Homework 3

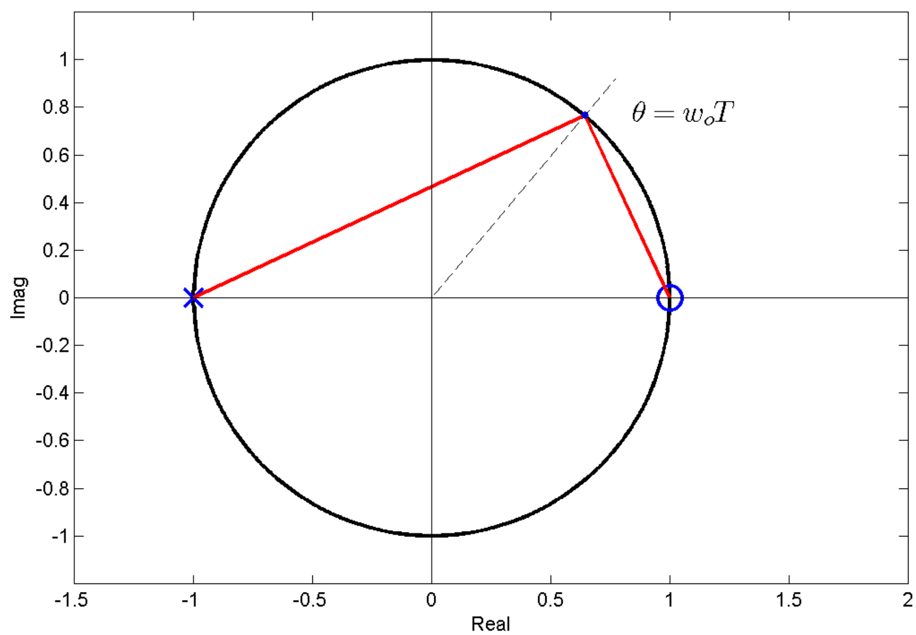
Due: **MONDAY**, February 4, 2012, **at 5 pm**, in the homework boxes outside Harold Frank Hall 3120A.

Problem 1. Tustin with pre-warp.

In this problem, we consider a geometric interpretation of the Tustin transformation with pre-warp. Recall that the goal of Tustin with pre-warp is to match gain and phase of the frequency response of the resulting DT TF to that of the CT TF at one, particular frequency. Here, the Laplace operator s is approximated via a scaled version of the trapezoid rule as $s \to K\frac{(z-1)}{(z+1)}$. In a CT frequency response, $s \to j\omega_o$. This expression has phase +90 (deg), and magnitude ω_o. In a DT frequency response, we plug in $z \to e^{j\omega_o T}$.

a) Show that for any complex number $e^{j\omega_o T}$ on the unit circle, the phase of $K(z-1)/(z+1)$ will in fact be exactly +90 degrees, matching that of $s \to j\omega_o$, as desired.

b) Use the geometry below to derive the value of K that will yield a magnitude of ω_o when $z = e^{j\omega_o T}$, and show this result matches the standard Tustin with pre-warp formula.



Problem 2. Step response for a DT TF.

For a CT linear system responding to a constant (step) input, the steady state response will be (as you may recall from 147A) take the form: $y(t) = K + c_1 e^{s_1 t} + c_2 e^{s_2 t} + \ldots + c_n e^{s_n t}$, where K is the final value (for a stable system) and the constants give the unique solution matching the (n) initial conditions at t=0 for an nth-order system.

For a DT linear system, the unit step response takes the form: $y(k) = K + c_1 z_1^k + c_2 z_2^k + \ldots + c_n z_n^k$.

a) For $\frac{Y(z)}{U(z)} = \frac{0.83z}{z^2 - 1.6z + 0.81}$, find the poles of the system and write the difference equation relating u and k.

b) For a unit step input, calculate by hand i) the final value, ii) y(n_1) (i.e., at k=n_1=1), and iii) y(n_2) (for k=n_2=2).

c) Solve for values of *K*, c_1, and c_2 of the unit step response and write the complete expression defining *y(k)* for $k \geq 0$. (Hint, *K* should be trivial, and you can set up a matrix equation of the form $Ax = b$, where A is a 2x2 matrix with $A(1,1) = z_1^{n_1}$, etc., and b is a 2x1 vector with $b(1) = y(n_1) - K$, etc. The MATLAB solution `x=A\b` will then give coefficients c_1 and c_2 in vector *x*.)

d) Repeat a) through c) for $\frac{Y(z)}{U(z)} = \frac{2z - z - 0.17}{z^2 - 1.6z + 0.81}$. Note the numerator affects the solutions for *K*, c_1, and c_2, but that the long-term response still has the same general form, which is determined solely by the poles.

Problem 3. Pole-zero matching approximation.

In MATLAB, one method for converting between DT and CT transfer functions is the 'matched' option. This is a heuristic approach. Since we know the poles are related as, $z \rightarrow e^{sT}$ this approach attempts to match zeros via the same transformation. There is one caveat: when converting from CT to DT, additional zeros are added at *z*=-1 (if necessary) to ensure the order of the denominator does not exceed the order of the numerator by more than one. (Intuitively, this ensures the system responds at *k*=1.) The steady state gain is (typically) selected so that the DC gains of each TF are equal.

a) For $\frac{Y(z)}{U(z)} = \frac{0.83(z+1)}{z^2 - 1.6z + 0.81}$, calculate the corresponding "matched" CT transfer function, assuming T=0.1 seconds. (Perform the calculation by hand, but check your answer using MATLAB's c2d function with the 'matched' option…)

b) To compare the step responses for the "matched" DT and CT systems, calculate the analytic expression for $y_{DT}(k)$ for the DT system and for $y_{CT}(t)$ for the CT one. Provide a MATLAB plot showing the *difference* between the two ($y_{CT}(t)$-$y_{DT}(k)$) for t=0:.01:10.

c) Calculate a new DT transfer function (with same T=0.1s) for which the output to a unit step will *exactly* match the output of the CT TF from part "a)". (Hint: Use intuition from Problem 2, and just focus on matching the two responses at k=1, when t=0.1, and at steady state, i.e., at the final value.)

Problem 4. Root locus and z-plane pole characteristics.

The rules for drawing a root locus for a DT system are the same as for a CT one. The particular "meaning" of pole locations will of course correspond to the z plane rather than the s plane, however.

a) For the system, $G(z) = \frac{(z - 0.625)}{(z - 1.25)(z - 1.25)}$ with T=0.05s, sketch the root locus by hand on the z plane, clearly illustrating the unit circle on your plot. Calculate *by hand* the precise value for and clearly label on your plot the values of the z-plane poles where the root locus crosses the unit circle. For what values of *K* is the closed-loop system stable?

b) For the system $H(z) = \dfrac{1}{(z-0.6)(z-0.6)(z-0.6)}$, with T=0.02s, sketch the root locus by hand on the z-plane, clearly illustrating the unit circle on your plot. For what value of *K* will one of the poles be exactly at z=0? Where will the other two closed-loop poles be in this case? For the corresponding, underdamped s-plane poles, what is the damped natural frequency, ω_d, and what is $\sigma = \zeta\omega_n$?

c) For the system in part "b)", now plot the root locus using MATLAB. Label where, for some positive value of *K*, the complex conjugate pair of poles will correspond to $\sigma = 5$rad/s. Label where the complex pole pair would correspond to s-plane poles with $\zeta = 0.7$. Finally, label where the root locus for this pole pair would have a damped natural frequency of $\omega_d = 25$ rad/s. (Hint: Use MATLAB to help draw construction lines on the root locus, as appropriate…)

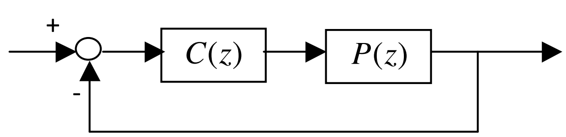
Problem 5. Frequency response for a DT TF.

a) For the transfer function $G(z) = \dfrac{1}{(z-0.8)}$, with T=0.1s, sketch the pole-zero plot by hand on the z plane, and show *graphically* at what location on the unit circle the resulting value $z = e^{j\omega_a T}$ would result in -90 degrees of phase when evaluated in *G(z)*. At what corresponding frequency, ω_a, is the phase -90 deg?

b) For the transfer function, $G(z) = \dfrac{1}{(z-0.8)(z-0.8)}$ assume we wish to design a controller to provide an additional 45 degrees of phase at the same frequency, ω_a, you solved for in part "a)". Show graphically at what location one must put a single zero to achieve this. For a controller of the form $C(z) = K(z+a)$, solve by hand for values of *K* and *a* that make ω_a the crossover frequency and that achieve a phase margin of exactly 45 degrees.

Problem 6. Direct Design.

a) Given the plant $P() = \dfrac{z}{z^2 - 1.2z + 1.9}$, design a controller of the form $C(z) = K(z+a)$ such that the closed-loop poles will be $z_1 = 0.95$, $z_2 = 0.4$.



b) Repeat this for desired poles $z_1 = 0.95$, $z_2 = 0.01$.

b) Explain why one cannot put either of the desired poles at the origin using this controller form.

c) Design a new controller that will put all closed-loop poles at z=0. Recall from lab, this is known as a deadbeat controller. (Hint: Try to determine the minimum number of poles and zeros required for C(z).)

Problem 7. Ripple.

a) Calculate an analytic expression for the response (*y(k)*) for a unit step input to the transfer function $P(z) = \dfrac{1}{z^3 + 0.8}$. Your methods should follow those used in Problem 2.

b) Use MATLAB to create a plot of your results at discrete values (k=0, 1, 2, …, 20) of k. Assume T=0.1 seconds for the DT system.

c) Now, overlay a plot of your analytic result for *y(t)* "in between" samples. Specifically, use the time vector `t=[0:.001:2]` in MATLAB (i.e., plug in the appropriate non-integer values of k for the expression in part "a").