Today: Frequency Response Methods

Lecture 16

ECE 147B
Digital Control

Reading: RS-13 (!!!), FPW 7.4 <or> FV (6.4), 6.5, (6.6) (RS-1)

Topics:
- Reduced-Order Estimator (Review)
- Frequency Response Methods
- Reference Tracking

Next Class:

- Reference Tracking (continued)

- HW 7 (Practice Final) due Tues, 3/12
- See "annotated" Lect. 15 notes on Web

(if Handout hard to read!)

Reduced-order Estimator

What if you only want to "estimate" a subset of the entire state vector? e.g., You only want to estimate "unmeasured" states, which are not inside y. For example, in LAB:

Position: encoder - measured, x

Velocity: Not measured! ← unmeasured, $\dot{x}$ ← Estimate ONLY these!

Let's partition the states, $x(k) = \begin{bmatrix} x_a(k) \\ x_b(k) \end{bmatrix}$ a: directly measured

b: not measured ← only estimate $\hat{x}_b$ w/ red-ord. est.

we will get this directly from measurement

$$\underbrace{\begin{bmatrix} x_a(k+1) \\ x_b(k+1) \end{bmatrix}}_{\text{"}X_{k+1}\text{"}} = \underbrace{\begin{bmatrix} A_{aa} & A_{ab} \\ A_{ba} & A_{bb} \end{bmatrix}}_{A} \begin{bmatrix} x_a(k) \\ x_b(k) \end{bmatrix} + \underbrace{\begin{bmatrix} B_a \\ B_b \end{bmatrix}}_{B} u(k)$$

← x_b to be estimated.

states ↳ $\boxed{\#1}$ $\underbrace{x_b(k+1)}_{\text{unknown}} = \underbrace{A_{bb}\, x_b(k)}_{\text{unknown}} + \underbrace{\left(A_{ba}\, x_a(k) + B_b\, u(k)\right)}_{\text{Known}}$ (instead of just "Bu").

Error dynamics

estimates ↳ $\boxed{\#2}$ → write an eqn for $\hat{x}_b(k+1)$ ... & then $\boxed{\#3}$ is calculate by $\boxed{\#1}$ minus $\boxed{\#2}$

error = states - estimates.

Reduced-order Estimator

from last page, (MATRIX EQN)

* $\boxed{x_a(k+1) = A_{aa}x_a(k) + A_{ab}x_b(k) + B_a u(k)}$

Think of this as "Cx" in "y=Cx"...

measured

This looks a lot like

$$\boxed{\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + \cancel{Du(k)} \end{aligned}}$$

$\rightarrow$ ① $x_b(k+1) =$ ② $A_{bb}x_b(k) + (A_{ba}x_a(k) + B_b u(k))$ ③

$\rightarrow (x_a(k+1) - A_{aa}x_a(k) - B_a u(k)) = A_{ab}x_b(k)$

④ "y(k)" $=$ ⑤ $Cx(k)$

...if we make the following SUBSTITUTIONS:

Was $\rightarrow$ Now

1) ① $x(k) \rightarrow$ $x_b(k)$ ← states TO BE ESTIMATED - by definition.

state dyn.

2) ② $A \rightarrow$ A_{bb} ← a sub-part of the full "A" matrix.

← x_a & u both known @ k.

3) ③ $Bu(k) \rightarrow$ $(A_{ba}x_a(k) + B_b u(k))$ ← "Known inputs now include BOTH "Bu" and also terms depending on $x_a(k)$ [known/measured].

4) ④ $y(k) \rightarrow$ $(x_a(k+1) - A_{aa}x_a(k) - B_a u(k))$ ← What is left on LHS of * once "$A_{ab}x_b(k)$" (like "Cx") is moved to RHS.

5) ⑤ $C \rightarrow$ A_{ab} ← "$A_{ab}x_b(k)$" is now analogous to "Cx"

measurements

GOAL is to match THIS form:
(But estimate ONLY $x_b(k)$...)

$$\boxed{\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + L(y(k) - C\hat{x}(k))}$$

Use piece ① through ⑤ above!
Estimator eqn (w/ feedback).

Reduced-order Estimator

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + L(y(k) - C\hat{x}(k))$$ ← Traditional (not "reduced") form, which we wish to match.

$\boxed{\#2}$ $\hat{x}_b(k+1) = A_{bb}\hat{x}_b(k) + (A_{ba}x_a(k) + B_b u(k)) + L_r(\underbrace{[x_a(k+1) - A_{aa}x_a(k) - B_a u(k)]}_{= A_{ab}x_b(k)} - A_{ab}\hat{x}_b(k))$

↑ estimate

We have a new estimator equation, as desired, but there is a PROBLEM! We do not have access to "$x_a(k+1)$" yet! (Causality...)
(To be discussed NEXT PAGE...)

Uh oh!

Don't want "k+1"!

from eqn $\boxed{*}$, last page...

So, use "place" with
$\left\{ \begin{array}{l} \text{"A"} = A_{bb} \\ \text{"C"} = A_{ab} \end{array} \right\}$ ← or "acker"

Recall:

$\boxed{\#1}$ $x_b(k+1) = A_{bb}x_b(k) + (A_{ba}x_a(k) + B_b u(k))$

↑ dynamics.

Error Dynamics? $e_{xb} \equiv x_b - \hat{x}_b$, So, take $\boxed{\#1}$ minus $\boxed{\#2}$

$$x_b(k+1) - \hat{x}_b(k+1) = A_{bb}(x_b(k) - \hat{x}_b(k)) - L_r(A_{ab}x_b(k) - A_{ab}\hat{x}_b(k))$$

$\therefore$ $\boxed{e_b(k+1) = [A_{bb} - L_r A_{ab}]\, e_b(k)}$ ← poles set by $\boxed{\text{"}A_{bb} - L_r A_{ab}\text{"}}$

↑ "A" − ↑ "LC"

Reduced-order Estimator

Issue: How can we included "$x_a(k+1)$" in our actual implementation of the reduced-order estimator?

Ans: Let $\boxed{x_c(k) \equiv \hat{x}_b(k) - L_r x_a(k)}$

Get $\hat{x}_b(k+1)$ from #2

And note, $x_a(k+1) = A_{aa}x_a(k) + A_{ab}x_b(k) + B_a u(k)$

$$\therefore x_c(k+1) = \hat{x}_b(k+1) - L_r x_a(k+1)$$

cancels out!

$$= \left(A_{bb}\hat{x}_b(k) + (A_{ba}x_a(k) + B_b u(k)) + L_r(\cancel{A_{ab}x_b(k)} - A_{ab}\hat{x}_b(k)\right) -$$

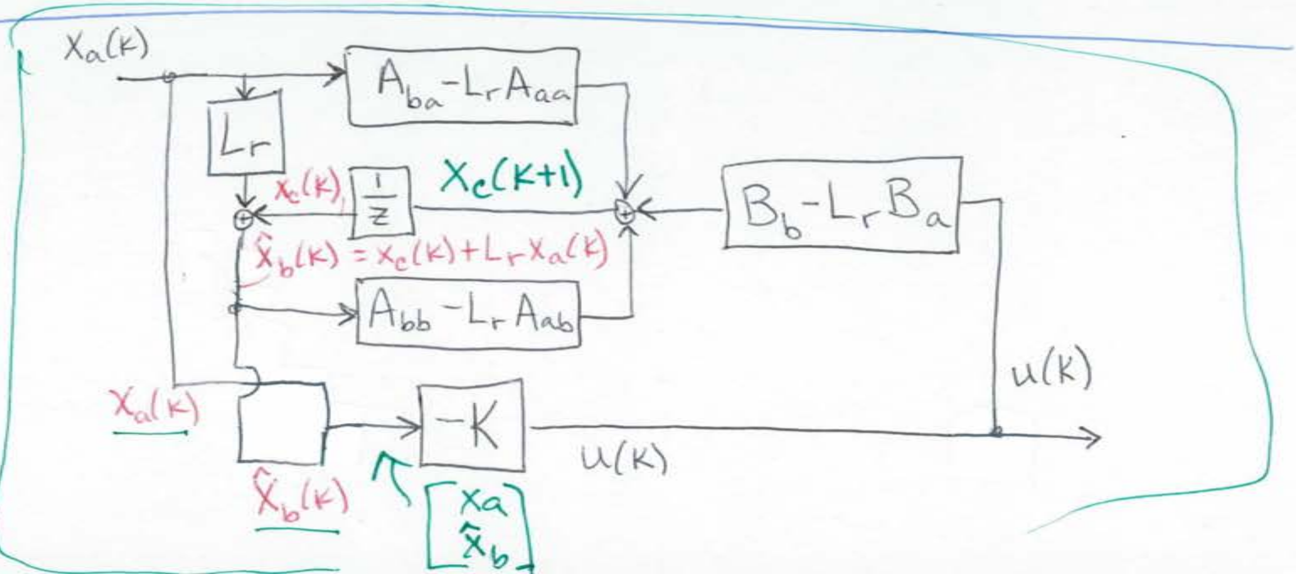
$$L_r(A_{aa}x_a(k) + \cancel{A_{ab}x_b(k)} + B_a u(k))$$

Regrouping:

$$\boxed{x_c(k+1) = (A_{bb} - L_r A_{ab})\hat{x}_b(k) + (A_{ba} - L_r A_{aa})x_a(k) + (B_b - L_r B_a)u(k)}$$

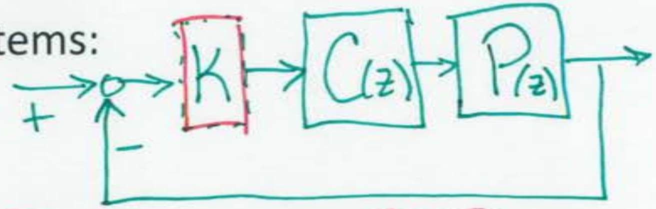
And then, $\hat{x}_b(k) = x_c(k) + L_r x_a(k)$

$$u(k) = -K\begin{bmatrix} x_a(k) \\ x_b(k) \end{bmatrix}$$



Frequency Response Methods

For what "K" is the system stable?



4. More details on NYQUIST CRITERION for DT systems:

e.g. $N_{cw} = 1$ here

$$\boxed{Z = P + N_{CW}}$$

Unstable closed-loop poles of "$\frac{KCP}{KCP+1}$"

i.e., # RHP zeros in "KCP+1"

unstable open-loop poles in "KCP"

clockwise encirclements of "−1" on a Nyquist plot in "loop transmission"

5. Rules to sketch a Bode plot for a discrete-time (DT) system, G(z), BY HAND.
   a) Use "z=1" to evaluate DC gain and phase.
   b) Use "z=-1" to evaluate gain and phase at NYQUIST frequency.
   c) Use "$z=e^{sT}$" mapping: $z = e^{sT}$ $\qquad s = \frac{1}{T}\log(z)$
   d) For poles and zeros "near" z=1 (within .15 "or so"):

$$z \approx 1 + sT \quad \text{approx.}$$

$$\boxed{s \approx \frac{1}{T}(z-1)}$$

to map z-plane poles & zeros to s-plane.

   e) Don't confuse "half-sample" delay stuff into all this! That is for the SPECIFIC CASE of taking a CT plant, G(s), and preceding it with a ZOH and then sampling the output…

Freq. Resp. - Examples

Use MATLAB to check these!!

$$G_1(z) = \frac{K}{(z-0.9)(z-0.9)} = \frac{K}{z^2 - 1.8z + 0.81} \Rightarrow$$

char. eqn. after "closing the loop":

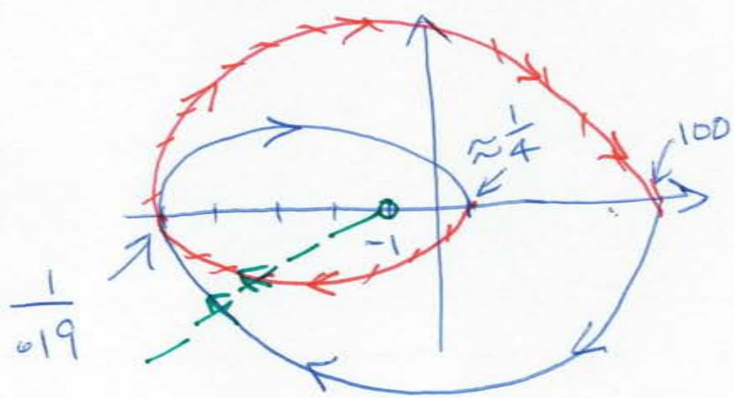
$$\boxed{z^2 - 1.8z + 0.81 + K = 0}$$

(1) Stable for what "K" values?

$\therefore\ ? < K < 0.19$

$\boxed{\text{Say } T = .01 \text{ sec}}$

(3) Sketch Nyquist for "K=1" case.



$\boxed{N_{CW} = 2}$ $\quad P = \emptyset$ (open-loop)

$\therefore\ Z = 2 + 0 = 2$
(# unstable CL poles for K=1.)

(2) Sketch Bode for "K=1" case.

DC gain? $(z=1)$ $\quad \frac{1}{(1-.9)(1-.9)} = \left(\frac{1}{.1}\right)^2 = 100$

DC phase? $+100 \Rightarrow$ so $0°$ phase

Gain @ Nyquist? $(z=-1)$ $\quad \frac{1}{(-1-.9)(-1-.9)} \approx \left(\frac{1}{-2}\right)^2 \approx \frac{+1}{4}$

$\left(\frac{1}{-1.9}\right)^2$, exactly...

* Phase @ ω_{Ny}? $0°$? or $-360°$?

Just look @ $\frac{1}{(z-0.9)}$ @ $z=-1 \Rightarrow \frac{1}{-1-.9} = \frac{1}{-1.9} \approx \frac{-1}{2}$

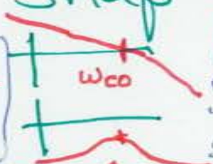
$\boxed{-180° \text{ phase}}$

$\therefore\ \boxed{G_1(z) \text{ is @ } -360° \text{ @ } z=-1}$

Frequency Response Methods

e.g. "Loop Shaping"

① High gain @ Low Freq. ∴ Low SS error



② "Gentle" slope @ c.o.

③ "Nice" phase margin. ∴ Low "overshoot"

We focus more on what is "New" about control design for discrete-time systems (as compared with design for continuous-time systems…)

HOWEVER, "loop shaping" is still a valid approach! (i.e., gain and phase margin…)

Below are Quick Tips, summarizing the key ideas to apply toward DT systems.

1. Recall how to evaluate frequency response for some transfer function, "G(z)":

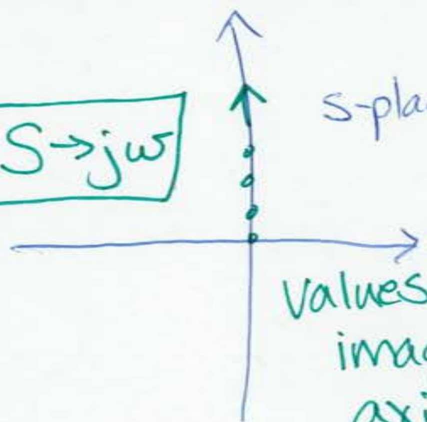
CT

$G(s)$

$s \to ?$

$\left(\text{e.g. } G(s) = \frac{K}{(s+a)}\right)$

$\boxed{s \to j\omega}$



Values on imag. axis.

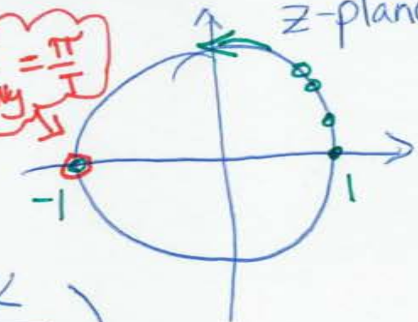
DT

$\boxed{z \Rightarrow e^{jwT}}$

$\omega_{Ny} = \frac{\pi}{T}$

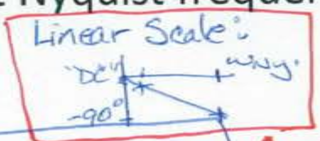
$G(z)$

$\left(\text{e.g., } G(z) = \frac{K}{z-a}\right)$



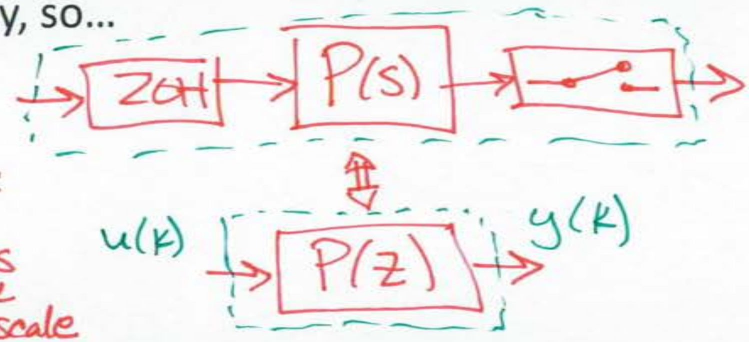
2. Primary effect of a ZOH on frequency response is the HALF-SAMPLE delay:
-90 degree change at Nyquist frequency, so…

$\frac{-\pi}{2} @ \omega_{Ny} = \frac{\pi}{T}$



General rule: $\boxed{\Delta\phi \cong \frac{-\omega T}{2}}$

↑ Note: Bode plots have $\log x$ scale



Frequency Response Methods

3. Only SOME rules for CT Bode (and Nyquist) plots STILL apply (for DT plots):

CT	DT
a. Can plot gain and phase BY HAND easily.	a. No. (well, not "directly"…) First, try to "map" $G(z)$ to some $G(s)$
b. Can take an EXPERIMENT Bode plot. → Can be used for Loop Shaping.	B. Yes! $\rightarrow \boxed{P(z)=?} \rightarrow$ {similar to "pole-zero" mapping}
c. Nyquist stability criterion applies… $\boxed{Z = P + N_{cw}}$	c. Yes – but be careful [See next page…] Use "N_{cw}" to define "encirclements".
d. Gain and phase margins can be used to interpret closed-loop response.	d. Yes! (Loop Shaping)
e. Error constants (Kp or Kd) can be read DIRECTLY from Bode plot.	d. Yes!
f. Gain and phase curve computed easily for extra compensator poles or zeros.	e. No… X asymptotes no longer apply…
g. Effect of pole or zero in compensator easy to predict via Bode sketch.	f. No… X

All the same issue.

Freq. Resp. - Examples

Use MATLAB to check these!!

Sketch Bode plot BY HAND: $G_2(z) = \dfrac{10(z-0.98)}{(z-0.9)}$, w/ $\boxed{T = 0.01 \text{ sec}}$

$s \approx \frac{1}{T}(z-1)$ ← s-plane approximate equivalents.

zero first,

Zero: $z = +0.98$, $s \approx \frac{1}{.01}(.98-1) = \frac{-.02}{.01} \approx -2 \left(\frac{rad}{s}\right)$

Pole: $z = +0.9$, $s \approx 100(.9-1) = -10 \,(rad/s)$

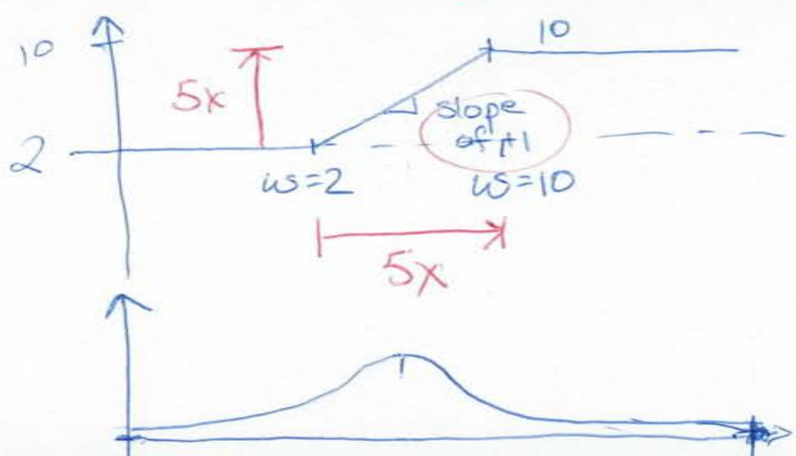
← ...then pole.

∴ LEAD controller (adds phase)

@DC ($z=+1$), $10\left(\frac{1-.98}{1-.9}\right) = 10\left(\frac{.02}{.1}\right) = +2$ ← Gain = 2, ← Phase = 0°

@ω_{Ny} ($z=-1$), $10\left(\frac{-1-.98}{-1-.9}\right) \approx 10$ } not quite... Gain = 10, Phase = 0° (or -360°?)

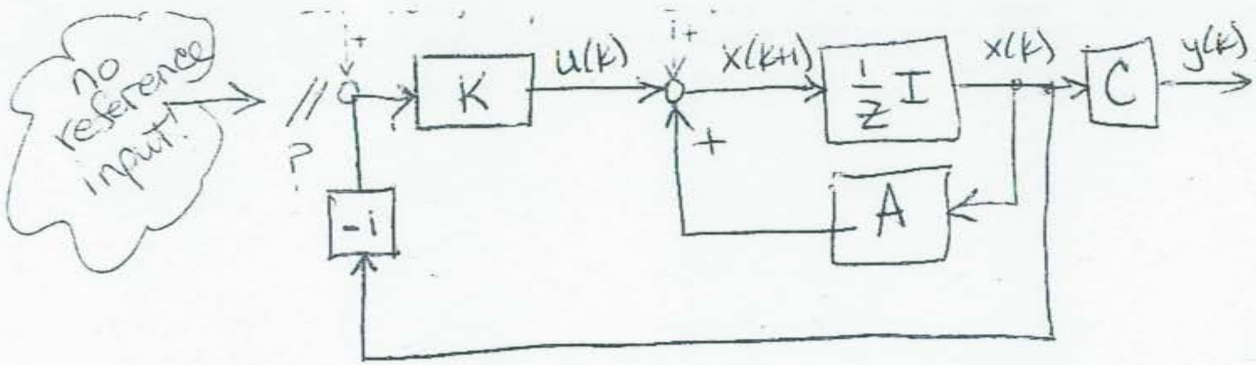
Bode



$\omega_{Ny} = \frac{\pi}{T} = 314 \text{ rad/sec.}$

Reference Tracking

So far, we have only considered state space control for a “regulator”:



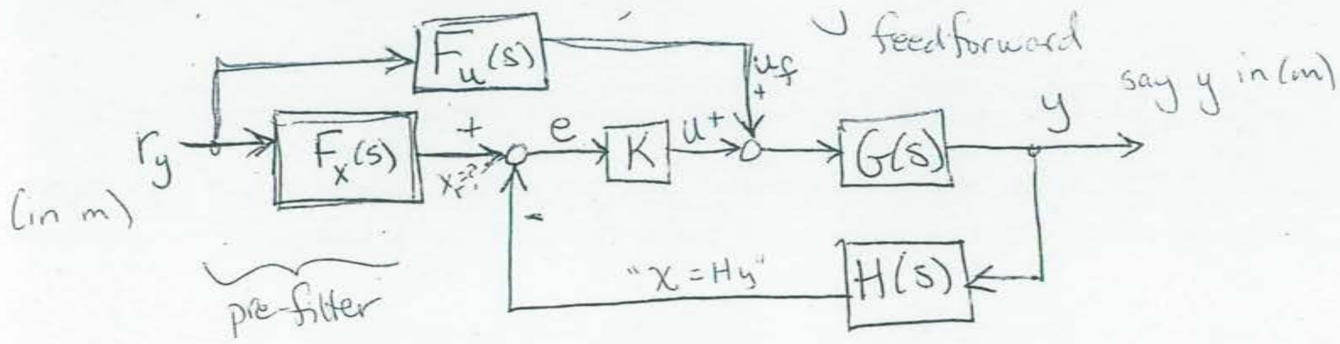
Questions: How can we

1) Compare x(k) (or an ESTIMATE of x(k)) with a desired REFERENCE, $x_r(k)$?

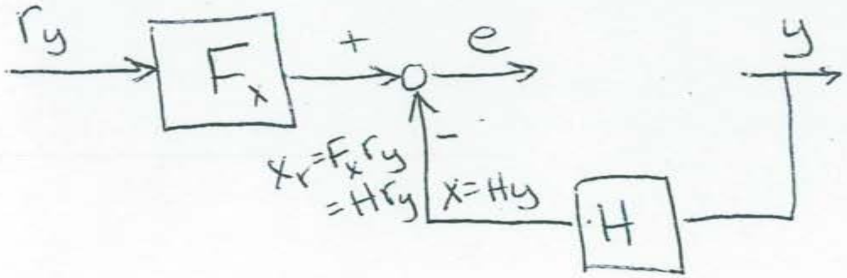
2) Generate a feedforward term, $u_f(k)$, to eliminate steady state error?

Reference Tracking

First, consider the familiar "SISO" system case:



F_x should scale output "y" to match (in units) the feedback term:

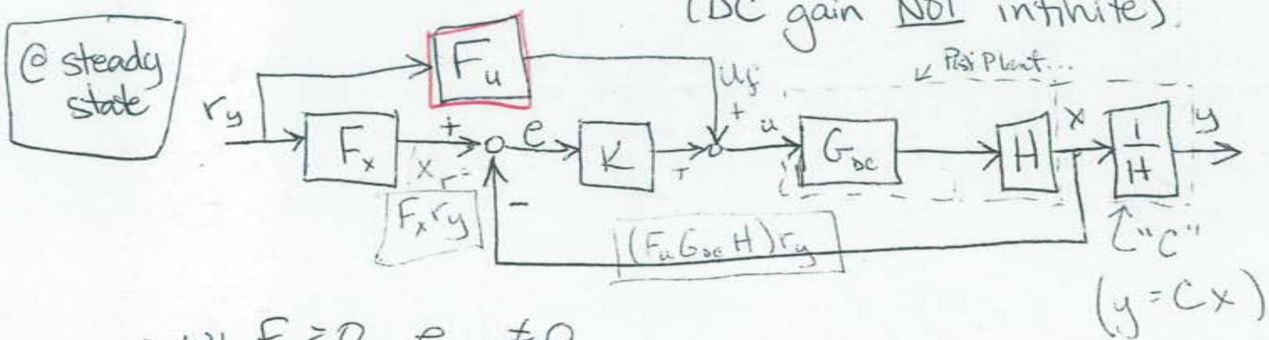


$\boxed{F_x = H}$

Reference Tracking

...And Fu should result in NO FEEDBACK ERROR at steady state:

(DC gain NOT infinite)



$$\boxed{F_u = \frac{1}{G}}$$

- w/ $F_u = 0$, $e_{ss} \neq 0$.
- If we want $e_{ss} = 0$, $F_x r_y - (F_u G_{DC} H) r_y = 0$

MIMO equivalent:

Now, consider a MIMO (DT) state space system form: