Today: State (Space) Feedback

Lecture 11

ECE 147B

Digital Control

Reading: FPW 7.2, 8.1

or: FV 6.5, (9.1), 9.2

Topics:

- Quick, last-minute REVIEW! ("Pop Quiz")
- "Direct Design" revisited: unstable poles in G(z)?
- State space block diagrams
- Modeling the ZOH via ss equations
- State feedback: u=-Kx
- Control Canonical Form

Next Class: MIDTERM EXAM.

- You may bring one, single-single, self-prepared 8 ½ x 11 sheet of notes. No calculators, cell phones, etc.

Say $X_A = \frac{z}{z+1}$ $\quad X_B = \left(\frac{z-1}{z}\right) X_A$

$X_B = 1$ $\quad$ then $Y_B = \left(\frac{z-1}{z}\right) Y_A$ $\quad \left(\frac{z-1}{z}\right) = 1 - \frac{1}{z}$

(if X_A yields Y_A, & X_B yields Y_B.)

Pop Quiz!

1. If the final value to a unit step input to G(z) is 5, then the final value to an IMPULSE input is:
   a. 5
   b. 0
   c. ∞
   d. There is not enough information to determine this.

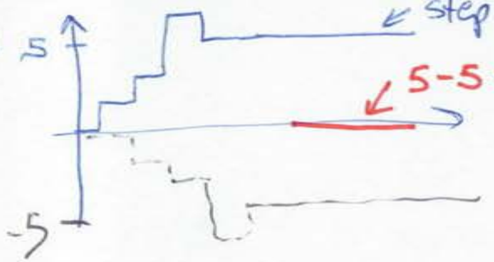
$$G(z) \equiv \frac{Y(z)}{X(z)}, \quad X(z) = \frac{z}{z-1}$$

X an impulse means $X_{tot} = 1$

$$X_1 + X_2 = X_{tot} = 1$$

$\uparrow X_2 = -\frac{1}{z} X_1$ } Time shift & sign flip

①



2. If the final value to an impulse input to G(z) is +5, then the final value to a unit step input is:
   a. 5
   b. 0
   c. +∞
   d. There is not enough information to determine this.

Pop Quiz!

3. The Final Value Theorem says:
   a. Just blindly plug in "z=0", always.
   b. Just blindly plug in "z=1", always.
   c. Given transfer function G(z)=Y(z)/X(z) has some input X(z),

      $1-\frac{1}{z}$

      the final value of Y(z) will be: $\lim_{z\to 1}\left(\frac{\boxed{z-1}}{z}\cdot X(z)\cdot G(z)\right)$ ← z is going to 1, so this denominator doesn't matter.
   d. Use definition in "c.", but only apply IF POLES ARE STABLE!

      otw., the "final value" is not well defined.

For 1. & 2 (last page),

$$\boxed{\lim_{z\to 1}\left(\frac{X_B(z)}{X_A(z)}\right)\cdot \text{F.V. of } Y_A(z) = \text{F.V. of } Y_B}$$

1. $X_A = \frac{z}{z-1}$ (step), $X_B = 1$ $\therefore \frac{X_B}{X_A} = \frac{z-1}{z}$ } goes to 0 as $z \to 1$

---

2. $X_A = 1$ (impulse), $X_B = \frac{z}{z-1}$ $\therefore \frac{X_B}{X_A} = \frac{z}{z-1}$ } goes to ∞ as $z\to 1$

   $z-1 \to 0$

Pop Quiz!

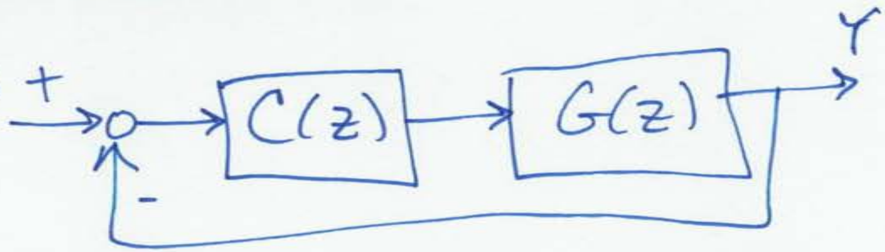
4. For "Direct Design" of controller C(z) of an $\boxed{\text{open-loop plant, } G(z),}$ to yield some desired closed-loop plant, P(z), which statements are TRUE:

Yes! ✓ a. P(z) cannot respond faster than G(z), since C(z) is causal.

Yes! ✓ b. Zeros of G(z) outside the unit circle cannot be "canceled" using poles of C(z), since C(z) can't have unstable poles.

No... c. Unstable poles of G(z) should just become zeros in C(z), using "pole-zero cancellation".

Recall:



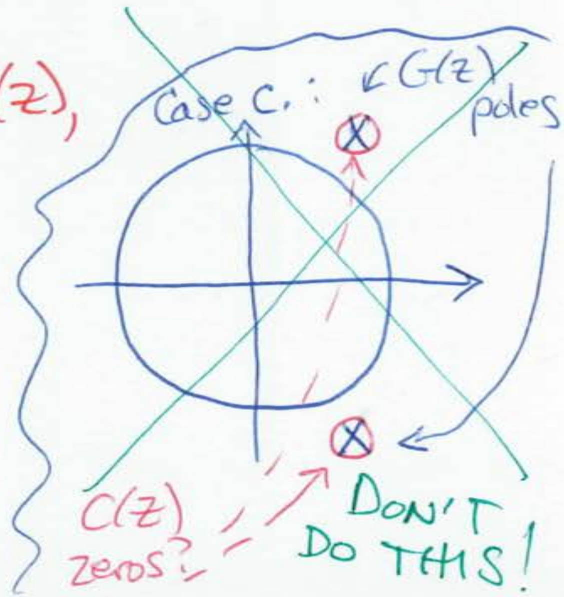
$$P(z) = \frac{Y(z)}{X(z)}$$

$$\boxed{P = \frac{CG}{1+CG}}$$

So, to get a desired P(z),

$$\boxed{C = \frac{1}{G}\left(\frac{P}{1-P}\right)}$$

Say G(z) has unstable poles...
We cannot just put zeros of C here to "cancel" them.



Direct Design: One last thing!

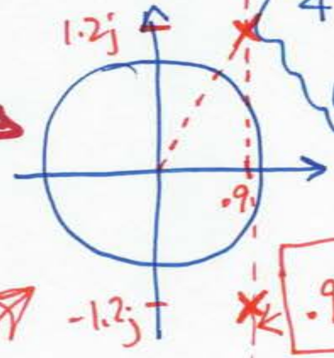
for an impulse input, $y(k=0)=0$
$y(k=n-m)=\frac{a}{b}$
$y(k=n-m)$

Example: $G(z) = \dfrac{z}{z^2 \underbrace{-1.8z}_{\text{"}-2x\text{"}} + \underbrace{\frac{9}{4}}_{\text{"}r^2\text{"}}} = \dfrac{az^m + \ldots}{bz^n + \ldots}$ ← so $y(k=1)=1$

Open-loop poles of $G(z)$?

$\boxed{z = +0.9 \pm 1.2j}$ ← UNSTABLE

UNSTABLE →

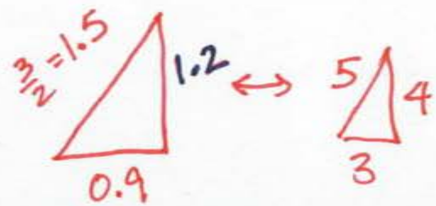


$\frac{9}{4} > 1$
$\therefore$ UNSTABLE

$\boxed{.9 = -\left(\frac{-1.8}{2}\right)}$

$r^2 = \frac{9}{4}$

$\boxed{\therefore r = \frac{3}{2}}$



Instead of putting zeros of $C(z)$ "on top of" $G(z)$,

Let: $\boxed{G = \frac{b(z)}{a(z)} = \frac{b}{a} \quad C = \frac{c}{d} \quad P = \frac{e}{f}}$

So, $C = \frac{1}{G}\left(\frac{P}{1-P}\right)$

$C = \frac{\boxed{a}}{b}\,\frac{e}{\cancel{f}}\,\frac{\cancel{f}}{\boxed{(f-e)}}$

To "cancel out" a, we can let: $\boxed{a = f-e}$

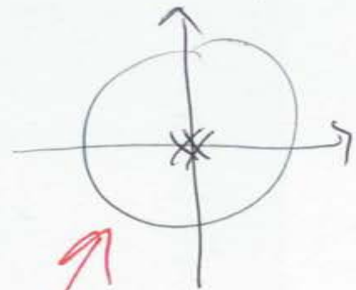
So unstable poles of $G(z)$ become zeros of $\boxed{1-P}$.

$\therefore \boxed{1-P = \frac{f}{f} - \frac{e}{f} = \frac{f-e}{f}}$

Say $G(z) = \dfrac{z}{(z^2 - 1.8z + \frac{9}{4})}$

$\therefore$ we <u>want</u> $P(z)$ w/ <u>poles</u> @ $z=0$, $z=0$.

"$z^2 = 0$" ← char. eqn.



Deadbeat control wanted.

Now, the zeros of $P(z)$ will be set using the "trick" on last page...

$P = \frac{e}{f}$, $C = \frac{c}{d}$, $G = \frac{b}{a}$

$f = z^2$

$e = f - a = z^2 - (z^2 - 1.8z + \frac{9}{4})$

$\therefore e = 1.8z - 9/4$

$$C = \frac{\cancel{a}}{b} \cdot \frac{e}{(\cancel{f-e})} = \frac{e}{b} = \boxed{\frac{1.8z - 2.25}{z} = C(z)}$$

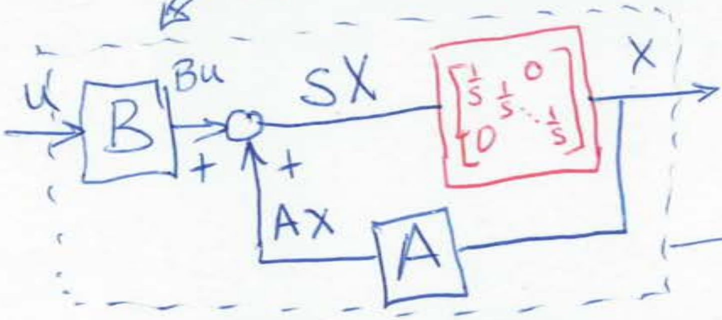
cancel, by design.

State space: Block diagram of equations

1. Dynamics. (What internal system does, "open loop".)

CT → $\dot{x} = Ax + Bu$ ⟶ or $zX = A_d x + B_d u$ (DT case)

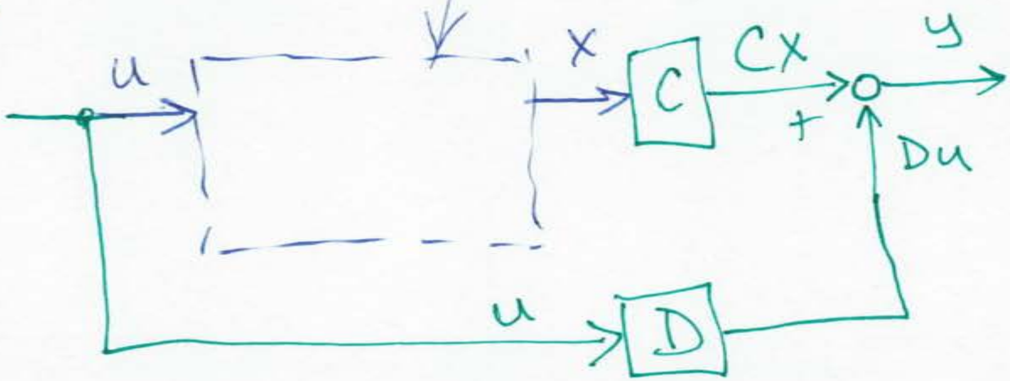
$sX = Ax + Bu$



$x = (\frac{1}{s} I) sX$ } open-loop dynamics

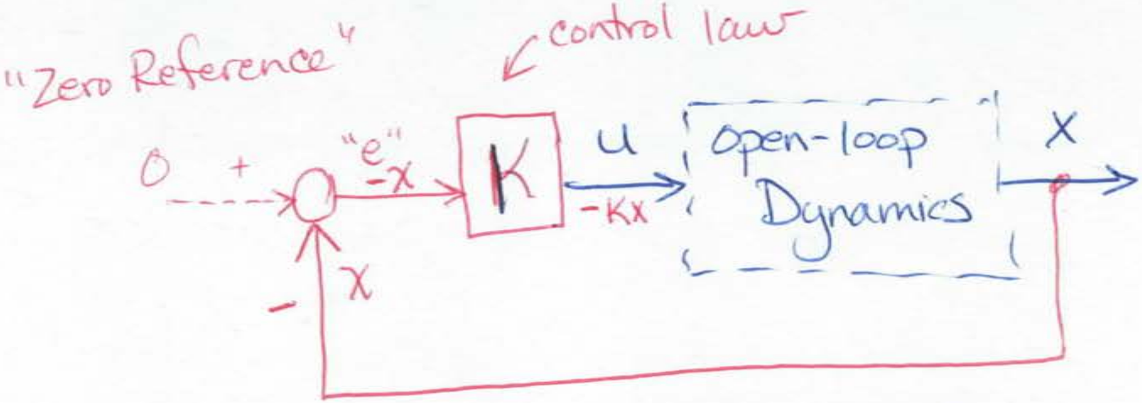
2. Outputs. (Things we can MEASURE.)

$y = Cx + Du$



State space: Block diagram of equations

1. Control: A way to MODIFY the dynamics. i.e., set u as a function of X, via a "control law", e.g., a "PD controller" (for example)...



n : states
m : outputs
k : actuations

Control Law : $\boxed{\underset{(k\times 1)}{u} = -\underset{(k\times n)}{K}\,\underset{(n\times 1)}{X}}$

Consider $X = \underset{(2\times 1)}{\begin{bmatrix} \dot{x} \\ x \end{bmatrix}}$ & u some actuation

$k=1 \rightarrow 1\times 1$ (scalar)

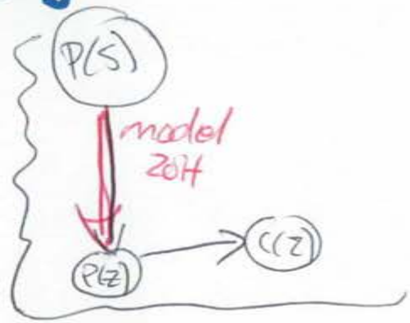
$\therefore \underset{\substack{(k\times n)\\(1\times 2)}}{K} = [K_1 , K_2]$

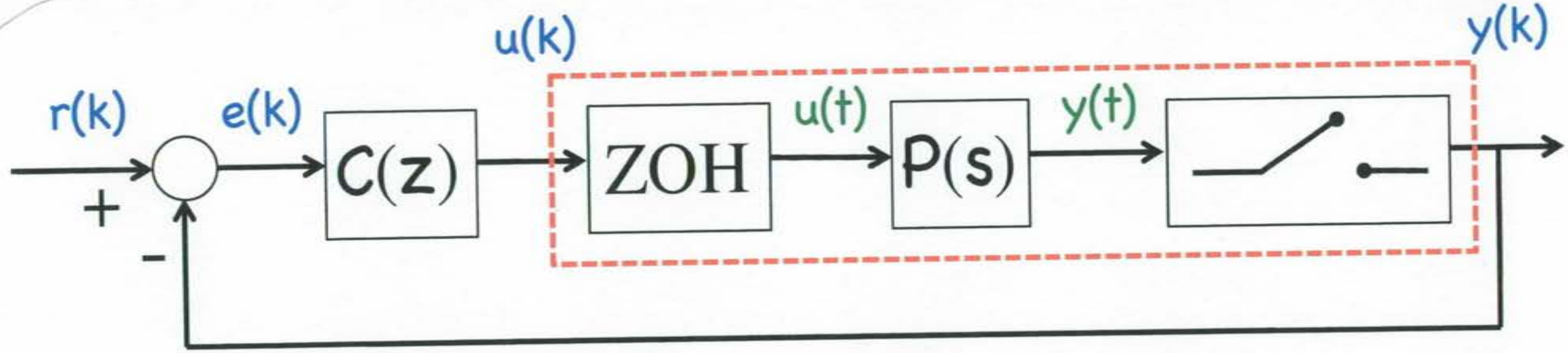
$$u = -KX = -K_1 \dot{x} - K_2 x \qquad x_1 = \dot{x} \qquad x_2 = x$$

$$= \underbrace{-K_d \dot{x} - K_p x}_{\text{a lot like PD!}}$$

"Case 2": Modeling the ZOH plus sampling

Given some C.T. system, with A, B, C & D...





$$\boxed{A_d = e^{AT}} = \text{expm}(AT) \neq \text{exp}(AT)$$

"matrix exponential"

"element by element"

$$\boxed{B_d = \int_0^T e^{AT}\, dt\, B = A^{-1}(A_d - I)B}$$

"Case 2": Modeling the ZOH plus sampling

The Matrix Exponential:

$$e^M = \sum_{k=0}^{\infty} \frac{1}{k!} M^k$$

Properties:

$$e^{M\times 0} = I$$

$$e^{M(s+t)} = e^{Ms} e^{Mt}$$

$$e^{-Mt} e^{Mt} = I$$

$$\frac{de^{Mt}}{dt} = Me^{Mt}$$

Example 1: M=1

(M just a 1x1 matrix, i.e., a scalar...)

$$e^1 = 1^0 + 1^1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$
$$= 2.718281828 \text{ (etc...)}$$

Example 2: $\ddot{x} + 5\dot{x} + 10x = u \quad (T = 1\text{sec})$

$$X = \begin{bmatrix} \dot{x} \\ x \end{bmatrix}, \quad \dot{X} = AX + Bu$$

$$\ddot{x} = -5\dot{x} - 10x + u$$

$$\dot{x} = \dot{x} \quad (?) \leftarrow 0*u$$

$$\left(\frac{d}{dx} x_2 = x_1\right)$$

$$A = \begin{bmatrix} -5, & -10 \\ 1, & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

In MATLAB: exp(A) vs expm(A)?

Approximations, for "small T"...

Note: Analogous approximation trick applies for "e^{AT}" when "AT" has "small" entries as for "e^{sT}" when "sT" has small magnitude:

Example: $\ddot{x} + 2\dot{x} + 5x = u$ $\underline{(T = 0.01\text{sec})}$

If $X = \begin{bmatrix} \dot{x} \\ x \end{bmatrix}$, Then $A = \begin{bmatrix} -2, & -5 \\ 1, & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

w/ ZOH:

$A_d = \text{expm}(A*.01) = \begin{bmatrix} .97995, & -.0495 \\ .0099, & .99975 \end{bmatrix}$, $B_d = ?$

similar to last page!

$$e^{\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}T} = \begin{bmatrix} A_d & B_d \\ 0 & I \end{bmatrix}$$

Forward Euler approximation:

$$\boxed{A_d = I + AT}$$

↑ for "small" T, ZOH answer is "close" to this.