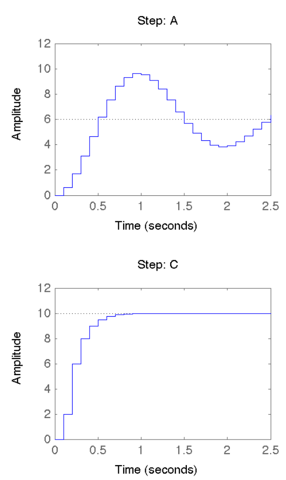
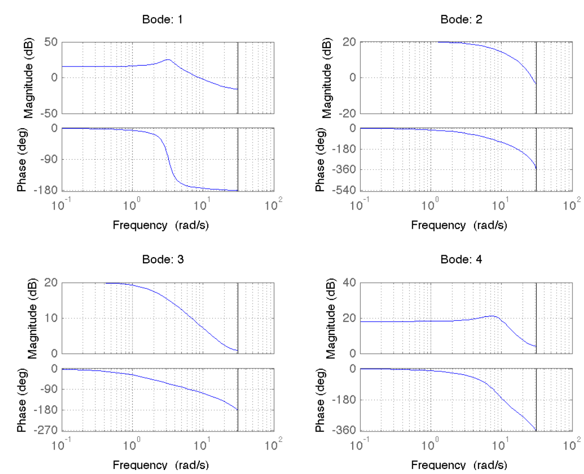
Homework 2 - due **MONDAY**, January 28, 2013, **at 5 pm**, in the homework box (outside 3120A)

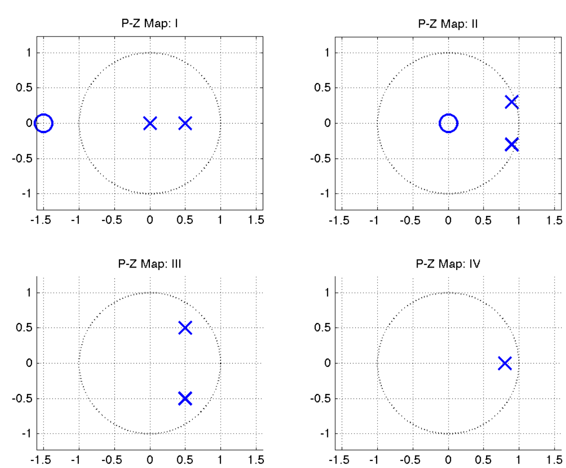
Problem 1. Discrete-time dynamics. (Hint: You might wish to do Problems 2 and 3 fir, for intuition…) Below are 4 step responses (a,b,c,d), 4 Bode plots (1,2,3,4), and 4 pole-zero plots (I,II,III,IV).

a) Match these up. (i.e., list the 4 correctly-matched triplets. e.g., if A, 1, and I match, write "A-1-I".)

b) Estimate the discrete-time transfer function for each system. If all cases, sampling time is T=0.1 sec.







Problem 2. Final Value Theorem (FVT).

This is really a "tutorial" problem. i.e., the problem itself should be relatively easy, and the preamble is given to provide a derivation of the FVT, only for your own understanding. (In fact, you could probably just skip to the top of page 3, if you wish…)

[Tutorial] Imagine you are given a discrete-time transfer function, F(z). You wish to calculate what the final value will be, given a unit pulse input, *for a STABLE (all poles in the unit circle) system*.

Intuitive approach.
We anticipate the method will be analogous to what holds for continuous-time (CT) TF's.
CT FVT: $\lim_{t\to\infty} x(t) = \lim_{s\to 0}(s\cdot X(s))$
A CT zero at the origin of the s-plane corresponds to a DT zero at z=1, so instead of (s-0)*X(s), we might expect to evaluate (z-1)*X(z). Similarly, instead of evaluating the limit as s goes to 0 on the s-plane, we might expect to evaluate the DT limit as z goes to +1 on the z-plane. Intuitively, this gives:

DT FVT: $\boxed{\lim_{k\to\infty} x(k) = \lim_{z\to 1}((z-1)\cdot X(z))}$

Mathematical approach.
Let us be more formal. First, note that:

1. $Z[x(k)] = X(z) = \sum_{k=0}^{\infty} x_k z^{-k} = x_0 z^0 + x_1 z^{-1} + x_2 z^{-2} + \ldots x_{ss} z^{-\infty}$
2. $Z[x(k+1)] = \sum_{k=0}^{\infty} x_{k+1} z^{-k} = x_1 z^0 + x_2 z^{-1} + x_3 z^{-2} + \ldots x_{ss} z^{-\infty}$
   $= z\cdot X(z) - x_0 z^{+1}$

Here, x_{ss} is the final value of x(k), i.e., as k goes to infinity.

Now, take equation 2. minus 1. to get:

3. $(z\cdot X(z) - x_0 z^{+1}) - X(z) = \sum_{k=0}^{\infty}\left[x_{k+1} z^{-k} - x_k z^{-k}\right]$

Now, imagine that we reach a steady state, where you can write the difference equation such that all x_k equal x_{ss}. If so, there is no point distinguishing among powers of z. (For a CT differential equation, we set derivatives to zero at steady state; here we x_k converge to a single value, at steady states…) This corresponds to the limit as all z^k are simply equal to z^0=1. So, letting z=1, we first take equation 3,

$$(z\cdot X(z) - x_0 z^{+1}) - X(z) = \sum_{k=0}^{\infty}\left[x_{k+1} z^{-k} - x_k z^{-k}\right]$$

Rearranging, and taking the limit as ,

$$\lim_{z\to 1}\left((z-1)\cdot X(z) - x_0 z\right) = \lim_{z\to 1}\sum_{k=0}^{\infty}\left[x_{k+1} z^{-k} - x_k z^{-k}\right]$$

In the equation above, we will need to check if "(z-1)" can be factored out of the denominator of X(z) before evaluating the limit. (This is analogous to leaving "s*X(s)" in the CT FVT…) Simplifying:

$$\lim_{z\to 1}\left((z-1)\cdot X(z)\right) - x_0 = \left[x_1 - x_0\right] + \left[x_2 - x_1\right] + \ldots + \left[x_{\infty-1} - x_\infty\right]$$

$$\lim_{z\to 1}\left((z-1)\cdot X(z)\right) - x_0 = \left[-x_0\right] + \left[x_\infty\right]$$

$$\lim_{z\to 1}\left((z-1)\cdot X(z)\right) = \left[x_\infty\right] = x_{ss} = \lim_{k\to\infty} x(k)$$ **DT FINAL VALUE THEOREM**

For a UNIT STEP RESPONSE, recall that the transfer function for a unit step is z/(z-1). For a continuous-time system, the final value to a unit step input is:

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0}\left(s\cdot \frac{1}{s} X(s)\right)$$

Similarly, for a discrete-time system, **the final value to a unit step input** is:

4. $$\lim_{k \to \infty} x(k) = \lim_{z \to 1}\left((z-1)\cdot \frac{z}{(z-1)} X(z)\right)$$

So, the final value to a unit step (above) just amounts to plugging in "z=1" in the transfer function F(z).

Using equation 4, for each system (a-c) below, assume sample time T=1 second and:
i) (Try to) evaluate the final value for a unit step input for the DT TF.
ii) Solve for the pole locations on the z-plane.
iii) Sketch the unit step response by hand for time steps 0<=k<=20. (I suggest you use MATLAB to check the step response, but then sketch a plot by hand and label axes clearly.)

a) $\dfrac{z}{(z-.6)}$ b) $\dfrac{z}{(z-.6)(z-1)}$ c) $\dfrac{z}{(z^2-2\cos(\pi/5)+1)}$ d) $\dfrac{z}{(z^2-2.2\cdot\cos(\pi/5)+1.1)}$

e) What happens when you try to use the DT FVT for a system with a pole at z=1? (Why?) Mathematically, how can you calculate the steady-state *slope* of the unit step response in this case? Hint: what can you say about the final value of an IMPULSE response? And why?

f) What happens for systems with poles elsewhere on the unit circle? (i.e., not at z=1, but at a distance exactly 1 from the origin on the z-plane.) What info does the FVT give in this (marginally stable) case?

g) What happens if a pole is outside the unit circle? Is the Final Value Theorem still valid in this case?

Problem 3. Unit step response characteristics for DT TF.

We have mentioned in class that there is a correspondence between poles on the s-plane at a given "s" (where s is a complex number) and poles on the z-plane at "e^{sT}". Below, use T=0.1 seconds in all cases.

Show that this is the case for each of the three systems (a-d) below. Specifically,

a) $K_a \dfrac{z}{z-0.9}$ b) $K_b \dfrac{z}{z+0.9}$ c) $K_c \dfrac{z}{z^2-0.81}$ d) $K_c \dfrac{z}{z^2+0.81}$

i) Solve for the poles on the z plane (roots of the denominator) for each DT TF.

ii) Find values of K_a, K_b, K_c, and K_d such that each response will have a DC gain of 10. (That is, a unit step response should go to a final value of 10.) [Show work!!]

iii) For each system (a-d), write the DIFFERENCE EQUATION represented by the DT transfer function, and calculate the first 6 values of a unit step response BY HAND, in a table. Use MATLAB to plot the DT unit step response for each system, using your values for K.

iv) For each system, find corresponding poles on the s-plane, such that z=e^{sT}, and derive a CT TF with these poles and with the same DC gain of 10. Use MATLAB to plot the DT and analogous CT unit step response for each system (a-d), using 4 separate plots (one for each pair of DT and CT signals).
[*Note: Do* *not* *worry about including any zeros in the CT TFs in this problem, just the poles! Recall that z-plane zeros essentially map to time delays in the step response.*]

Problem 4. Tustin with pre-warp.

Assume we wish to match both the magnitude and phase of the continuous-time controller below, C(s), when evaluated at the frequency $\omega_o = 100$ rad/sec. Use a sampling time of T = 0.01 seconds.

$$C(s) = \frac{(s+20)}{(s+100)}$$

a) By hand, plug in $s = j\omega_o = 100j$ rad/sec, and calculate gain and phase at this frequency.

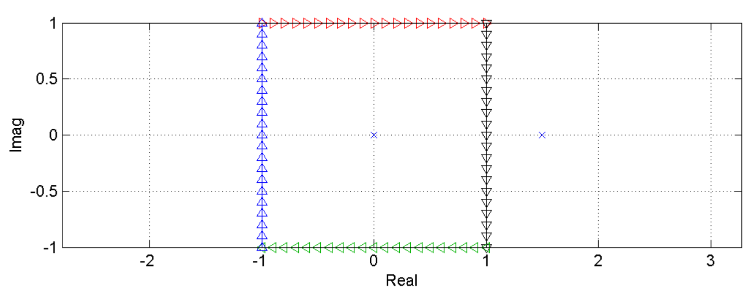
b) By hand, solve for $C_b(z)$ by replacing s with $\frac{2(z-1)}{T(z+1)}$, which is the standard Tustin transformation. Evaluate magnitude and phase for $z = e^{sT} = e^{j\omega_o T}$.

c) Now, solve for $C_c(z)$ by replacing s with $\alpha\frac{(z-1)}{(z+1)}$, where alpha is selected via the Tustin with pre-warp transformation, $\alpha = \omega_o / \tan(\omega_o T/2)$, for $\omega_o = 100$ rad/sec. By hand, show magnitude and phase do in fact match the result in part a), as desired.

d) At what frequency, if any, do both the magnitude and phase of $C_b(z)$ and $C(s)$ match? (i.e., using standard Tustin, without pre-warp. Hint, think about what α=2/T would correspond to…) If they never match, explain why.

Problem 5. Nyquist intuition tutorial.
By now, you should basically know how Nyquist works. This problem is designed in hope of giving a slightly more intuitive about "why" it works.

Using MATLAB, write a script to do the following, toward creating two labeled output plots (a and b):
a-i) Create a set of roughly 100 points on the complex plane, evenly spaced and <u>going clockwise</u> around the box shown below on the s-plane. (Instead of enclosing the righthand plane, our contour encloses some other, arbitrary area… Each point on the contour is a complex number.)



a-ii) Using MATLAB, evaluate the function $G_a(s)$=1/s for each point from part i above, and plot the resulting closed curve. Clearly label where each of the 4 "sides" (blue/left; red/top; black/right; green/bottom) has been mapped to on your new plot. (Use a different color or line type or set of symbols for each edge.) Explain the direction of encirclement (CW vs CCW) of the origin.

a-iii) For what values of K will all pole(s) of $KG/(1+KG)$ all be inside the contour above? (Why?) For what values of K will all pole(s) of $KG/(1+KG)$ all be outside the contour above? (Why?)

b) Repeat steps ii and iii for $G_b = 1/(s(s-1.5))$ (instead of for $G_a(s)$)