Today: **Optimal Control and Estimation**

Lecture 18

ECE 147B
Digital Control

Reading: FPW 9.3.5, 9.4.1 <or> FV 10.2-10.4
<AND> RS-16

N_x & N_u

$\frac{K_I}{z-1}$ (DT) OR $\frac{K_I}{s}$

Topics:

- Review: Reference Tracking and Integral Control
- Optimal Control: Linear Quadratic Regulator (LQR)
- Optimal Estimation: The Kalman Filter
- MATLAB Examples

Goal: understand meaning (& effect) of **Q** and **R** matrices, for each case (control vs estimation).

Next Class:

- Review and Q & A

*Review!

Reference Tracking (for State Space Control), Review

n states in x (n×1)
m inputs in u (m×1)
ℓ outputs in y (ℓ×1)

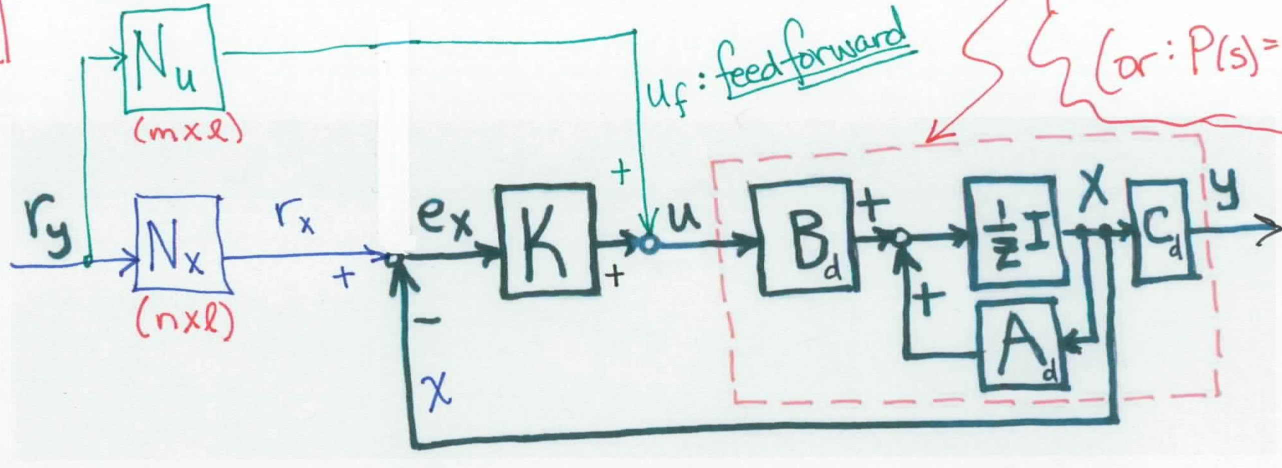
r_y already "matches" y (desired size & values).

$$\underset{(n\times 1)}{\underline{\underline{r_x}}} = \underset{(n\times \ell)}{\boxed{N_{\underline{x}}}}\ \underset{(\ell\times 1)}{r_y}$$

$$\underset{(m\times 1)}{\underline{u_f}} = \underset{(m\times \ell)}{\boxed{N_{\underline{u}}}}\ \underset{(\ell\times 1)}{r_y}$$

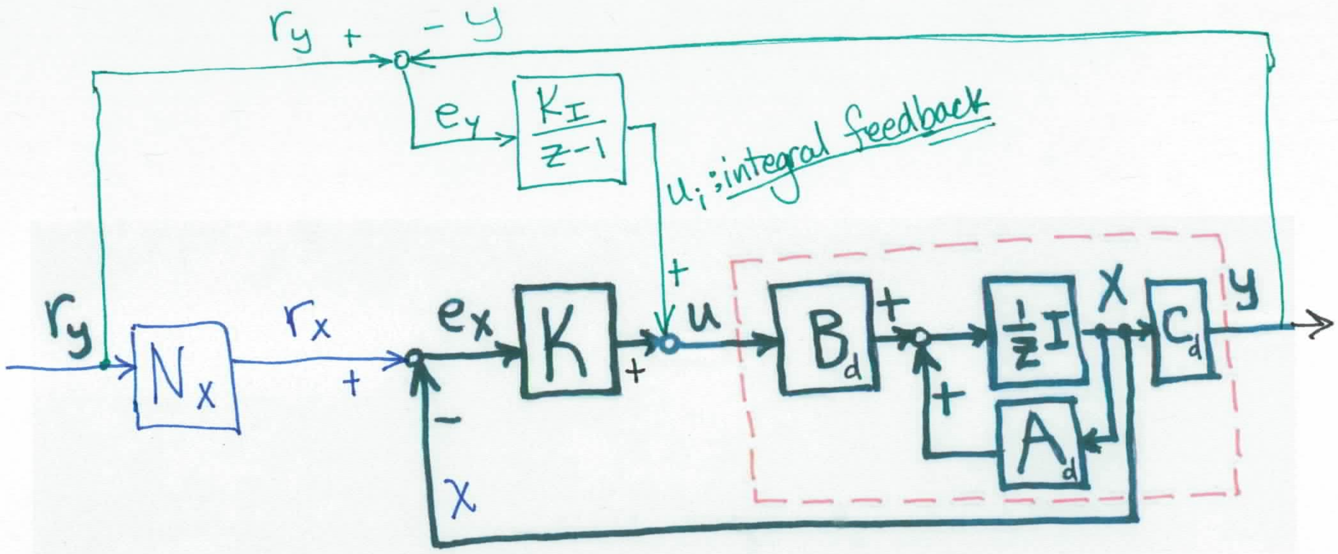
For a SISO system, this is: $P(z) = \frac{Y(z)}{U(z)}$

(or: $P(s) = \frac{Y(s)}{U(s)}$)



Integral (State Space) Control, Review

r_x must still "match" x



one-page summary!!

"$u = -Kx$"

Optimal Control: Linear Quadratic Regulator

Linear: Linear System ($xz = A_d x + B_d u$) ← Linear Difference Equations (or linear differential, for CT case...)

Quadratic: Squared (quadratic) cost to be minimized ("x^2" (states) & "u^2" (actuations) terms)

Regulator: Regulate all states to zero (no reference), as steady state aim.

Goal is to minimize a quadratic COST FUNCTION over all (future) time, which penalizes both:

(1) ERRORS (in the states, x) and

(2) EFFORTS (in the inputs, u):

$$J_{LQR} = \sum_{k=0}^{\infty} \underbrace{\Big[\underset{(1\times n)}{x_k^T} \underset{(n\times n)}{Q} \underset{(n\times 1)}{x_k} + \underset{(1\times m)}{u_k^T} \underset{(m\times m)}{R} \underset{(m\times 1)}{u_k} \Big]}_{(1\times 1):\ J \text{ is a scalar.}}$$

Increase Q → makes K larger, poles FASTER.

Increase R → K gets smaller, poles SLOWER. (closed-loop)

Where our CONTROL is the usual STATE FEEDBACK:

$$\boxed{u = -K_{LQR}\, x}$$

Usually, Q & R are DIAGONAL.

Then: $J = \sum \Big[Q_{11} \underbrace{x_{1,k}^2}_{(x_1(k))^2} + Q_{22} x_{2,k}^2 + \ldots + R_{11} u_{1,k}^2 + \ldots \Big]$

Q and R can be set using BRYSON'S RULE:

Bryson's rule "normalizes" x & u by the "largest acceptable values" for each:

$$Q = \begin{bmatrix} Q_{11} & & 0 \\ & Q_{22} \ddots & \\ 0 & & Q_{nn} \end{bmatrix} \rightarrow Q_{ii} = \left[\frac{1}{\underbrace{\max(x_i(k))}_{\text{max acceptable value}}} \right]^2 ; \quad R = \begin{bmatrix} R_{11} & & 0 \\ & \ddots & \\ 0 & & R_{mm} \end{bmatrix} \rightarrow R_{ii} = \left[\frac{1}{\max(u_i(k))} \right.$$

Note: Q & R matrices again... but different meaning.

one-pager!

Optimal Estimation: Linear Quadratic Estimation

K: Kalman

time step k

This optimal estimator is also called a "Kalman filter".

The full Kalman filter is a 5-step procedure, where $L_K(k)$ has a different optimal solution at each time step, k... and CONVERGES to $L_{K,ss}$

↖ steady state

Goal is to minimize VARIANCE in the estimated states.

We now assume there is both:

(1) PROCESS noise, in the state dynamics: $x(k+1) = A_d x(k) + B_d u(k) + Gw(k)$

↖ process noise

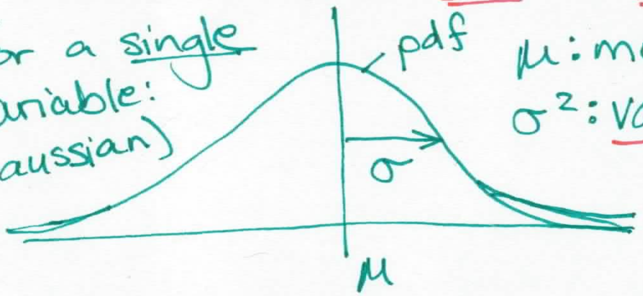
(2) MEASUREMENT noise, affecting y(k): $y(k) = C_d x(k) + \cancel{D_d u(k)} + v(k)$

↖ measure noise

(a) (b)

We will assume both w(k) and v(k) are Gaussian, white noise sources.

(a) For a single variable: (Gaussian)



μ: mean

σ^2: variance

(b) No "history" for white noise.

No time correlation (for samples).

Zero mean, ∴ $\mu = 0$

But the different states $(x_1(k), x_2(k) \ldots)$ can have covariance:

$\boxed{Q = E(w^T w);\ R = E(v^T v)}$

Optimal solution for estimator gains, L, depends of total current variance of the state estimates, before a measurement (P^-), compared with measurement variance (R):

For a single variable (again)

$$\boxed{L = \frac{\sigma_{p^-}^2}{\sigma_{p^-}^2 + \sigma_m^2}}$$

← As $\sigma_p \uparrow$, L goes ↓...

total variance of estimate a priori (p^-) the measurement

variance of the measurement noise

$$\boxed{\frac{P^-_{\text{a priori}}}{P^-_{\text{a priori}} + R} = L}$$

measurement variance

variance in estimate BEFORE measurement

$= (1 - L_K) P^-$

$$\boxed{P^+_{\text{a posteriori}} = \frac{P^- \cdot R}{P^- + R}}$$

Var. in est. After measurement

More on covariance matrices Q and R...

Both the optimal Control and optimal Estimation assume noise is Gaussian and white.
For Both problems:

Q (nxn) is a covariance matrix for STATES, x

R (ℓxℓ) is a covariance matrix for OUTPUTS, y ← ABSOLUTE Q & R can be scaled! → "The cost function "J" is still minimized with the SAME SOLUTION when Q & R are each multiplied by the same, scalar CONSTANT!"

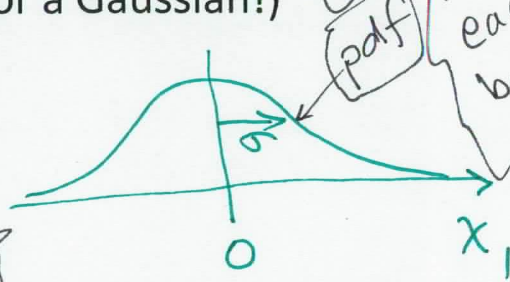
Q and R must therefore be symmetric, positive semi-definite matrices.
(You can't have a "negative" standard deviation or variance for a Gaussian!)

"sigma1" "sigma2" use: N=1e3, to plot

MATLAB examples.

```
mu1 = 2; mu2 = 1; N=1e6;
x1 = normrnd(0,mu1,1,N);
x2 = normrnd(0,mu2,1,N);
figure(1); clf; subplot(231)
plot(x1,x2,'b.'); axis equal
```

zero mean ($\mu=0$)
σ = "mu1"
{bad variable name choice!}



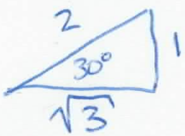

```
A = 30*pi/180;
x1a = cos(A)*x1 - sin(A)*x2;
x2a = sin(A)*x1 + cos(A)*x2;
subplot(232);
plot(x1a,x2a,'b.'); axis equal
```

New variables are a linear combo of old ones (x1 & x2)

```
x1b = 4*x2;
x2b = 3*x2;
subplot(233); plot(x1b,x2b,'b.'); axis equal
```

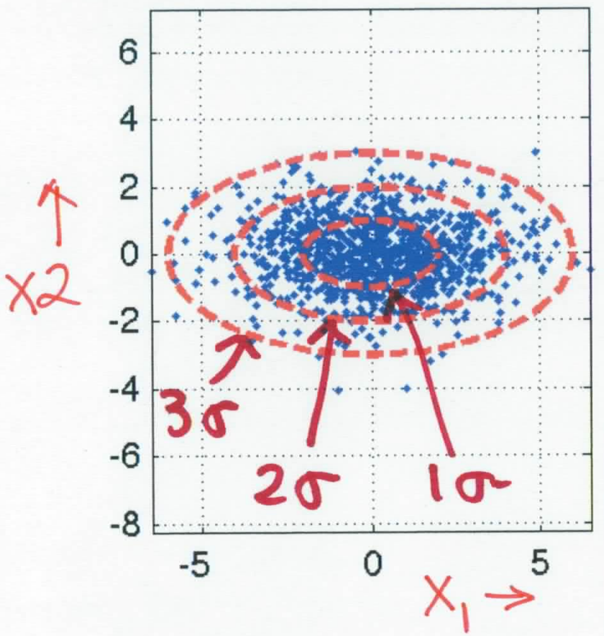
Each a scaled version of just x2.

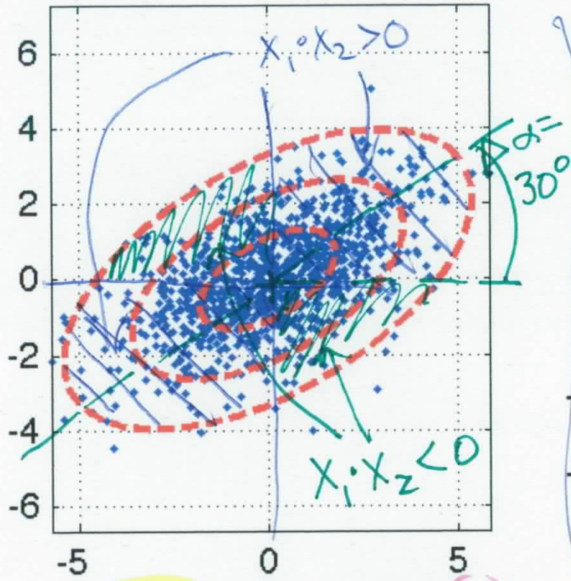
More on covariance matrices Q and R...

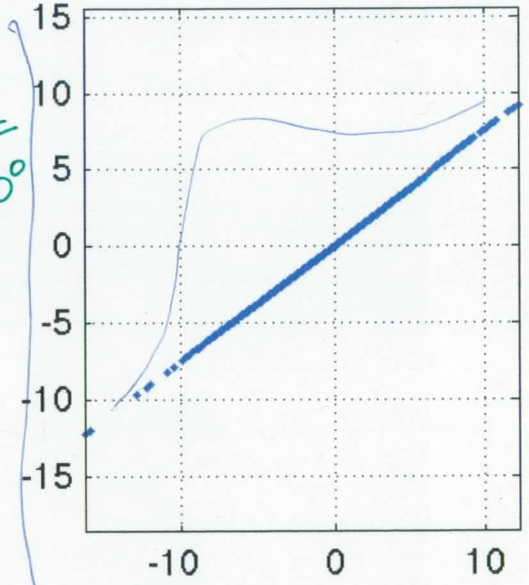


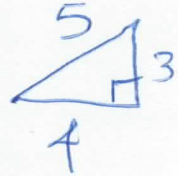
$\cos\alpha = \frac{\sqrt{3}}{2}, (\cos\alpha)^2 = \frac{3}{4}$

$\sin\alpha = \frac{1}{2}, \sin^2\alpha = \frac{1}{4}$









X_1 & X_2 are **independent** Gaussians.

Covariance?

∴ zero offdiag. values

$$Q = \begin{bmatrix} \sigma_{x_1}^2 & 0 \\ 0 & \sigma_{x_2}^2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

check with MATLAB:

$\boxed{Q = \text{cov}(X1, X2)}$

$X1a = \cos\alpha \cdot X_1 - \sin\alpha \cdot X_2$

$X2a = (\sin\alpha) \cdot X_1 + (\cos\alpha) X_2$

$\cos(\frac{\pi}{6})\sin(\frac{\pi}{6}) = .433 * (2)^2 = 1.732$

$= -.433 * (1)^2 = -.433$

$Q_{11} = \cos^2\alpha \cdot \sigma_{x_1}^2 + \sin^2\alpha \cdot \sigma_{x_2}^2$

$= (\frac{3}{4}) \cdot 4 + (\frac{1}{2}) \cdot 1$

$E(x_{1a}^2) = 3.25$

$Q_{22} = 1.75$

$Q_{12} = Q_{21} = 1.299$

covariance: $(E(x_1 * x_2))$

$$Q = \begin{bmatrix} 3.25, & 1.299 \\ 1.299, & 1.75 \end{bmatrix}$$

$\boxed{Q = \text{cov}(x1a, x2a)}$

$$Q = \begin{bmatrix} 4^2 & 3 \cdot 4 \\ 3 \cdot 4 & 3^2 \end{bmatrix} = \begin{bmatrix} 16, & 12 \\ 12, & 9 \end{bmatrix}$$

$\boxed{Q = \text{cov}(x1b, x2b}$

LQR Toy Example [Optimal Control Gains]

Let us consider a simple, SISO system, where K will be a SCALAR value.

$$\frac{Y(s)}{U(s)} = \frac{X(s)}{U(s)} = \frac{1}{s+1} \Rightarrow$$

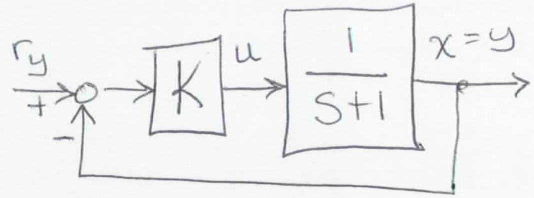
$$\frac{Y(s)}{R_y(s)} = \frac{K}{s+1+K}$$

$X(s+1) = u$

$\dot{X} + X = u$

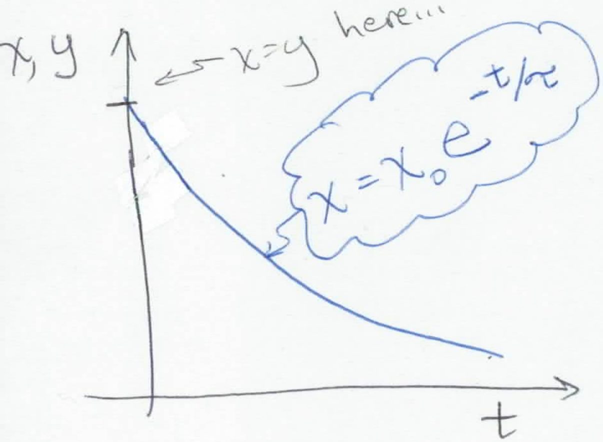
$\left.\begin{aligned} sX &= -X + u \\ &= AX + Bu \end{aligned}\right\}$ $A = -1$, $B = 1$

$\left.\begin{aligned} y &= X \\ &= CX \end{aligned}\right\}$ $C = 1$, $D = 0$



$\therefore \frac{Y(s)}{R_y(s)} = \frac{K}{s+(1+K)}$

pole: $\boxed{\left.\begin{aligned} s &= -(1+K) \\ \tau &= \frac{1}{K+1} \end{aligned}\right\}}$ first-order closed-loop system...



For a continuous-time (CT) system,

$$J = \int_0^\infty \left[Qx^2 + Ru^2 \right] dt = Q\int_0^\infty x^2 dt + R\int_0^\infty u^2 dt$$

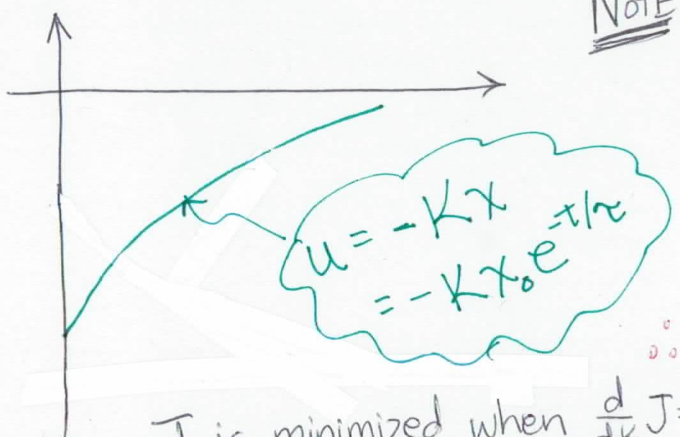
NOTE: x & u are scalar variables, so Q & R are each (1×1).

$$\int_0^\infty y^2 dt = \int x^2 dt = \int \left[e^{-t/\tau}\right]^2 dt = \int_0^\infty e^{-2t/\tau} dt = \frac{-\tau}{2} e^{-\frac{2t}{\tau}}\Big|_0^\infty = \frac{+\tau}{2}$$

$-\frac{\tau}{2}(0-1)$

$$\int_0^\infty u^2 = \int_0^\infty (-Kx)^2 dt = \int K^2 x^2 dt = K^2 \int x^2 dt = K^2 \frac{\tau}{2}$$

Recall, $\boxed{\tau = \frac{1}{K+1}}$



$$\therefore J = Q\left(\frac{\tau}{2}\right) + R\left(K^2 \frac{\tau}{2}\right) = \frac{1}{2}\left(Q + RK^2\right)\left(\frac{1}{K+1}\right)$$

J is minimized when $\frac{d}{dK}J = 0 = \frac{d}{dK}\left(\frac{Q+RK^2}{K+1}\right) = 2RK(K+1)^{-1} - (Q+RK^2)(K+1)^{-2} = \frac{2RK(K+1) - (Q+RK^2)}{(K+1)^2} = 0$

So, solve this quadratic eqn for "K". Two solutions...? ← $\left\{\begin{aligned} &2RK^2 + 2RK - Q - RK^2 = 0 \\ &\boxed{RK^2 + 2RK - Q = 0} \end{aligned}\right.$

* Take the solution for K with STABLE closed-loop poles!!

LQR Toy Example [Optimal Control Gains]

$\boxed{\tau = \frac{1}{K+1}, \therefore \tau = 1 \text{ for } K = 0 \text{ (open-loop)}}$

pole @ $s = \frac{-1}{\tau}$

Check this in MATLAB:

only R changes...

```
A=-1; B=1; C=1; D=0;
Q=1; R=1;
Kb = lqr(A,B,Q,R)
```

$= 0.4142$

```
pb = eig(A-B*Kb)
```

$= -1.4142$ ← closed-loop pole

$= -1 - K_b$

```
R=0.1;
Ka = lqr(A,B,Q,R)
```

$= 2.3166$

```
pa = eig(A-B*Ka)
```

$= -3.3166 = -1 - 2.3166$

```
R=10;
```

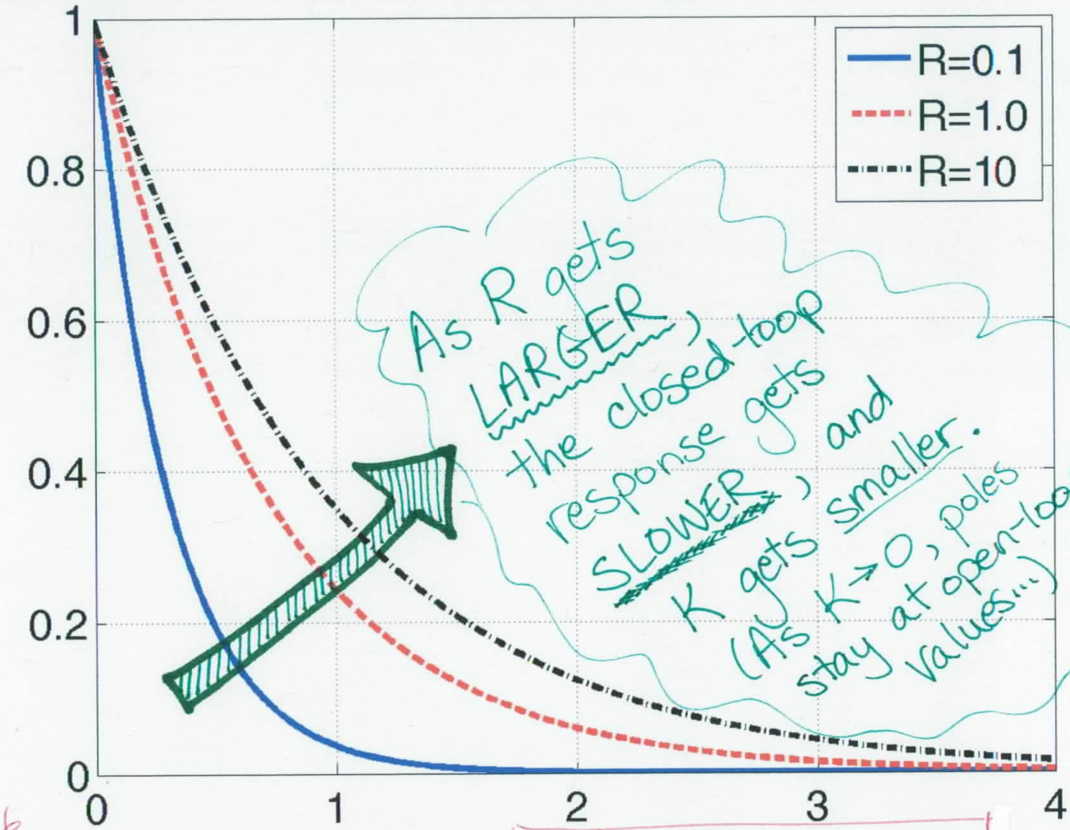
← R is "large"...

```
Kc = lqr(A,B,Q,R)
```

$= 0.0488$ ← ∴ K is "small"

```
pc = eig(A-B*Kc)
```

$= -1.0488$ ← with "large" R penalty, closed-loop pole ≈ −1 (open-loop pole)



$\boxed{RK^2 + 2RK - Q = 0}$

```
Ktest = -roots([R  2*R  -Q])
```

	R=0.1	R=1.0	R=10
K_{test} (2 values for "roots")	$\cancel{-4.3166}$ ← unstable and 2.3166 ← stable	$\cancel{-2.4142}$ and 0.4142	$\cancel{-2.0488}$ ← Results in an unstable pole! and 0.0488
K, from MATLAB LQR / τ_{CL} (closed-loop)	2.3166 / $\tau = .3015$	0.4142 / $\tau = .707$	0.0488 / $\tau = .9535$

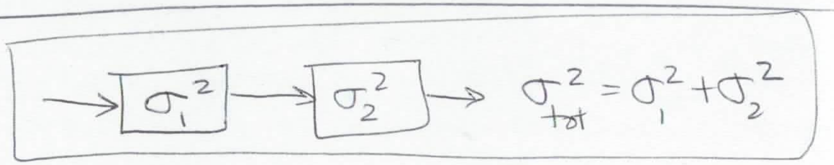
Kalman Filter Toy Example [Optimal Estimation]

The elegance of the Kalman filter depends on the assumption of GAUSSIAN, WHITE NOISE. ← zero mean; not correlated over time.

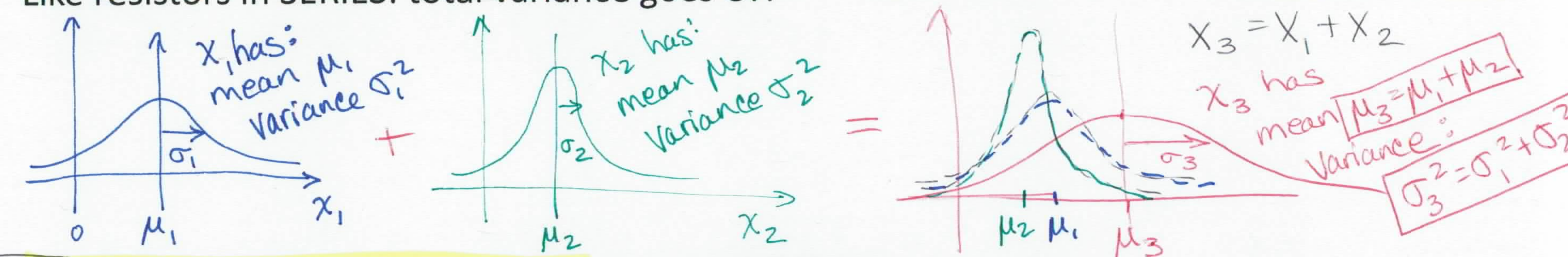
↑ Only μ & σ^2 ← (cov matrix) needed for "bookkeeping"...

There are two way to combine Gaussians...

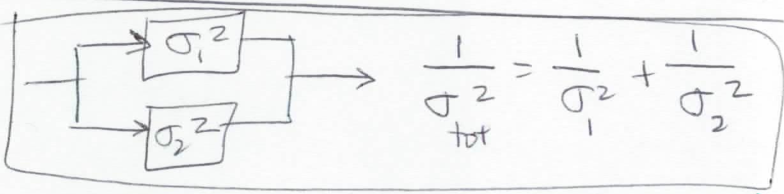
(1) Two noisy variables get ADDED to one another.
Like resistors in SERIES: total variance goes UP.



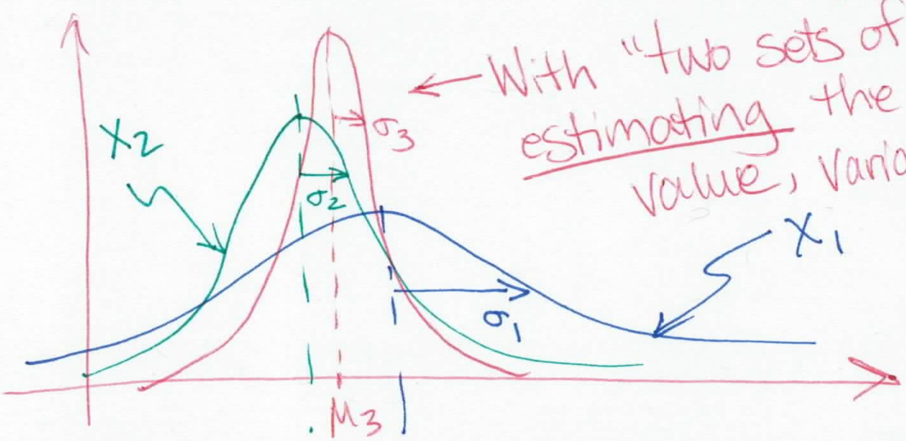
$$\sigma_{tot}^2 = \sigma_1^2 + \sigma_2^2$$



(1) Two noisy estimates are combined for a total estimate.
Like resistors in PARALLEL: total variance goes DOWN.



$$\frac{1}{\sigma_{tot}^2} = \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}$$



← With "two sets of eyes" estimating the same value, variance goes DOWN, & μ_3 is closer to the lower variance variable (here, $\sigma_2 < \sigma_1$, so σ_3 closer to σ_2...)

$$\boxed{\sigma_3^2 = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}}$$

$$\boxed{\mu_3 = \left(\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)\mu_1 + \left(\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)\mu_2}$$

Consider: $\boxed{X_{k+1} = X_k + W_k}$ and $\boxed{Y_k = X_k + V_k}$. W_k has variance **Q**. V_k has variance **R**.

Kalman Filter Toy Example [Optimal Estimation]

Recall:

$$P^+ = \frac{P^- R}{P^- + R} = (1-L)P^-$$

← "Resistors in parallel"

$$\sigma_3^2 = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

← Two independent, Gaussian-noise ESTIMATES. Variance goes down.

$$\underbrace{P^- = P^+ + Q}$$

← "Resistors in series"

$$\sigma_3^2 = \sigma_1^2 + \sigma_2^2$$

← Two Gaussian processes... Variance goes up.

$$L = \frac{P^-}{P^- + R}$$

↑ Optimal Estimator Gains

$$P_{ss}^+ = \frac{\boxed{(P_{ss}^+ + Q)}R}{\boxed{(P_{ss}^+ + Q)} + R}$$

$$P_{ss}^- = P_{ss}^+ + Q$$

$$\therefore P_{ss}^+ (P_{ss}^+ + Q + R) = (P_{ss}^+ + Q)R$$

$$\boxed{(P_{ss}^+)^2 + Q(P_{ss}^+) - QR = 0}$$

→ $\boxed{\texttt{roots([1, Q, -Q*R])}}$

↑ to solve for P_{ss}^+, and then: $P_{ss}^- = P_{ss}^+ + Q$

$$\boxed{L_{ss} = \frac{P_{ss}^-}{P_{ss}^- + R}}$$

MATLAB example, DT system:

```
A=1; B=1; C=1; D=0; T=1;
ss1=ss(A,B,C,D,T)
Q=1; R=1;
[Kest,Lss,P] = kalman(ss1,Q,R);
```

$L_{ss} = 0.618$

Transfer functions: Control Canonical Form systems

$(n=1)$
$l=1$
$m=1$

A quick summary…

Control canonical form example, with one input (u) and output (y), i.e., "SISO":

$$\underset{\text{DT}}{z}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A_d}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{B_d} u \quad \longleftrightarrow \quad zX = AX + Bu$$

"big" X

$$(zI - A)X = Bu$$

$$X = (zI - A)^{-1}Bu$$

$$y = CX$$

$$y = C(zI - A)^{-1}Bu$$

$$X \equiv \begin{bmatrix} z^2 x \\ zx \\ x \end{bmatrix}$$

"little" x.

$$(z^3 + a_1 z^2 + a_2 z + a_3)x = 1u$$

$$\therefore \frac{x}{u} = \frac{1}{z^3 + a_1 z^2 + a_2 z + a_3}$$

$$\boxed{y = \underbrace{\begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}}_{C_d} X}$$

And

$$\boxed{\frac{Y(z)}{u(z)} = \frac{c_1 z^2 + c_2 z + c_3}{z^3 + a_1 z^2 + a_2 z + a_3}}$$