time system time response analysis. Given a second-order system, P(z), with sampling time T=0.1 seconds and poles on the z-plane at

$$z = 0.99 \pm 0.03j$$

answer the following questions about a ***unit step*** response for the system:

a) Approximately how many samples are there in one oscillation, and what is the time from one peak to the next?





$\omega_d T \cong .03$

$\omega_d \approx 0.3 \ \text{rad/s}$

Here, $\theta \approx .03$ rad.

"$\pi = \theta$" would be Nyquist (2 pts per oscillation),

$\therefore N = 2\left(\frac{\pi}{.03}\right)$

$\cong \left(\frac{628}{3}\right)$

$\boxed{N \approx 209 \text{ samples per oscillation}}$

a) $\boxed{NT \cong 20.9 \text{ sec from one peak to the next.}}$

b) What is the time constant of the decay envelope?

$$\tau = \frac{1}{\sigma} = \frac{1}{\zeta\omega_n}$$

$-\sigma T = -.01$, $T = .1$ sec

$\sigma = 0.1$

b) $\therefore \boxed{\tau \approx 10 \text{ seconds}}$

c) Assume the system has a single zero at $z = -1$. Write an expression for (and/or estimate numerically) the percent overshoot of a unit step response. (i.e., the percentage by which the response exceeds the final value.)

$$M_p = 100\% * \left(e^{-\frac{t_p}{\tau}}\right)$$

$$= 100\% * \left(e^{-\frac{10.45}{10}}\right)$$

$$\approx 100\% * \left(\frac{1}{e}\right)$$

c) $\boxed{M_p \approx 36\% \text{ (roughly)}}$

4) For the discrete-time transfer function with T=0.01 seconds, shown below:

$$G(z) = \frac{4}{(z-0.9)(z-0.96)}$$

a) What are the DC gain and phase?

Plug in $z=1$, $\frac{4}{(1-.9)(1-.96)} = \frac{4}{(.1)(.04)} = \frac{4}{.004}$

$G(z=1) = 1000$

$\therefore \boxed{|G| = 1000, \quad \angle G = 0°}$

60dB

b) What is the Nyquist frequency?

$\omega_{Nyquist} = \frac{\pi}{T} \approx \frac{3.14}{.01} \approx \boxed{314 \text{ rad/s}}$

c) What are the (approximate) gain and (exact) phase at the Nyquist frequency?

Plug in $z=-1$: $\frac{4}{(-1-.9)(-1-.96)} =$

$\frac{4}{(-1.9)(-1.96)} = \frac{4}{(1.9)(1.96)} \approx \frac{4}{3.7}$ or $\frac{4}{4} \approx 1$

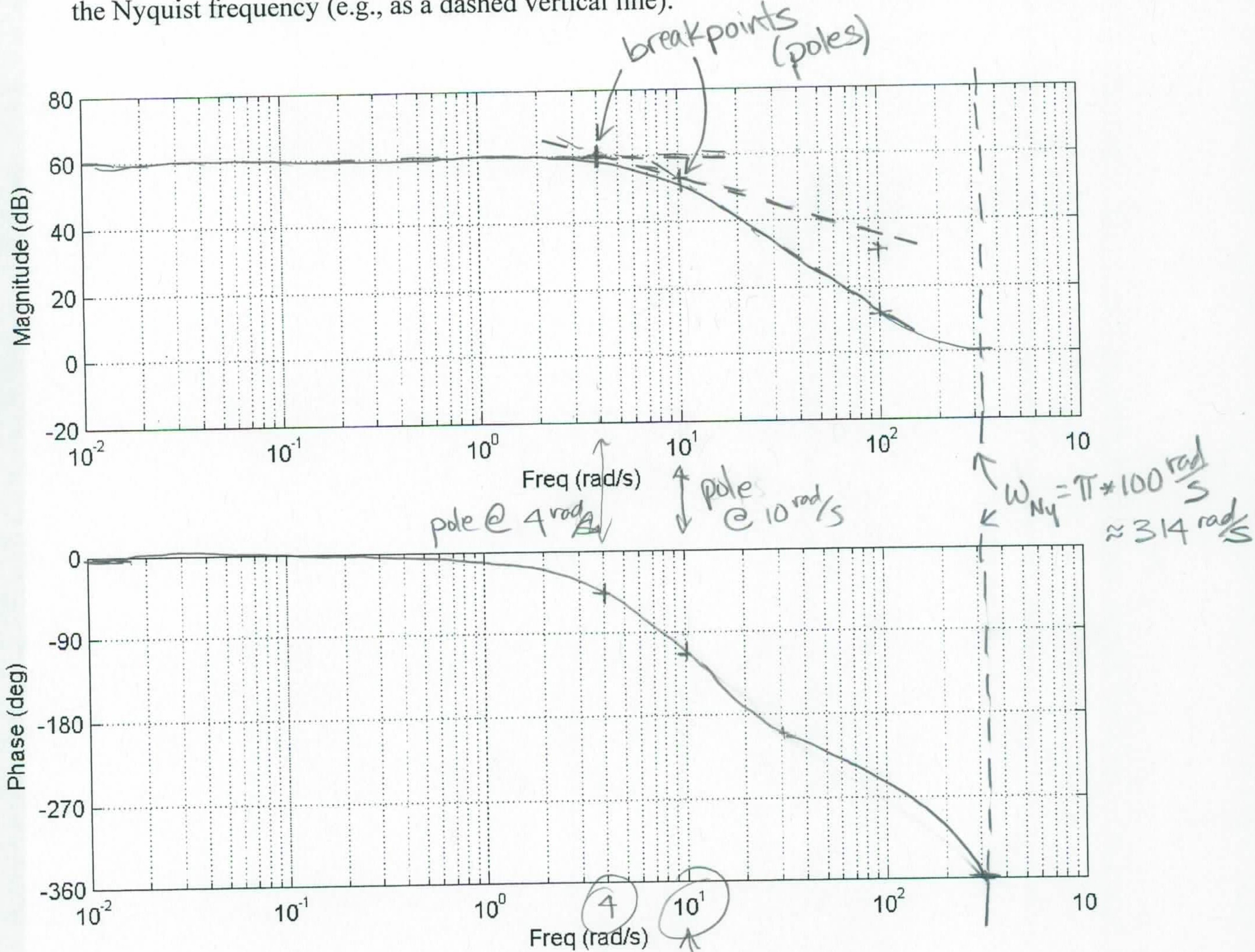
≈ 1.1

$\boxed{|G| \approx 1, \quad \angle G = -360° = 0°}$

0 dB

either answer OK here...

d) Sketch a Bode plot for G(z) on the axes provided. (Hint, you can break the TF into two parts, if helpful. Multiplying two transfer functions corresponds to adding their respective Bode plots.) Carefully label breakpoints for all poles and/or zeros, and show the Nyquist frequency (e.g., as a dashed vertical line).



$Z_1 = .9 \cong 1 - s_1 T$

$s_1 \approx \frac{-.1}{.01} = 10 \frac{rad}{sec}$ (pole here)

$Z_2 = .96 \cong 1 - s_2 T$

$s_2 \approx \frac{-.04}{.01} = 4 \frac{rad}{sec}$ (pole here)

5) Consider the discrete-time state space system,

$$x_{k+1} = A_d x_k + B_d u_k$$
$$y_k = C_d x_k + D_d u_k$$

with the following matrices:

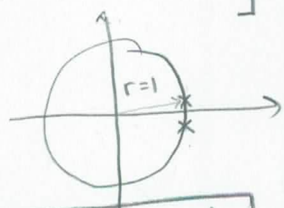
$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad A_d = \begin{bmatrix} 0.99 & -1 \\ 0.01 & 1 \end{bmatrix} \qquad B_d = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C_d = [0 \quad 1] \qquad D_d = [0]$$

a) What is the characteristic equation for the DT SS system shown? Is the system stable, unstable, or marginally stable – and why?

$$|zI - A| = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} .99 & -1 \\ .01 & 1 \end{bmatrix} = \begin{bmatrix} (z-.99) & 1 \\ -.01 & (z-1) \end{bmatrix} = 0$$

$$(z-.99)(z-1) - (-.01)(1) = 0$$



$$z^2 - 1.99z + .99 + .01 = 0$$

$$z^2 - 1.99z + 1 = 0 \leftarrow \boxed{\text{marginally stable!}}$$

$\boxed{\text{Poles on unit circle @ } z = .995 \pm \sqrt{1-(.995)^2}\, j}$

b) Find the four matrices A, B, C, and D for a continuous-time state space representation that maps to this DT SS system via the forward Euler transformation, with T=.01 seconds. (As partial hints, recall for this transformation that: $A_d = I + AT$, $B_d = BT$, ...)

$$\begin{bmatrix} .99 & -1 \\ .01 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + AT, \quad \therefore \quad AT = \begin{bmatrix} -.01 & -1 \\ .01 & 0 \end{bmatrix} = \tilde{A}$$

$$\boxed{B = \frac{1}{T} B_d = \begin{bmatrix} 100 \\ 0 \end{bmatrix}} \qquad \boxed{A = \frac{1}{T}\tilde{A} = \begin{bmatrix} -1 & -100 \\ 1 & 0 \end{bmatrix}}$$

$$\boxed{C = C_d = [0 \quad 1]} \qquad \boxed{D = D_d = [0]}$$

[continues next page…]

c) Solve for the control gains, K, to place all poles at the origin (i.e., deadbeat control) for the discrete-time state space system. (Not the answer to part b!)

$$A_d - B_d K = \begin{bmatrix} (.99-K_1) & (-1-K_2) \\ (.01) & (1) \end{bmatrix}$$

$$(z-(.99-K_1))(z-1)+(.01)(1+K_2)=0$$

$$z^2+\underbrace{(-1-(.99-K_1))}_{=0}z+\underbrace{(.99-K_1)+.01(1+K_2)}_{=0}=0$$

$$1+.99-K_1=0$$

$$\boxed{\therefore K_1=1.99}$$

$$(.99-K_1)+.01(1+K_2)=0$$

$$(.99-1.99)\times 100+.01(1+K_2)\times 100=0$$

$$-100+1+K_2=0$$

$$\boxed{K_2=99}$$

$$\therefore K=[1.99,\ 99]$$

↑ (1x2) matrix of gains...

EXTRA CREDIT: Kalman filtering / optimal state estimate for noisy systems.

a) For a single-input, single-out (SISO) system, we wish to combined a model-update estimate $\bar{x}^-$ which has an uncertainty characterized by variance $P^- = \sigma_a^2$ and a measurement y with variance $R = \sigma_b^2$. The total uncertainty of the new weighted-average estimate, $\bar{x}^+ = Ly + (1-L)\bar{x}^-$, will be:

$$P^+ = \sigma_{tot}^2 = L^2 R + (1-L)^2 P^- = L^2\sigma_b^2 + (1-L)^2\sigma_a^2$$

Solve for the value of L which will minimize the resulting variance, P^+. Specifically, take a partial derivative of the equation above with respect to L, and set this to zero. (Your answer should look familiar, from the lectures on Optimal Estimation. However, you must show work here for full credit.)

$$\frac{\partial}{\partial L}(P^+) = 2LR + (-1)\cdot 2(1-L)P^- = 0 \leftarrow \text{to find min.}$$

$$0 = LR - P^- + LP^-$$

$$= L(R+P^-) - P^-$$

$$\therefore \boxed{L = \frac{P^-}{R+P^-} = \frac{\sigma_a^2}{\sigma_a^2+\sigma_b^2}} \leftarrow \text{can be written in either form.}$$

b) Now, solve for Lss (the steady-state Kalman gain, as k goes to infinity), P_{ss}^+ (the a posteriori covariance of the estimate), and the process noise covariance matrix, Q, when:

$$P_{ss}^- = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \quad R = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}.$$

Assume above that the system is discrete-time, with

$$A = \begin{bmatrix} \sqrt{0.5} & 0 \\ 0 & 0.5 \end{bmatrix}, \quad B = C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \ \text{T=1 sec.}$$

(This sounds complicated, but calculations of L and P_{ss}^+ are actually simple extensions of part a.) To calculate Q, you must recall that to get P_{ss}^-, first the covariance P_{ss}^+ is "reshaped" by the A matrix and then Q is added in.

SS eqns are completely DECOUPLED here!

$$L_{ss} = \begin{bmatrix} \left(\frac{1}{1+3}\right) & 0 \\ 0 & \left(\frac{4}{4+1}\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{4}{5} \end{bmatrix} = \boxed{\begin{bmatrix} .25 & 0 \\ 0 & .8 \end{bmatrix}} \ L_{ss}$$

$$\boxed{P_{ss}^+ = \begin{bmatrix} \frac{1\cdot 3}{1+3} & 0 \\ 0 & \frac{4\cdot 1}{4+1} \end{bmatrix} = \begin{bmatrix} .75 & 0 \\ 0 & .8 \end{bmatrix}}, \quad P_{ss}^- = \begin{bmatrix} (\sqrt{.5})^2(.75), & 0 \\ 0 & , (.5)^2(.8) \end{bmatrix} + Q_{ss} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

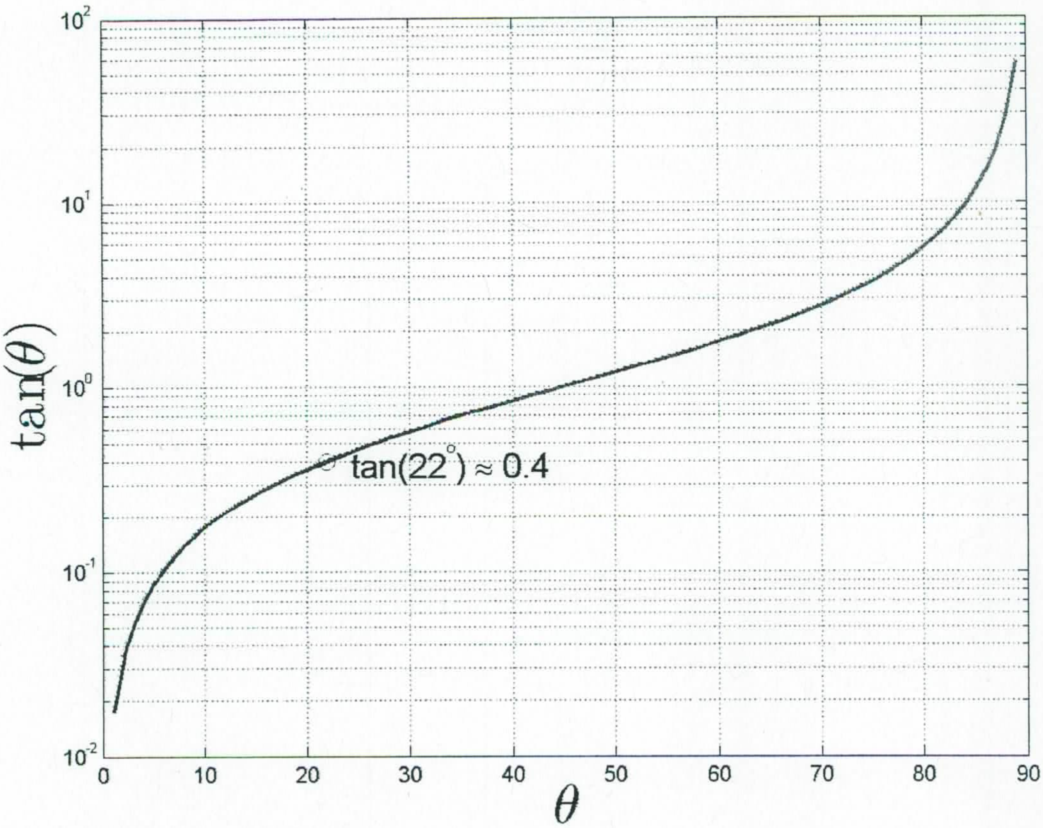
(Extra space below…)

$$P^-_{ss} = \begin{bmatrix} (.5)(.75), & 0 \\ 0 & , (.25)(.8) \end{bmatrix} + Q_{ss} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\boxed{Q_{ss} = \begin{bmatrix} (1-.375) & 0 \\ 0 & (4-.2) \end{bmatrix} = \begin{bmatrix} .625 & 0 \\ 0 & 3.8 \end{bmatrix}}$$

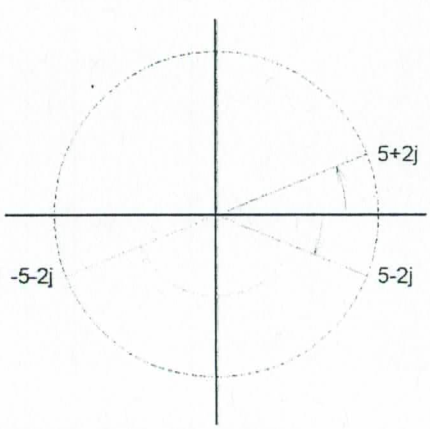
End of exam. Have a great Spring Break!

Appendix A: The tangent function.



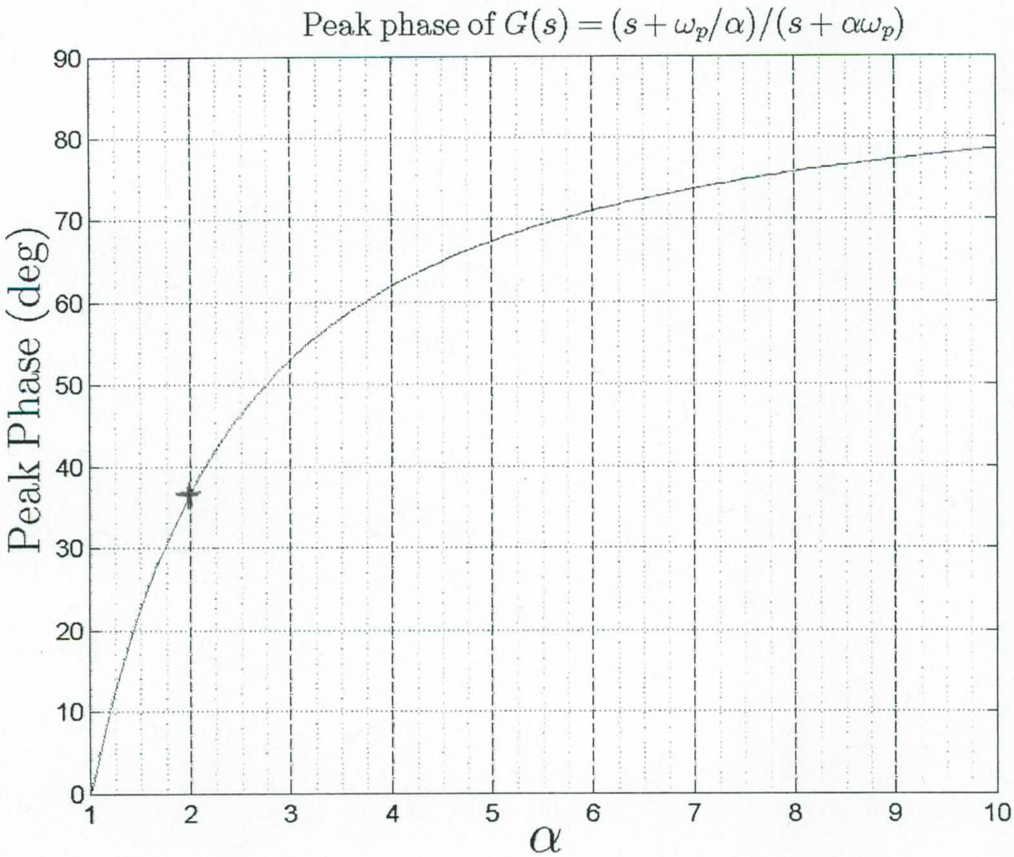
The examples and figure below illustrate how to use semilog plot above:

- The complex number 5+2j has phase: $\text{atan}\left(\frac{2}{5}\right) \approx 22^\circ$.
- The complex number 5-2j has phase: $\text{atan}\left(\frac{-2}{5}\right) \approx -22^\circ$.
- The complex number -5-2j has phase: $\text{atan}\left(\frac{-2}{-5}\right) \approx -180^\circ + 22^\circ = -158^\circ$.



Appendix B: Lead and Lag phase shift prediction.

The figure below is similar to Figure 2.17 in FPW (page 41), except that the definition of "alpha" below is different. (Below, it is the square root of the value in Fig. 2.17 – which makes design a bit easier, since you can avoid taking a square root by using the figure below…)



Example usage. Say we desire a continuous-time compensator to provide 30 degrees of phase. Find 30 degrees on the y axis, and this corresponds to alpha=1.75 (approximately). Say we wish this peak phase to occur at 500 rad/s, then:

$$G(s) = \frac{(s + \frac{500}{1.75})}{(s + 500 \cdot 1.75)} = \frac{(s + 500\frac{4}{7})}{(s + 500\frac{7}{4})} = \frac{(s + \frac{2000}{7})}{(s + \frac{3500}{4})} \approx \frac{(s + 300)}{(s + 900)}$$

For discrete-time controller design, this can then be converted using Tustin (or Tustin with pre-warp, as appropriate/required).