Today: # Separation Principle

Lecture 14
ECE 147B
Digital Control

$\boxed{u_k = -K\hat{x}_k}$, instead of $u_k = -Kx_k$

Reading: FPW 8.2.5, 8.3
or: FV 9.5.2, 9.6 (RS-13)

where $\hat{x}_k$ is an estimate of the true state, x_k

Topics:
- State estimation: feedback approach
- Estimator design via pole placement
- Observability (compared to Controllability)
- Separation Principle
- Reduced-Order Estimator

Next Class: Frequency Response Methods.
- FPW 7.4 <or> FV (6.4), 6.5, (6.6) (RS-1)

State Estimation – "feedback" approach

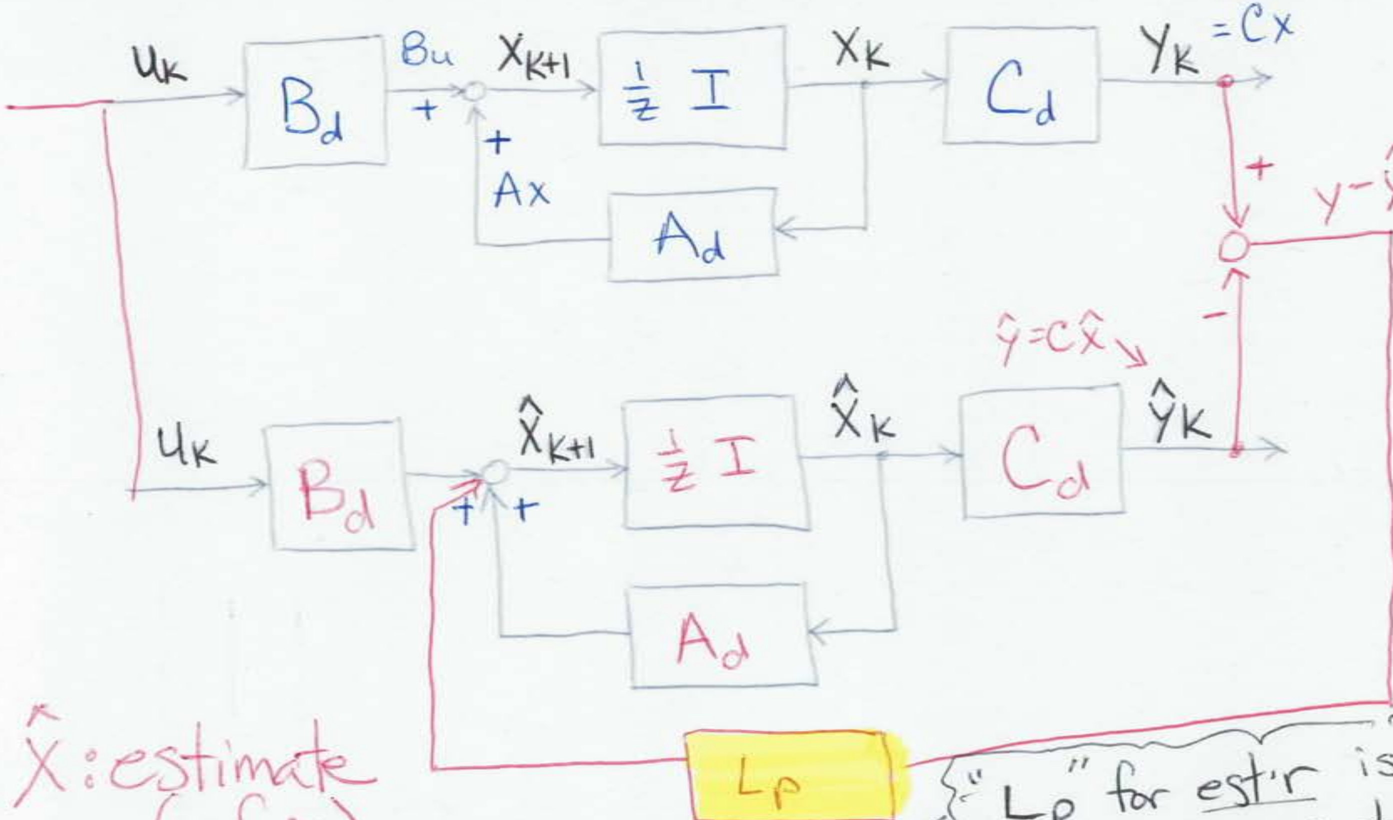
n states
ℓ outputs

Start with some "initial value" as an ESTIMATE of the states…
…use BOTH a MODEL ("Ax+Bu") and a MEASUREMENT ("y=Cx+Du").

Estimate of state WITH MEASUREMENT FEEDBACK : $\hat{x}(k+1)$

measurements

But how do we "feed back" MEASUREMENT info?



$\hat{X}$: estimate (of x)

"L_p" for est'r is analogous to "K" for controller.

State Estimation - "feedback" approach

(Continued...)

$$\text{state: } \overset{zx}{X_{k+1}} = A_d X_k + B_d u_k$$

$$\text{estimate: } \hat{X}_{k+1} = A_d \hat{x}_k + B_d u_k \underbrace{+ L_p C_d (x_k - \hat{x}_k)}_{\text{feedback part}}$$

(cancel out in subtraction)

Error dynamics?

$$\tilde{x} \rightarrow e_x(k+1) = \underbrace{x_{k+1} - \hat{x}_{k+1}}_{e} = A_d \underbrace{(x_k - \hat{x}_k)}_{} - 0 - L_p C_d \underbrace{(x_k - \hat{x}_k)}_{}$$

$$e_x(k+1) = A_d e_x(k) - L_p C_d e_x(k)$$

$$e_x(k+1) = \underbrace{(A_d - L_p C_d)}_{\text{"closed-loop" dynamics, set by } \boxed{\mathrm{eig}(A_d - L_p C_d)}} e_x(k)$$

← poles of the estimator.

Looks similar to pole placement?

K: Controller feedback drives the states to zero

Let's compare:

- pole placement for state feedback

Recall $x_{k+1} = A_d x_k + B_d u_k$

where $\boxed{u_k = -Kx_k}$

$$\therefore \boxed{x_{k+1} = (A_d - B_d K)x_k}$$

← We pick K

- pole placement for estimator design

Now

$$\boxed{e_{x,k+1} = (A_d - L_p C_d)e_{x,k}}$$

← & L_p

L_p: Observer "feedback" drives the error in state estimate to zero.

MATLAB commands?

or "acker"

Let's compare:

- pole placement for state feedback

$$\underset{(n\times n)}{(A} - \underset{(n\times m)}{B}\underset{(m\times n)}{K)} \longrightarrow \boxed{K = \text{place}(A, B, p\,des)}$$

n states
m inputs

- pole placement for estimator design

$$\underset{(n\times n)}{(A} - \underset{(n\times \ell)}{L_p}\underset{(\ell\times n)}{C)} \Rightarrow$$

Note

$$\det|zI-(A-LC)| = \det|zI-(A-LC)^T|$$

where $(AB)^T = B^T A^T$

ℓ outputs

$$= \det|zI - (A^T - C^T L^T)|$$

So: $\boxed{L_p = (\text{place}(A', C', p))'}$ ← 3 "transpose" operators

← desired est. poles.

Observability (vs Controllability)

→ When can we set estimator poles arbitrarily?
"Controllability" question is ANALOGOUS to "Observability"

$$\mathcal{C} \equiv [B, AB, A^2B, \ldots, A^{n-1}B]$$

controllability matrix

$C_o = \mathrm{ctrb}(A, B)$

$K = \mathrm{acker}(A, B, pc)$

* Full rank means $x_i \to x_f$

(of C_o) – "controllable"

(of Ob) – "observable"

Observability matrix

$$\mathcal{O} \equiv \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

$Ob = \mathrm{obsv}(A, C)$

$L_p = \mathrm{acker}(A', C', po)'$

$L_c = \mathrm{acker}(A', A' * C', po)'$

P = "predictor"

"current" estimator.

$\boxed{L_p : \text{"predictor estimator"}}$

Observability

Example Say $C = [0, 1]$
measure position, only

CT: $A = \begin{bmatrix} -1, & -1 \\ 1, & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$X = \begin{bmatrix} \dot{x} \\ x \end{bmatrix}$, $\boxed{\ddot{x} = -\dot{x} - x}$

control canonical form

DT: $A_d \approx I + AT$
$B_d \approx BT$
$(T = 0.1s)$

$A_d = \begin{bmatrix} .9, & -.1 \\ .1, & 1 \end{bmatrix}$, $B_d = \begin{bmatrix} .1 \\ 0 \end{bmatrix}$

*Given n measurements (in y, $y_1 \to y_n$), can we mathematically calculate all n states in x_1? (" in x_n?)
(i.e., does this ~~look~~ like "n eqns", "n unknowns"?)

Say x_1 is not known.
But, u_K & y_K are known,
for each $K \geq 1$,

① $\boxed{y_1 = Cx_1}$ ← "x_1" $= x_{k=1} = \begin{bmatrix} \dot{x}_1 \\ x_1 \end{bmatrix}$

$y_2 = Cx_2 = C(A_d x_1 + B_d u_1)$

② $\boxed{(y_2 - CB_d u_1) = CA_d x_1}$

This IS the "Observability matrix"!

$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix}$

$\therefore \begin{bmatrix} y_1 \\ y_2 - CB_d u_1 \end{bmatrix} = \begin{bmatrix} C \\ CA_d \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ x_1 \end{bmatrix}$

e.g., $y_1 = 1$, $u_1 = 0$; $y_2 = 2$, $u_2 = 0$

$\underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0, & 1 \\ .1, & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ x_1 \end{bmatrix}}$

$\mathcal{O} = \begin{bmatrix} 0 & 1 \\ .1 & 1 \end{bmatrix}$

↓

$\mathcal{O}^{-1} = \begin{bmatrix} -10 & +10 \\ 1 & 0 \end{bmatrix}$

$1 = 0\dot{x}_1 + x_1$

$\therefore \boxed{x_1 = 1}$

$2 = .1\dot{x}_1 + \cancel{x_1}$ $x_1 = 1$

$\boxed{\dot{x}_1 = 10}$

$\Rightarrow$ if $\mathcal{O}$ is full rank... then we INVERT $\mathcal{O}$

$X_{k=1} = \mathcal{O}^{-1} \begin{bmatrix} y_1 \\ y_2 - CB_d u_1 \end{bmatrix}$

Knowing x_1, we can find x_n...

Note, $\boxed{X_{k=2} = A_d X_{k=1} + B_d u_{k=1}}$

Using the state estimate for feedback...

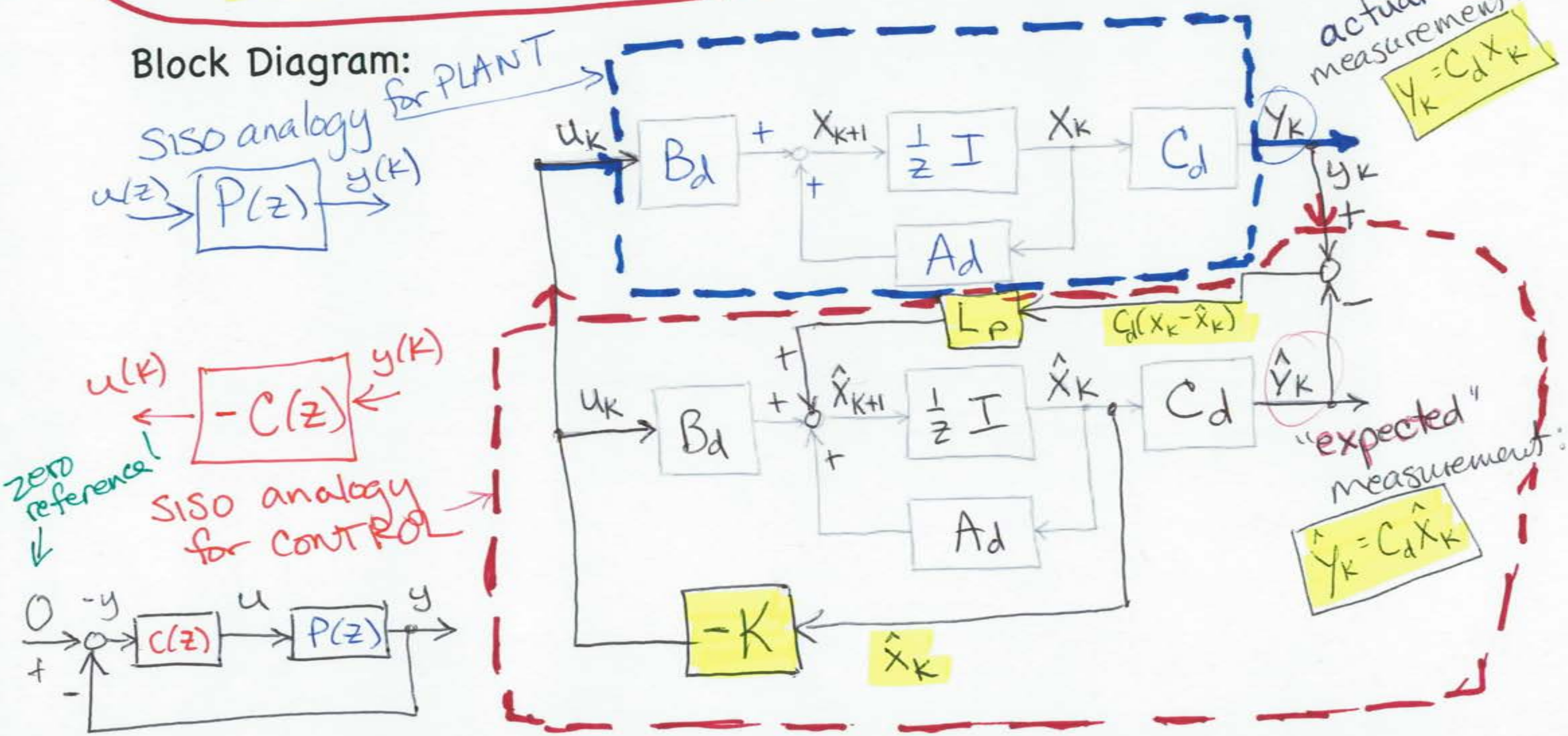
Recall STATE FEEDBACK Control uses the true state:

$$\boxed{u(k) = -Kx(k)}$$

What if we use the state ESTIMATE for feedback control, instead?

$$\boxed{u(k) = -K\hat{x}(k)}$$

Block Diagram:



Closed-loop Dynamics, for state (x) and error (e_x)

$y = Cx(k)$
$Ly(k) = LCx(k)$

$\leftarrow u = -K\hat{x}$

$e \equiv x - \hat{x}$

$\therefore \boxed{\hat{x} = x - e}$

$$x(k+1) = Ax(k) + Bu(k)$$
$$= \quad '' \quad - BK(x(k) - e(k))$$

(1)
$$\boxed{x(k+1) = (A - BK)x(k) + BKe(k)}$$

(A) $\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + L(y(k) - C\hat{x}(k))$
$$= (A - LC)\hat{x}(k) + Bu(k) + LCx(k)$$

(B) $(x(k+1) = Ax(k) + Bu(k))$

(B) − (A)
$$e(k+1) = [Ax + Bu] - [(A-LC)\hat{x} + Bu + LCx]$$
$$= (A - LC)(x(k) - \hat{x}(k))$$

(2)
$$\therefore \boxed{e(k+1) = (A - LC)e(k)}$$

↑ Error dynamics DO NOT depend on x_k

n^{th} order system (x) with
n^{th} order estimate ($\hat{x}$)
has 2n poles:
poles of "A−BK" } decoupled
poles of "A−LC"

MATRIX form for DYNAMICS:

(1)
$$\begin{bmatrix} x_{k+1} \\ e_{k+1} \end{bmatrix} = \begin{bmatrix} (A-BK) & BK \\ 0 & (A-LC) \end{bmatrix} \begin{bmatrix} x_k \\ e_k \end{bmatrix}$$
(2)

eigenvalues?

$$\text{eig}\left\{\begin{bmatrix} (A-BK), & BK \\ 0, & (A-LC) \end{bmatrix}\right\} = \ldots$$

"Upper Block Diagonal" form,

$$\ldots = \{\text{eig}(A-BK),\ \text{eig}(A-LC)\}$$

Example: continued

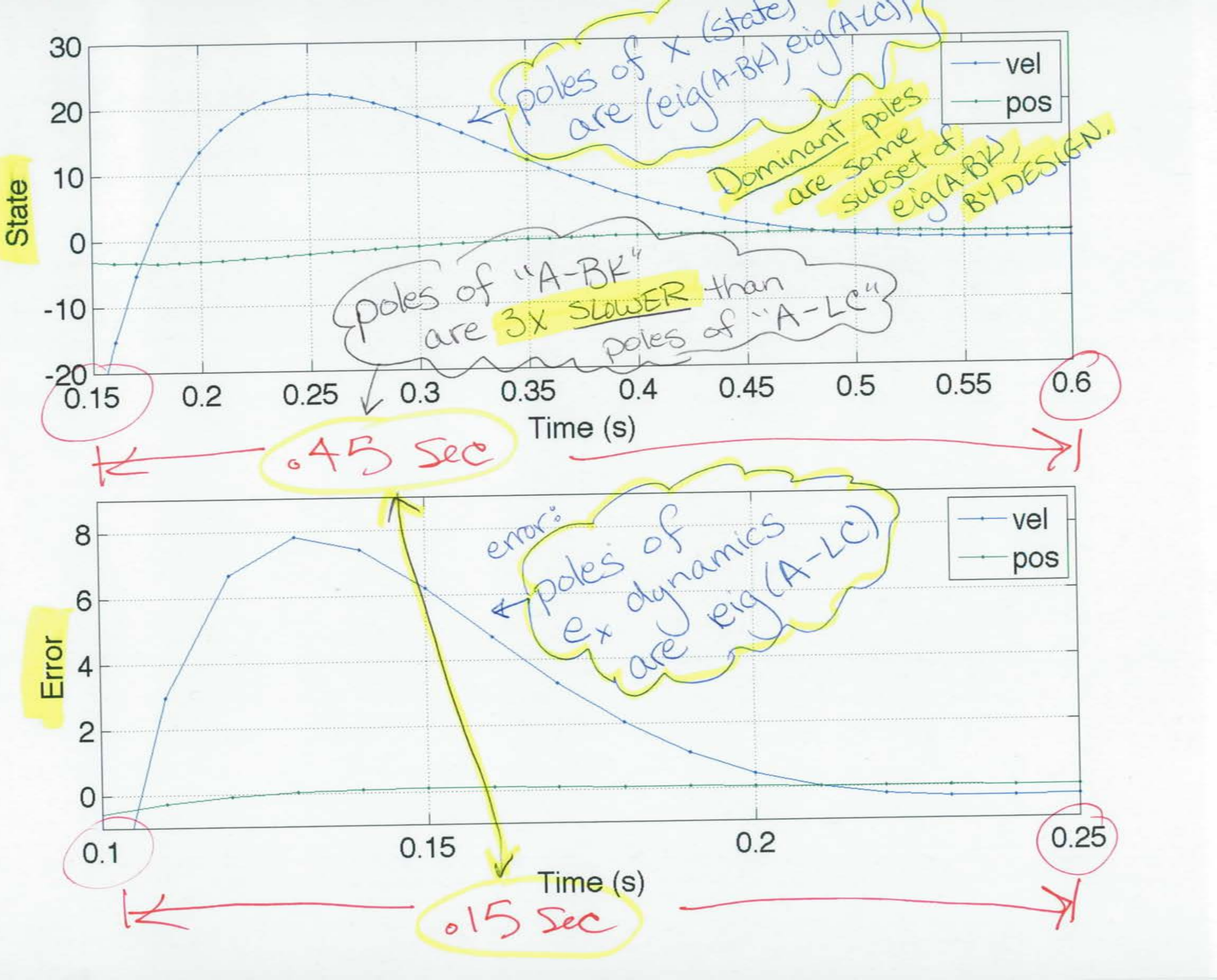
```
C = [0 1];
pe_ct = 3*pct;
pe_dt = exp(pe_ct*T);
Lpc = place(A',C',pe_ct)'
Lpd = place(Ad',Cd',pe_dt)'
Kd = place(Ad, Bd, pdt)

Xinit = [10;10];
Xest = [0;0];
N = 100
X = zeros(2,N); X(:,1) = Xinit;
Xe = X; Xe(:,1) = Xest;
u = zeros(1,N);
for n=1:N-1
    u(n) = -Kd*Xe(:,n);
    Xe(:,n+1) = (Ad-Lpd*C)*Xe(:,n) + Bd*u(n) + Lpd*C*X(:,n);
    X(:,n+1) = Ad*X(:,n) + Bd*u(n);
end
t = T*[0:N-1];
figure(1); clf; subplot(211)
plot(t,X,'.-'); subplot(212)
plot(t,X-Xe,'.-');
```

estimator poles "3x as fast" as controller poles

Start w/ error in estimate.

control laws ($u = -K\hat{x}$)

estimate update



State
Time (s)
vel
pos
poles of x (state) are {eig(A-BK), eig(A-LC)}
Dominant poles are some subset of eig(A-BK), BY DESIGN.
poles of "A-BK" are 3x SLOWER than poles of "A-LC"
.45 sec
Error
Time (s)
vel
pos
error: poles of e_x dynamics are eig(A-LC)
.15 sec

Separation Principle

For feedback w/ an estimator, $\begin{cases} \hat{x}_{k+1} = A_d \hat{x}_k + B_d u_k + L_p(y_k - \hat{y}_k) \\ u_k = -K\hat{x}_k \end{cases}$

① First pick (dominant) closed-loop poles via pole placement, for eig(A-BK).

② Then pick estimator poles, generally 2x - 4x "faster" than eig(A-BK)

est. poles are eig(A-LC)

③ TOTAL actual poles (2n poles for nth-order system) will be

$$\underbrace{(\text{eig}(A-BK), \text{eig}(A-LC))}_{\text{2 "separate", decoupled math problems!}}$$

How to place poles, to set "K" and Lp"?

Controller poles:

NEXT

TIME!

Estimator poles: