Midterm Review

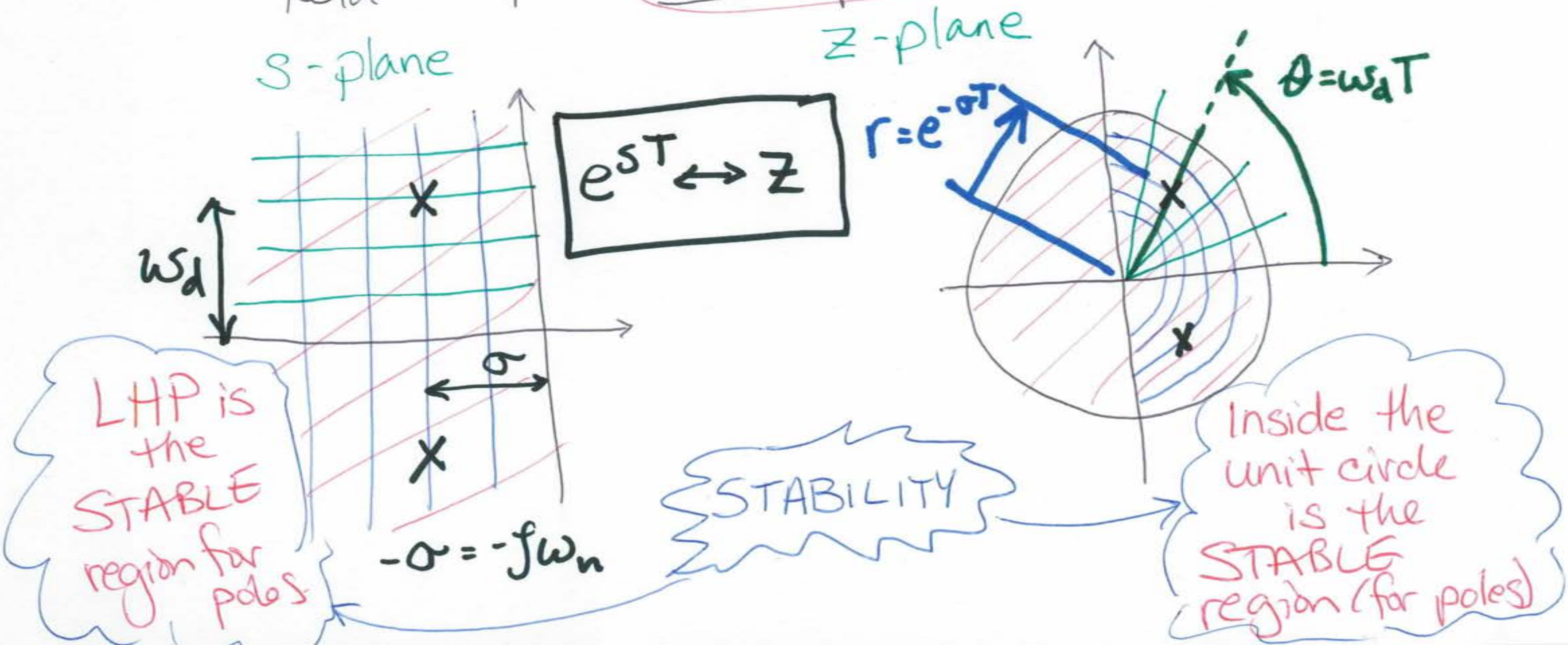
Lecture 10

ECE 147B

Topics to review:

1. Understand POLES!
   - s-plane
   - z-plane
   - Relationship to time response.

See also LAST PAGE!



2. When $|sT| \ll 1$ (small)

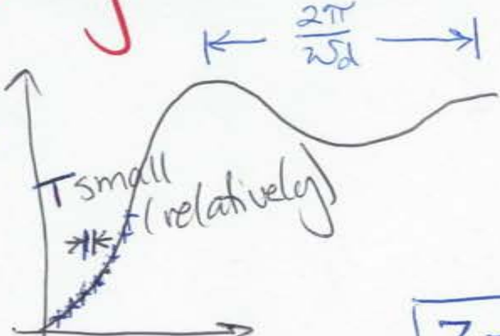
$z = \boxed{e^{sT} \approx \underline{1 + sT}}$ ← i.e., if we have many samples "per time constant" &/or "per cycle".

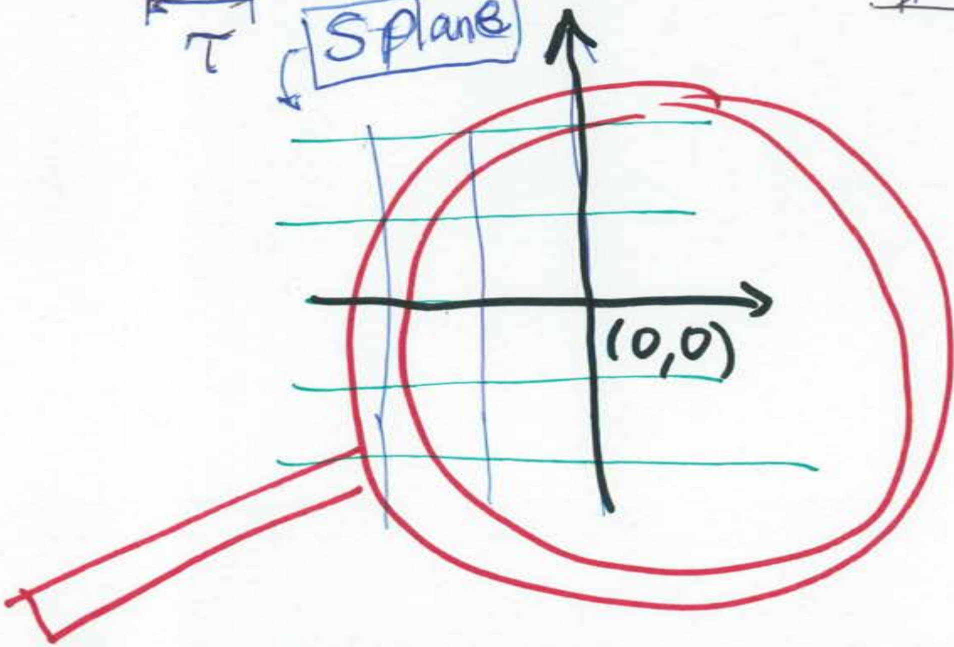
Note, this is actually the FORWARD EULER.

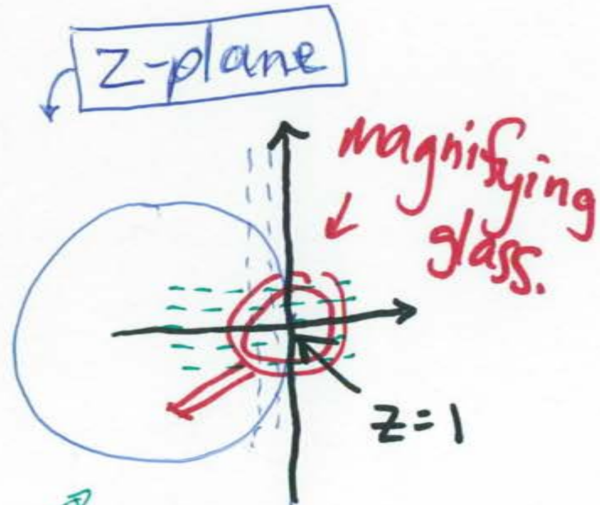
e.g.

Approx is good IF: Sample time "T" is small compared with τ &/or ω_d^{-1}

or







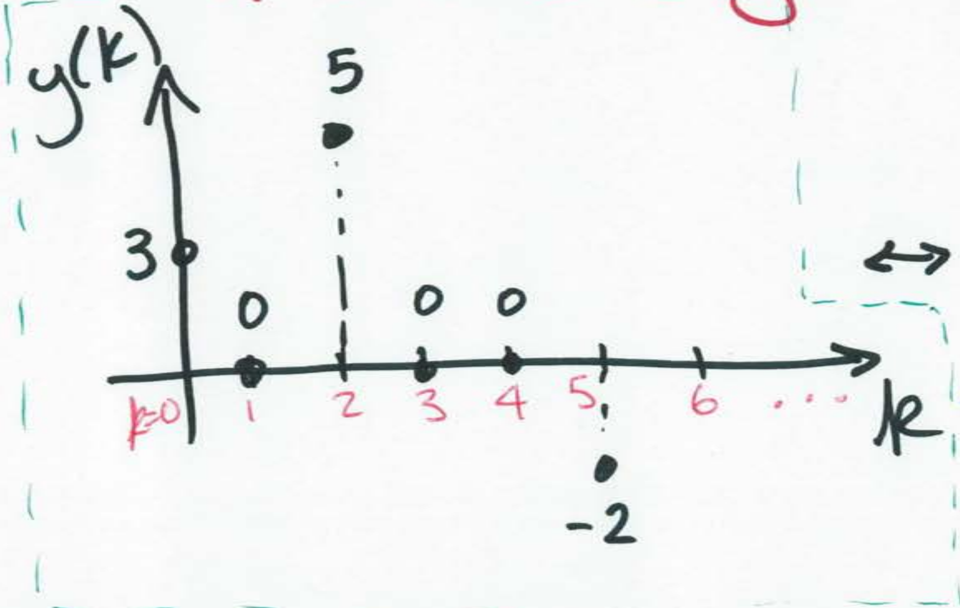
Looks close to a linear grid, when you "zoom in" near z=1.

3. Definition of Z transform...

$$Z(y(k)) = \sum_{k=-\infty}^{\infty} y(k) z^{-k}$$

... can be used directly in the case of a "finite impulse response":

Important! You should be able to map either way: Z xform $\longleftrightarrow$ a sketch: { • impulse resp. or • step resp }



$$\leftrightarrow Y(z) = 3z^0 + 5z^{-2} + -2z^{-5}$$

$$\leftrightarrow Y(z) = \frac{3z^5 + 5z^3 - 2}{z^5}$$

← z=0 for all 5 poles ("deadbeat")

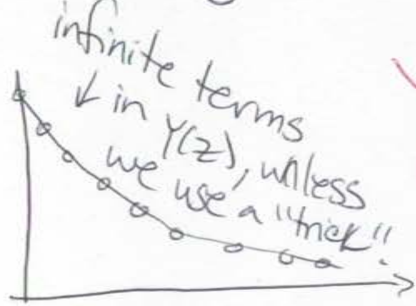
4. Z transform for first-order system.

(One pole & the pole is not at $z=0$.)

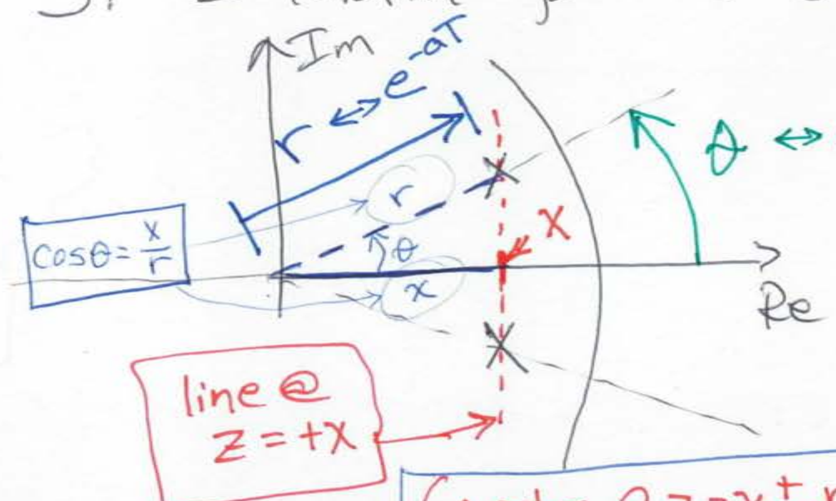
4. & 5. are the two classic "infinite impulse responses" you should know!

$$\boxed{\text{pole @: } z = r \quad (\text{or } z = -r)}$$

e.g. $$\text{“}\underbrace{\frac{Kz}{z-e^{-aT}}}_{\text{(as in example on last page of inclass H.O.)}}\text{”} \longleftrightarrow \left(\frac{Kz}{z-r}\right)$$



5. Z xform for a second-order system



$$\left(\therefore \text{poles @ } z = x \pm r\sin\theta\, j\right)$$

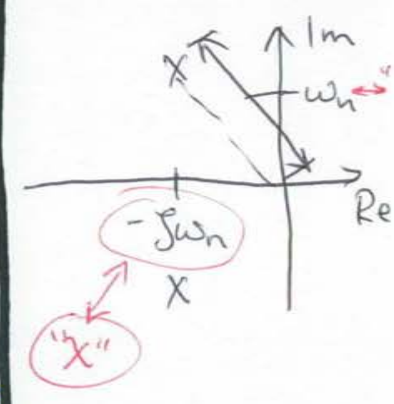
We will have something of the form:

$$\frac{K_1 z^2 + K_2 z + K_3}{z^2 - 2xz + r^2}$$

where $\theta = \operatorname{acos}\left(\frac{x}{r}\right)$

Analogy on s-plane:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$



Remember...

6. You can (sometimes) write **n** eqns for **n** unknowns (based on a time response, e.g., impulse or step response).

e.g. $G(z) = \dfrac{az + b}{z + c} = \dfrac{Y(z)}{X(z)}$ } ← i.e., assume either 1st-order or 2nd-order. Here, we say "1st-order", by intuition. Numerator is of order ≤ that of denom.

1st-order → $z + c$

$\therefore \boxed{a x_k + b x_{k-1} = y_k + c y_{k-1}}$

↑ write out for particular k values.

$a x_0 + b x_{-1} = y_0 + c y_{-1}$

$\therefore \boxed{a = y_0 = 8}$

$a x_1 + b x_0 = y_1 + c y_0$

$b = 4 + 8c$

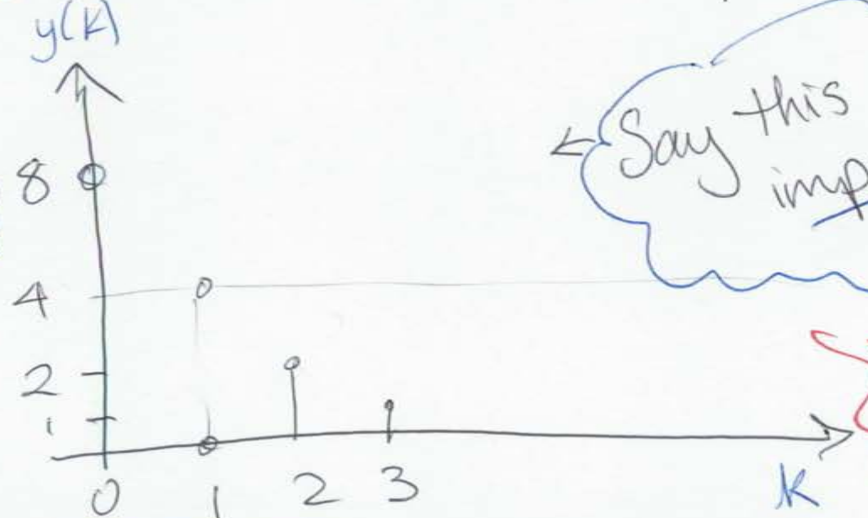
$a x_2 + b x_1 = y_2 + c y_1$

$0 = 2 + 4c$

$\therefore \boxed{c = \frac{-1}{2} = -r}$ $\boxed{b = 0}$

k	x	y
-1	0	0
0	1	8
1	0	4
2	0	2
3	0	1
⋮	⋮	⋮

$r = \frac{1}{2}$



← Say this is an impulse response... i.e., $x(k) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$

To do @ home...

Combining: $\boxed{G(z) = \dfrac{8z}{z - \frac{1}{2}}}$ – Given the impulse response above.

7. FVT & IVT (as $z \to \infty$). ← for Time Responses.
(as $z \to 1$) $\lim_{z \to 1}$

8. & D.C. gain involves using "$z = 1$". (plug into some $G(z)$).

Freq. Resp.

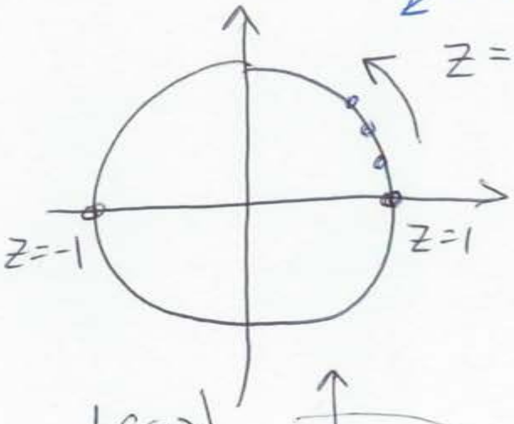
"High freq." makes no sense beyond

9. !!! → $\boxed{\omega_{Nyquist} = \frac{\pi}{T}}$ ↔ "$z = -1$"

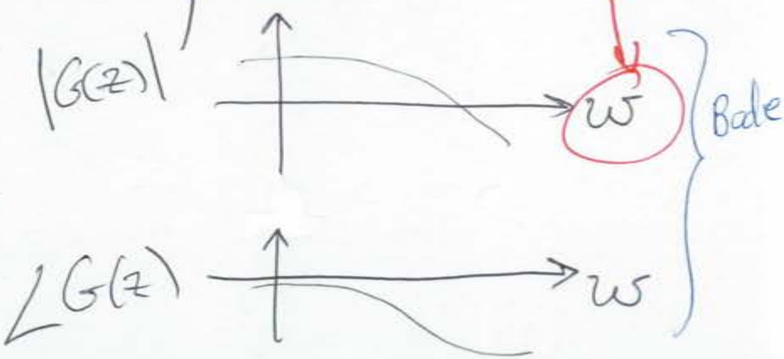
Only depends on sample time, T!

Values of z to "plug in" for Bode plot... ...all on the UNIT CIRCLE.

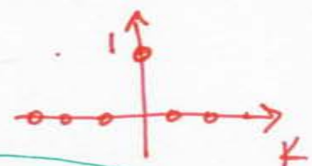
$z = e^{\omega T j}$



10. Unit step: $\frac{z}{z-1}$



11. Unit impulse: 1



~~Not a "normalized unit pulse"!~~

12. Numerical Approx'n:

- Same for TF's (CT $\leftrightarrow$ DT)
- And for state space (CT $\rightarrow$ DT)

For converting $C(s) \leftrightarrow C(z)$ (either way) Know all 5 methods in the TABLE below...

For an exam, only worry about "Forw. Euler" approx (for state space!)

$$\boxed{\begin{array}{ll} A_d = I + AT \overset{\text{similar to}}{\longleftrightarrow} \text{"}1+sT\text{"} & C_d = C \\ B_d = BT & D_d = D \end{array}}$$

Table to go from s to z (or z to s)
(differential to difference eqns)

For the first 4 methods, just replace "s" in C(s), directly.

Forward Euler	$s \leftrightarrow \left(\frac{z-1}{T}\right)$	$z \leftrightarrow 1+sT$	
Backward Euler	$s \leftrightarrow \left(\frac{z-1}{zT}\right)$	$z \leftrightarrow \left(\frac{1}{1-sT}\right)$	
Tustin (trapezoid, bilinear)	$s \leftrightarrow \frac{2}{T}\left(\frac{z-1}{z+1}\right)$	$z \leftrightarrow \left(\frac{2}{T}+s\right)/\left(\frac{2}{T}-s\right)$	
Tustin with prewarp	$s \leftrightarrow \alpha\left(\frac{z-1}{z+1}\right)$	$z \leftrightarrow (\alpha+s)/(\alpha-s)$	$\alpha \equiv \frac{\omega_0}{\tan\left(\frac{\omega_0 T}{2}\right)}$
Pole-zero matching	$s \leftrightarrow \frac{\log(z)}{T}$	$z \leftrightarrow e^{sT}$	See RULES, Lect. 7 Notes

where ω_0 is the "prewarp" frequency

*See Lecture 7 (page 15 of annotated notes)

13. Given a step response for some $G(z)$,

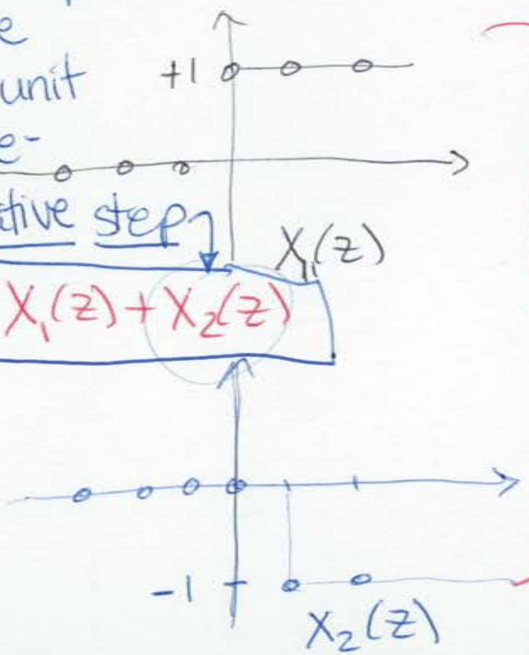
"$G(z)$" is the inverse z-transform of the output time response to a unit impulse input.

What would a unit impulse response be?

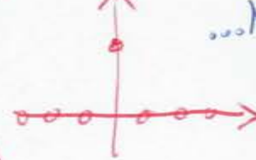
Consider an impulse input as the SUM of a unit step & a time-delayed negative step

$$\boxed{X_{imp}(z) = 1 = X_1(z) + X_2(z)}$$

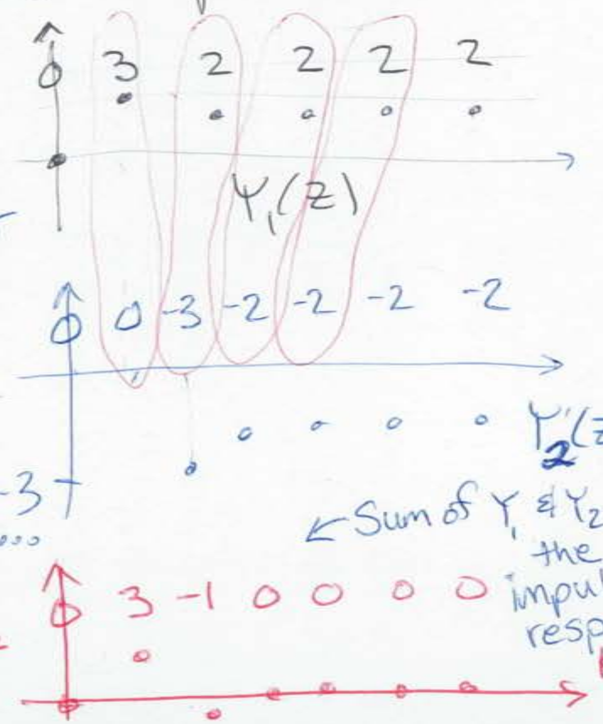
$1z^0$



$X_1(z)+X_2(z)$ gives an impulse



...Now, just add $Y_1(z)$ & a time-delayed & flipped $Y_2(z)$...



← Sum of Y_1 & Y_2 is the impulse response, ← $G(z)$

$$\therefore G(z) = \frac{3}{z} - \frac{1}{z^2}$$

$$G(z) = 3z^{-1} - z^{-2}$$

14. Know the Def'n of ZOH ... $\leftarrow$ $P_{ZOH}(z) = \left(\frac{z-1}{z}\right) \mathcal{Z}\left\{\frac{1}{s}P(s)\right\}$
(see example @ end of slides...)

$G(z)$: Open-loop (OL) plant
$P(z)$: Closed-loop (CL) plant
$C(z)$: Controller

15. Direct Design.

$$C(z) = \frac{1}{G(z)}\left(\frac{P(z)}{1-P(z)}\right)$$

A. G cannot have zeros outside unit circle! (B/C $C(z)$ would require unstable poles.)

B. $P(z)$ [your desired C.L.T.F.] cannot respond faster than $G(z)$ [your O.L. (open-loop) T.F.] i.e., $C(z)$ is CAUSAL.

General Form:
$$G(z) = \frac{fz^m + gz^{m-1} \dots}{az^n + bz^{n-1} \dots}$$

for either a unit impulse or unit step in x, the first non-zero value of y is: $\frac{f}{a}$
$y(k=n-m) = \frac{f}{a}$
$\boxed{y_{n-m} = \frac{f}{a}}$

$$\boxed{a y_{k+5} + b y_{k+4} + \dots = f x_{k+3} + g x_{k+2} \dots}$$

15. ↑ ↓ 16.

16. Consider $G(z) = \frac{fz^3 + gz^2 \dots}{az^5 + bz^4 + \dots} = \frac{Y(z)}{X(z)}$

initial response just depends on these terms

$X = \frac{z}{z-1}$, $\leftarrow$ unit step in.

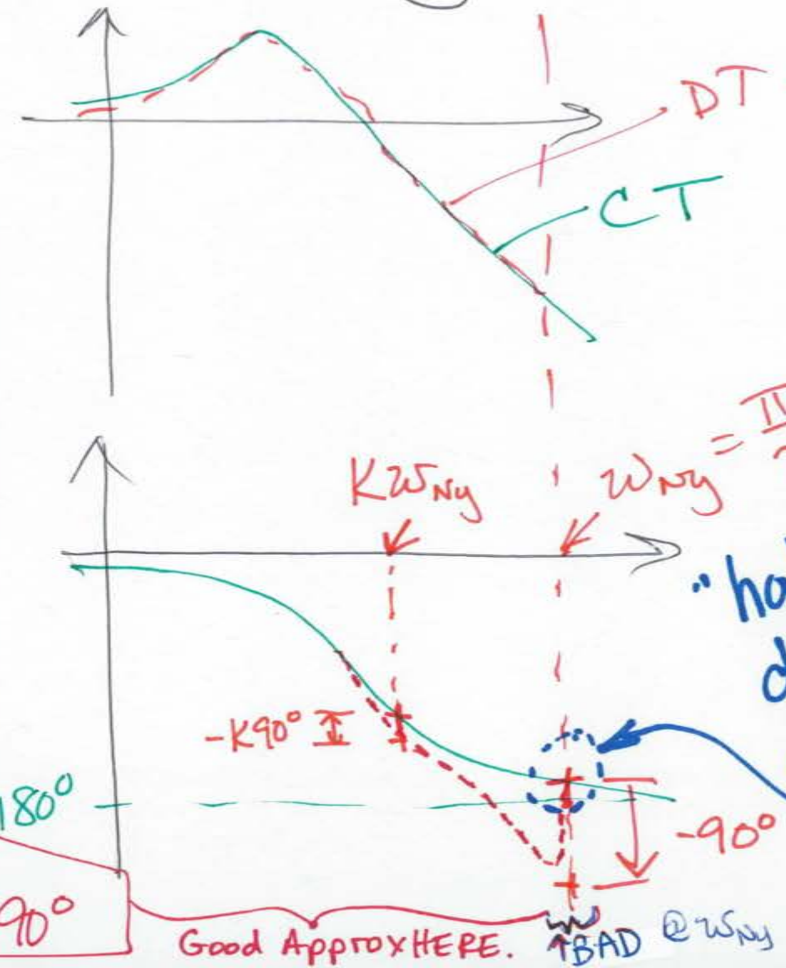
For a unit step response: A) First output that is non-zero for $Y(z)$ is $\frac{f}{a}$.

B) That first non-zero value happens @ $k = 5-3 = 2$.

17. The biggest issue (for control) in having a ZOH is (often) the

TIME DELAY.

i.e., a "half-sample delay" is a good approx... up until very close to $\omega_{Ny} = \frac{\pi}{T}$



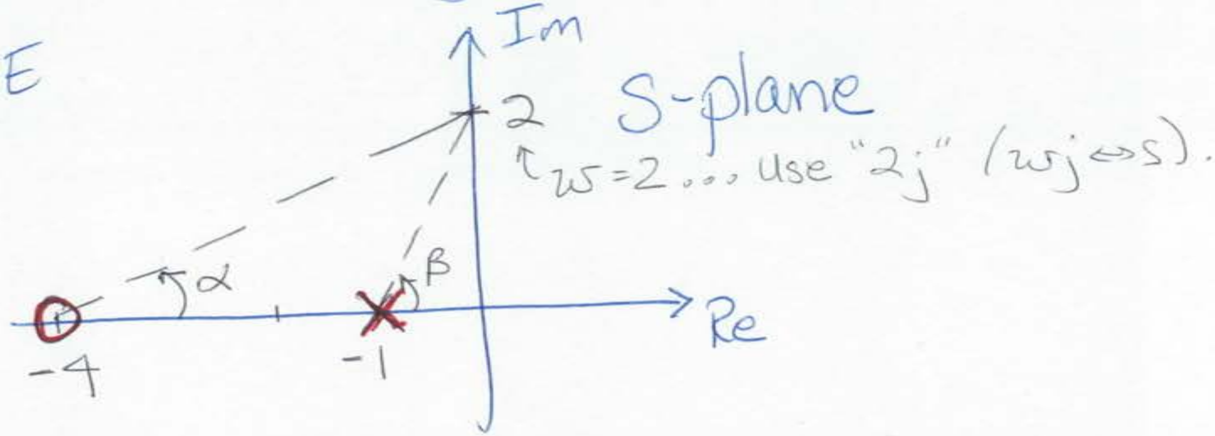
- Phase change due to "half-sample delay" approximation would be $-90°$ at Nyquist frequency ($\omega_{Ny} = \frac{\pi}{T}$)

$\therefore$ @ $\omega = K * \omega_{Ny}$, phase drop is $-K*90°$

"half-sample" delay is only an approx. @ ω_{Ny}, $G(z)$ must be REAL valued!

18. Use GEOMETRY to estimate PHASE for some $C(s)$ [or for $C(z)$, assuming you can convert to $C(s)$ using "$z \leftrightarrow e^{sT} \approx 1+sT$"]

EXAMPLE



$C(s) = K\dfrac{(s+4)}{(s+1)}$. @ $\omega = 2\left(\frac{rad}{s}\right)$, phase is $\Delta\phi = \alpha - \beta$

$$= \operatorname{atan}\left(\tfrac{2}{4}\right) - \operatorname{atan}\left(\tfrac{2}{1}\right)$$

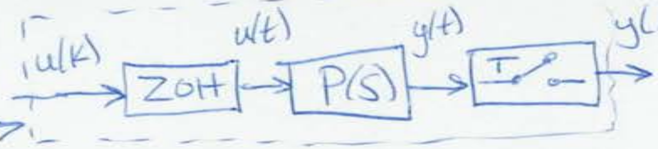
$$\approx (90 - 63^\circ) - 63^\circ$$

$$\boxed{\Delta\phi \approx -36^\circ}$$

19. Remember, Root Locus & Block Dia. Algebra work the same for z xform & z plane (as for Laplace & s-plane)

$\Longleftrightarrow \quad \dfrac{G}{GH+1} = \dfrac{Y}{X}$

14. ZOH equivalent.



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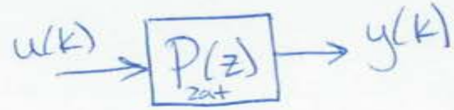


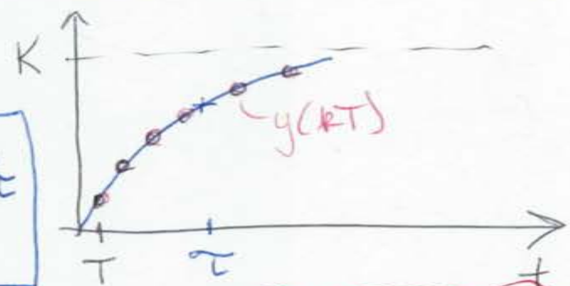
This "Sampled-data" config. is equivalent to:

$$\boxed{P_{ZOH}(z) = \left(\frac{z-1}{z}\right) \cdot \mathcal{Z}\left\{\frac{1}{s} \cdot P(s)\right\}}$$

Example: Say $P(s) = \frac{K}{\tau s + 1}$. ← It's impulse response is: $y(t) = \frac{K}{\tau} e^{-t/\tau}$

What is "$\mathcal{Z}\left(\frac{1}{s} \cdot P(s)\right)$". Well, "$\frac{P(s)}{s}$" corresponds to the time sequence that integrates: $\boxed{y(t) = \int_0^t \left(\frac{K}{\tau} e^{-t/\tau}\right) = K\left(1 - e^{-t/\tau}\right)}$

The corresponding Z transform for this signal, sampled @ time steps kT is: $\left(\frac{z}{z-1}\right) K \left(\frac{1-r}{z-r}\right)$, where $\boxed{r \equiv e^{-T/\tau}}$



* Think of this as a "unit step" times some G(z) resulting in Response, y(kT).

Numerator is simply set so signal has the same FINAL VALUE, K!

$$P_{ZOH}(z) = \left(\frac{z-1}{z}\right) \cdot \underbrace{\left(\frac{z}{z-1}\right) K \left(\frac{1-r}{z-r}\right)}_{y(kT)}$$

$$= K\left(\frac{1-r}{z-r}\right), \left\{r \equiv e^{-T/\tau}\right\}$$

1. More examples. Estimate $G(z)$ given a time response.

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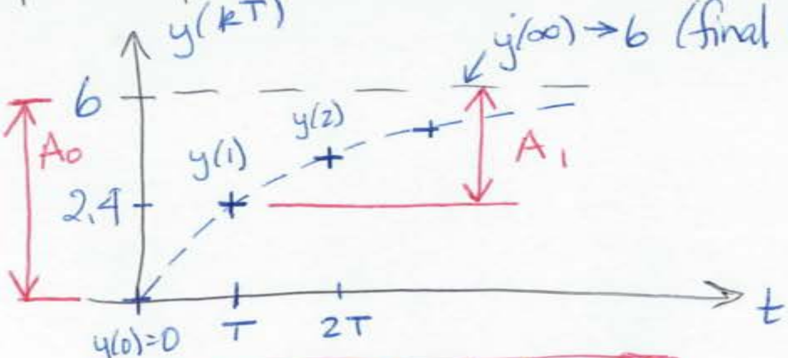
A. Say a unit step response is:



$$e^{-T/\tau} = \left[\frac{y(\infty) - y(1)}{y(\infty)}\right] = \frac{A_0}{A_1} = \frac{3.6}{6}$$

$$e^{-T/\tau} = \frac{3.6}{6} = 0.6$$

$$\therefore \boxed{\frac{-T}{\tau} = \log(0.6)} \to \boxed{G(z) = K\frac{z}{z-r}},$$

where $\boxed{r \equiv e^{-T/\tau}}$

r is the decay magnitude after one sample.

$\therefore r \equiv 0.6$

and Final Value $= 6 = K\frac{1}{1-r}$ $\boxed{K = 2.4}$

$\therefore K = 6\frac{1-r}{1} = 6\frac{1-.6}{1} = 6 * .4$

B. Say the unit step resp. for $H(z)$ is:

$$H(z) = \frac{Kz}{z^2 - 2xz + r^2}$$

$r = .98$

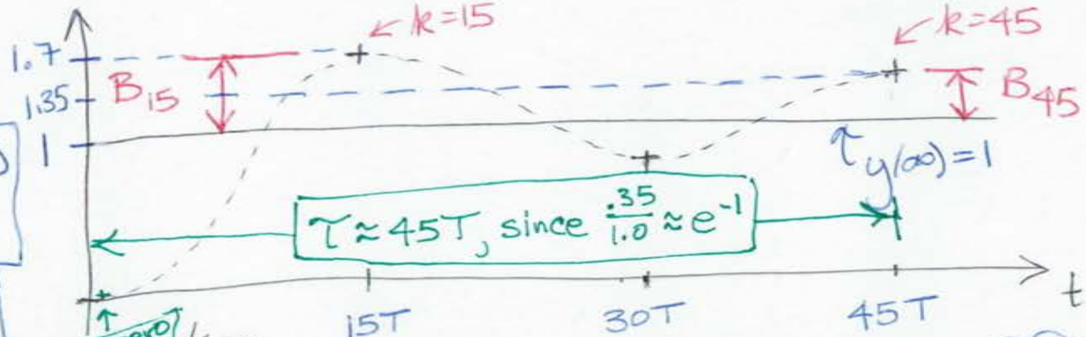
$x = r\cos(\frac{\pi}{15}) = r\cos(12°)$

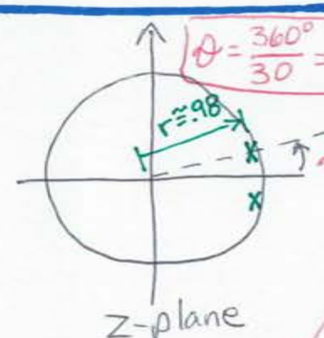
$K = 1 - 2x + r^2$

$$r^{30} = \frac{B_{45}}{B_{15}}$$

where $B_{45} \equiv y(45) - y(\infty) \cong 0.35$

$B_{15} \equiv y(15) - y(\infty) \cong 0.7$





$$\theta = \frac{360°}{30} = 12°$$

$(45T - 15T) = \frac{2\pi}{2\omega_d}$

$\omega_d = \frac{2\pi}{30T}$

30 samples per cycle (one cycle is 2π rad. or 360°)

So, $\theta = \frac{2\pi}{30}$

$$\therefore \boxed{r^{30} = \frac{.35}{.7} = \frac{1}{2}} \leftrightarrow \text{or } e^{\frac{-30T}{\tau}} = \frac{1}{2}$$

$$r \equiv e^{-T/\tau} \approx e^{\frac{-T}{45T}} = e^{-1/45} \approx 1 - 1/45 \approx 1 - .022$$

$$\therefore \boxed{r \approx .978}$$