Today: Pole Placement ^...for poles of the Controlled system dyn.

Lecture 12

ECE 147B
Digital Control

Reading: FPW 8.1

or: FV (9.1), 9.2

Topics:

- ✓ Modeling the ZOH / Review from last time
- ✓ State feedback: u=-Kx
- ✓ Control Canonical Form
- Pole Placement
- Controllability

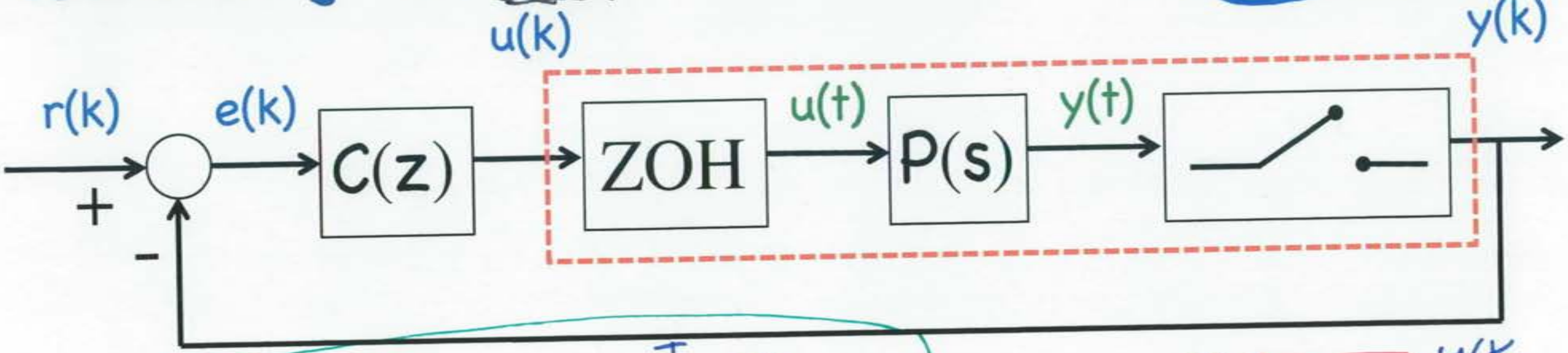
← Next Time!

Next Class: Estimator design via pole placement. ^ for poles of an observer, to estimate states.

- FPW 8.2.1-8.2.4 <or> FV 9.5.1, (RS-12)

Z.O.
← the "NON-Menu" part

Modeling the ZOH - REVIEW



$$\boxed{A_d \equiv e^{AT}, \quad B_d \equiv \underbrace{\int_0^T e^{At}\,dt\, B}}$$

$u(k) \rightarrow \boxed{P(z)} \rightarrow y(k)$

...can avoid this integral

The Matrix Exponential:

$$e^{M} = \sum_{k=0}^{\infty} \frac{1}{k!} M^{k}$$

expm command

A trick:

$$e^{\underbrace{\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} T}_{\text{"M"}}} = \begin{bmatrix} A_d & B_d \\ 0 & I \end{bmatrix}$$

e^{M} means expm(M) [in MATLAB]

Example: $\ddot{x} + 2\dot{x} + 5x = u$

$(T = 0.01\text{sec})$

$$X = \begin{bmatrix} \dot{x} \\ x \end{bmatrix}, \quad \boxed{A = \begin{bmatrix} -2, & -5 \\ 1, & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

Note that this is "Control canonical" form.

Forward Euler approximation:

$$A_d \cong \begin{bmatrix} .98, & -.05 \\ .01, & 1 \end{bmatrix} \leftarrow \boxed{A_d \approx I + AT, \quad B_d = BT}$$

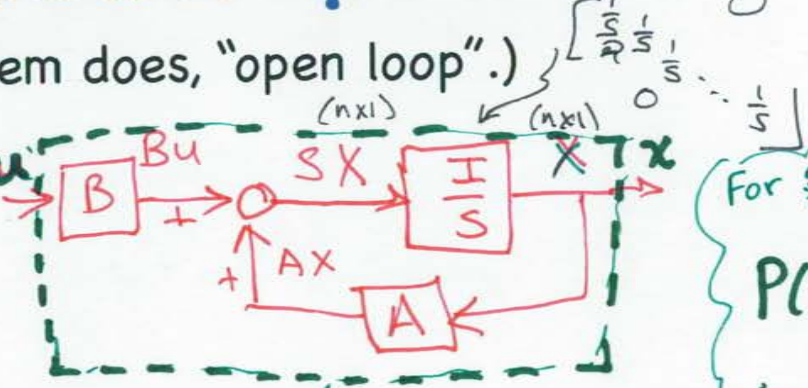
State space: Block diagram of equations

1. Dynamics. (What internal system does, "open loop".)

$\boxed{n \text{ states. (in } x)}$

open-loop dynamics → CT: $\boxed{sX = Ax + Bu}$

(Very analogous version for DT... or → [$zX = A_d X + B_d u$])



For SISO case: $P(z) \equiv \frac{X(z)}{U(z)}$ is inside the [dashed] box.

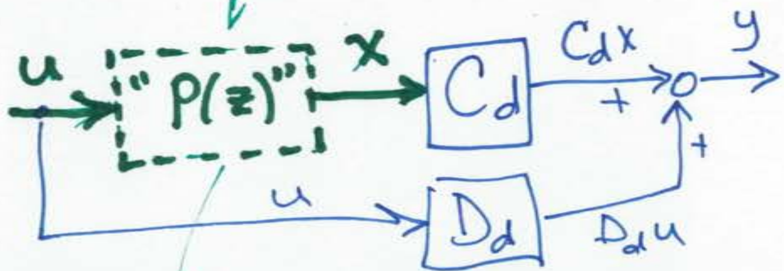
2. Outputs. (Things we can MEASURE.)

$\boxed{k \text{ outputs. (in } y)}$

Next class...

CT : $y = Cx + Du$

DT : $y = C_d x + D_d u$



3. Control: A way to MODIFY the dynamics. i.e., set u as a function of X, via a "control law", e.g., a "PD controller" (for example)...

$\boxed{m \text{ inputs (in } u)}$

closed-loop dynamics → $u = -Kx$

$sX = Ax + B(-Kx)$

$\boxed{sX = (A - BK)x}$

Note we have zero as a reference (for each x state)

negative feedback.

A Review...

State space: Poles and Zeros

1. Dynamics. (What internal system does, "open loop".) -> $\boxed{\text{POLES}}$

Solutions for $x \neq 0$, such that $u=0$? ← $u=0$. What $x(t)$ are still allowed?

$$sx = Ax + \cancel{Bu} \quad u=0 \qquad\qquad zx = A_d x + B_d u$$

$(sI-A)x=0$

$\therefore \boxed{\det(sI-A)=0}$ ← solutions are poles...

...and these are eig(A).

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2. Outputs. (Things we can MEASURE.) -> $\boxed{\text{ZEROS}}$

Solutions for $u \neq 0$, such that $y=0$?

$(sI-A)x = Bu$

$$Y = Cx + Du \qquad\qquad Y = C_d x + D_d u$$

$x = (sI-A)^{-1}Bu$

$\therefore y = C(sI-A)^{-1}Bu + Du$

$y = [C(sI-A)^{-1}B + D]u$

$y=0$ when $0 = [C(sI-A)^{-1}B + D]$

$\det[C(sI-A)^{-1}B + D] = 0$ ← solutions are the zeros.

can write as...

$$\det\begin{bmatrix} sI-A, & -B \\ C, & D \end{bmatrix} = 0$$

State space: CLOSED-LOOP Poles (and Zeros)

3. Control: A way to MODIFY the dynamics. i.e., set u as a function of X, via a "control law", e.g., a "PD controller" (for example)...

$\{K_d \dot{x} + K_p x\}$

State feedback: $u = -KX$

$X \equiv \begin{bmatrix} \dot{x} \\ x \end{bmatrix}$

$Kx = K_1 \dot{x} + K_2 x$

$$sX = Ax + Bu = Ax - BKx$$

$$(sI - (A - BK))x = 0$$

$$\boxed{\det\left[sI - (A - BK)\right] = 0}$$

$$zX = A_d x + B_d u = A_d x - B_d Kx$$

Same math... ...but "z-plane" meaning for pole locations.

$$\boxed{\det\left[zI - (A_d - B_d K)\right] = 0}$$

Solutions are closed-loop poles.

OR: $\boxed{\mathrm{eig}(A - BK)}$ $\longleftrightarrow$ i.e., "A" (open loop) is replaced by "A-BK"

State feedback example

$\ddot{x}_2 + 2\dot{x}_2 + 5x_2 = 0$

$\boxed{s^2 + 2s + 5 = 0}$

when $u=0$, to find poles

We can "read off" the coefficients from A

$m\ddot{x} + b\dot{x} + kx = f$

$X \equiv \begin{bmatrix} \dot{x} \\ x \end{bmatrix}$

$K = [?]$

- Say you are given the system dynamics and state variable definition at left. (m=1, b=2, k=5)
- What size must matrix "K" be? (1×2) (# inputs × # states)
- * What values in K would result in closed loop poles with ω_n=10 (rad/s) and ξ =1?

$K = [K_1 \quad K_2]$

$BK = \begin{bmatrix} K_1 & K_2 \\ 0 & 0 \end{bmatrix}$

$\ddot{x} = \frac{-b}{m}\dot{x} - \frac{k}{m}x + \frac{f}{m}$

$A = \begin{bmatrix} -2, & -5 \\ 1, & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\dot{X} = \begin{bmatrix} \ddot{x} \\ \dot{x} \end{bmatrix} = Ax + Bu$

$\frac{d}{dt}x_1 = \frac{-b}{m}x_1 - \frac{k}{m}x_2 + \frac{u}{m}$

$\ddot{x}_2 + \frac{b}{m}\dot{x}_2 + \frac{k}{m}x_2 = 0$ (for $u=0$)

$A = \begin{bmatrix} -\frac{b}{m}, & -\frac{k}{m} \\ 1, & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{m} \\ 0 \end{bmatrix}$

$\frac{d}{dt}x_2 = x_1$

$A - BK = \begin{bmatrix} -2-K_1, & -5-K_2 \\ 1, & 0 \end{bmatrix}$

$\therefore \boxed{2 + K_1 = 20 \rightarrow K_1 = 18}$

$\boxed{5 + K_2 = 100 \rightarrow K_2 = 95}$

* $\therefore$ We want: $\boxed{s^2 + 2\xi\omega_n s + \omega_n^2 = 0} \rightarrow \boxed{s^2 + 20s + 100 = 0}$

open loop dynamics:

$$\boxed{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n = 0}$$ ← Roots are o.L. poles!

"Control Canonical Form"

Note:

① negative of the coefficients of the OPEN-LOOP characteristic eqn.

$$A = \begin{bmatrix} -a_1 & -a_2 & & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ & \vdots & \ddots & 0 & 0 \\ 0 & 0 & & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad BK = \begin{bmatrix} K_1, & K_2, & \dots, & K_n \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & & & 0 \end{bmatrix}$$

$$K = \begin{bmatrix} K_1 & K_2 & \cdots & K_{n-1} & K_n \end{bmatrix}$$

② off-diagonal 1's: $\boxed{\frac{d}{dt} x_p = x_{p-1}}$, i.e., $x_p = \dot{x}_{p+1}$ → So, the states are related as derivatives of one another...

(for $p > 1$, since no "x_0" exists...)

$$\underset{(n \times 1)}{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

... And: X_1 is the highest derivative.

$\dot{x}_2 = x_1$
$\dot{x}_3 = x_2$
$\vdots$
$\dot{x}_n = x_{n-1}$

For "control canonical" form

negative of the coefficients of the CLOSED-LOOP char. eqn.

$$A - BK = \begin{bmatrix} -a_1 - K_1 & -a_2 - K_2 & & -a_{n-1} - K_{n-1} & -a_n - K_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ & \vdots & \ddots & 0 & 0 \\ 0 & 0 & & 1 & 0 \end{bmatrix}$$

Roots are closed-loop poles

$$\boxed{s^n + (a_1 + K_1) s^{n-1} + (a_2 + K_2) s^{n-2} + \dots + (a_n + K_n) = 0}$$

"Control Canonical Form"

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ ← One input, which affects only $\dot{x}_1$

$\left(\frac{d^n}{dt^n} x_n\right)$

$\dot{x}_1 = \ddot{x}_2 = \dddot{x}_3$

Example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

3rd-order: 3 states, 3 poles, (3 values in K.)

$$A - BK = A - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix}$$

closed-loop dynamics:

$$\dot{x}_1 = -(a_1+K_1)x_1 - (a_2+K_2)x_2 + \dots - (a_3+K_3)x_3 = \begin{bmatrix} -a_1-K_1, & -a_2-K_2, & -a_3-K_3 \\ 1, & 0, & 0 \\ 0, & 1, & 0 \end{bmatrix}$$

↑ unchanged

$$\begin{cases} \dot{x}_2 = x_1 \\ \dot{x}_3 = x_2 \end{cases}$$

$\dddot{x}_3 = \ddot{x}_2 = \dot{x}_1$

$$\therefore X = \begin{bmatrix} \ddot{x} \\ \dot{x} \\ x \end{bmatrix}$$

$\dot{x}_3 = x_2$

So, the char. eqn. is:

$$\left(\dot{x}_1 = \dddot{x} = -(a_1+K_1)\ddot{x} - (a_2+K_2)\dot{x} - (a_3+K_3)x\right)$$

$$s^3 + (a_1+K_1)s^2 + (a_2+K_2)s + (a_3+K_3) = 0$$

$\dot{x}_1 = \dddot{x}$

"Control Canonical Form"

Canonical Form makes it "easy" to solve for K....

Example:

$$\underline{\underline{A}} = \begin{bmatrix} 3 & -5 \\ 1 & 0 \end{bmatrix} \qquad \underline{\underline{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

For state feedback control "u=-Kx", solve for K such that closed-loop poles are at s=-3, s=-3.

...once a desired char. eqn. for the closed-loop system is known.

$$(s+3)(s+3) = 0$$

$$s^2 + 6s + 9 = 0$$

$$A - BK = \begin{bmatrix} 3-K_1 , & -5-K_2 \\ 1 , & 0 \end{bmatrix}$$

$$s^2 - (3-K_1)s + (5+K_2) = 0 \leftarrow \text{closed loop.}$$

$K_1 - 3 = 6 \rightarrow \boxed{\therefore K_1 = 9}$

$5 + K_2 = 9 \rightarrow \boxed{\therefore K_2 = 4}$

NOT
"Control Canonical Form"

Example:

still off-diagonal ones.

$$A = \begin{bmatrix} 0 & 1 \\ -5 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Note order is FLIPPED!

$$\dot{x} = \dot{x}_1 = x_2, \quad x = x_1$$

$$\ddot{x} = \dot{x}_2 = -5x_1 + 3x_2 + u$$

$$\ddot{x} = -5x + 3\dot{x} + u$$

$$\ddot{x} - 3\dot{x} + 5x = u$$

For state feedback control "u=-Kx", solve for K such that closed-loop poles are at s=-3, s=-3. HINT: Same dynamics but different definition of "X" here!

*Although this is **NOT** "control canonical" form, it is very similar...

Punchline:

*Negative values of Coefficients of char. eqn. are still entries in the A matrix...

But order they are listed is **FLIPPED**!

Gain matrix still defined as: $K \equiv [K_1, K_2]$ → So, K is also FLIPPED! $K_1 = 4$, $K_2 = 9$

"Control Canonical Form"

Example:

$$A_d = \begin{bmatrix} 0.6 & -1.2 \\ 1 & 0 \end{bmatrix} \qquad B_d = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Same math holds in the discrete-time (DT) case...

For state feedback control "u=-Kx", solve for K such that closed-loop poles are at z=0, z=0. This is "deadbeat" control!

$$zX = A_d X + B_d u$$

$$zX_1 = 0.6X_1 - 1.2X_2 + u$$

$$zX_2 = X_1 \quad (X_1 = zX_2)$$

$$\therefore zX_1 = z^2X_2 = 0.6zX_2 - 1.2X_2 + u$$

$$\boxed{z^2X_2 - 0.6zX_2 + 1.2X_2 = u}$$

For n^{th}-order system, characteristic eqn. is wrt X_n (the last element in X).

w/ $u=-Kx$; char. eqn. is...

$$\boxed{z^2 + (-0.6 + K_1)z + (1.2 + K_2) = 0}$$

$\therefore z^2 = 0$ REQUIRES $K_1 = 0.6$, $K_2 = -1.2$

In Control Canonical form,

$X_2(k+1) = X_1(k)$
$X_3(k+1) = X_2(k)$
$\vdots$
$X_n(k+1) = X_{n-1}(k)$

$$\boxed{\begin{aligned} zX_2 &= X_1 \\ zX_3 &= X_2 \\ &\vdots \\ zX_n &= X_{n-1} \end{aligned}}$$

So, $X_1(k) = X_2(k+1) = X_3(k+2) = X_n(k+n-1)$

All states are interrelated via the "Z" operator (time delay).

Pole Placement

NEXT CLASS

Comments:

1. By now, you should see (intuitively) how to solve for K such that "eig(A-BK)" [or "eig(A_d-B_dK)"] yield desired poles...
2. But our hand calculations would be TEDIOUS for large-dimensional, MIMO (multi-input multi-output) systems!

Is there a more efficient solution? → YES! This is "pole placement".

First, let's look at how to do this IN MATLAB:

```
A = rand(4,4); % some random, 4th-order open-loop system dynamics…
B = rand(4,1); % …including actuator inputs.
pdes = [-1 -2 -3 -4]; % desired poles (4 poles; 4th-order system…)
pnow = eig(A)  % open-loop poles
K = place(A,B,pdes) % "pole placement" command in matlab


K2 = acker(A,B,pdes) % alternate "pole placement" command


pact = eig(A-B*K)  % ACTUAL closed-loop poles!
```

Controllability NEXT CLASS

Given matrices A and B (for CT), or given A_d and B_d (for DT), the CONTROLLABILITY matrix is defined as:

$$C \equiv [B, \quad AB, \quad A^2B, \quad ..., \quad A^{n-1}B]$$

The system is CONTROLLABLE if and only if this matrix has FULL RANK (meaning it is INVERTIBLE).

Note the CHARACTERISTIC EQUATION for your desired CLOSED-LOOP (still nth-order) dynamics can be written as:

$$z^n + \alpha_1 z^{n-1} + \alpha_2 z^{n-2} + ... + \alpha_n$$

Then, the GAIN MATRIX "K" needed will be: $K = [0...0,0,1]C^{-1}\alpha_c(A)$

Where $\quad \alpha_c(A) \equiv A^n + \alpha_1 A^{n-1} + \alpha_2 A^{n-2} + ... + \alpha_n I$