Today: # Optimal Control and Estimation

Lecture 18
ECE 147B
Digital Control

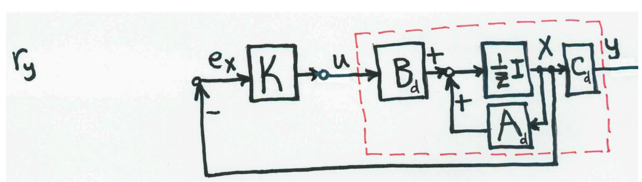
Reading: FPW 9.3.5, 9.4.1 <or> FV 10.2-10.4
<AND> RS-16

Topics:
- Review: Reference Tracking and Integral Control
- Optimal Control: Linear Quadratic Regulator (LQR)
- Optimal Estimation: The Kalman Filter
- MATLAB Examples

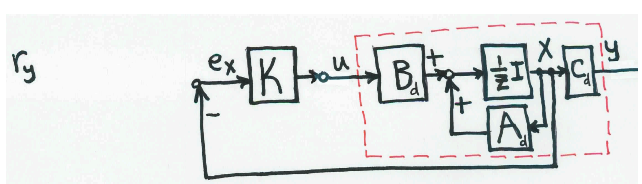
Next Class:

- Review and Q & A

Reference Tracking (for State Space Control), Review



Integral (State Space) Control, Review



Optimal Control: Linear Quadratic Regulator

Linear:

Quadratic:

Regulator:

Goal is to minimize a COST FUNCTION over all (future) time, which penalizes both:

(1) ERRORS (in the states, x) and
(2) EFFORTS (in the inputs, u):

Where are CONTROL is the usual STATE FEEDBACK:

Q and R can be set using BRYSON'S RULE:

Optimal Estimation: Linear Quadratic Estimation

This optimal estimator is also called a "Kalman filter".

Goal is to minimize VARIANCE in the estimated states.

We now assume there is both:

(1) PROCESS noise, in the state dynamics:

(2) MEASUREMENT noise, affecting y(k):

We will assume both w(k) and v(k) are Gaussian, white noise sources.

Optimal solution for estimator gains, L, depends of total current variance of the state estimates, before a measurement (P^-), compared with measurement variance (R):

More on covariance matrices Q and R…

Both the optimal Control and optimal Estimation assume noise is Gaussian and white.
For Both problems:

Q (nxn) is a covariance matrix for STATES, x

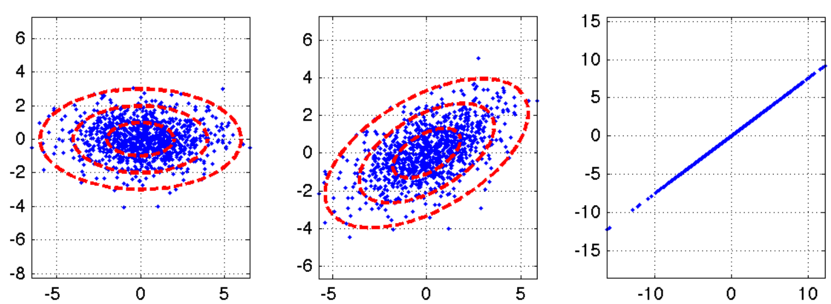
R ($\ell x \ell$) is a covariance matrix for OUTPUTS, y

Q and R must therefore be symmetric, positive semi-definite matrices.
(You can't have a "negative" standard deviation or variance for a Gaussian!)

MATLAB examples.

```
mu1 = 2; mu2 = 1; N=1e6;
x1 = normrnd(0,mu1,1,N);
x2 = normrnd(0,mu2,1,N);
figure(1); clf; subplot(231)
plot(x1,x2,'b.'); axis equal

A = 30*pi/180;
x1a = cos(A)*x1 - sin(A)*x2;
x2a = sin(A)*x1 + cos(A)*x2;
subplot(232);
plot(x1a,x2a,'b.'); axis equal

x1b = 4*x2;
x2b = 3*x2;
subplot(233); plot(x1b,x2b,'b.'); axis equal
```

More on covariance matrices Q and R…



LQR Toy Example [Optimal Control Gains]

Let us consider a simple, SISO system, where K will be a SCALAR value.

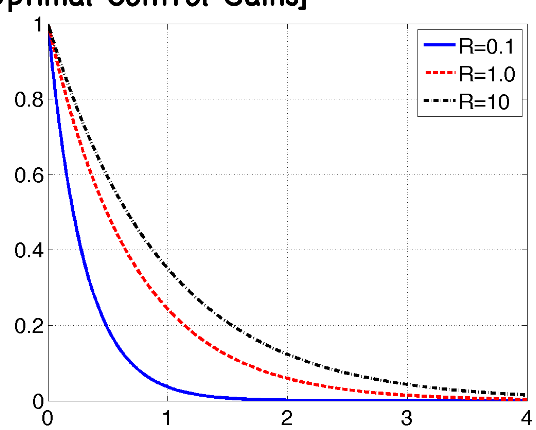
$$\frac{Y(s)}{U(s)} = \frac{X(s)}{U(s)} = \frac{1}{s+1}$$

$$\frac{Y(s)}{R_y(s)} = \frac{K}{s+1+K}$$

LQR Toy Example [Optimal Control Gains]

Check this in MATLAB:

```
A=-1; B=1; C=1; D=0;
Q=1; R=1;
Kb = lqr(A,B,Q,R)


pb = eig(A-B*Kb)



R=0.1;
Ka = lqr(A,B,Q,R)


pa = eig(A-B*Ka)


R=10;
Kc = lqr(A,B,Q,R)


pc = eig(A-B*Kc)
```




```
Ktest = -roots([R   2*R    -Q])
```

Kalman Filter Toy Example [Optimal Estimation]

The elegance of the Kalman filter depends on the assumption of GAUSSIAN, WHITE NOISE.

There are two way to combine Gaussians…

(1) Two noisy variables get ADDED to one another.
Like resistors in SERIES: total variance goes UP.

(1) Two noisy estimates are combined for a total estimate.
Like resistors in PARALLEL: total variance goes DOWN.

Kalman Filter Toy Example [Optimal Estimation]

Recall: $P^+ = \dfrac{P^- R}{P^- + R} = (1-L)P^-$ $\quad P^- = P^+ + Q$ $\quad L = \dfrac{P^-}{P^- + R}$

$$P_{ss}^+ = \frac{(P_{ss}^+ + Q)R}{(P_{ss}^+ + Q) + R}$$

MATLAB example, DT system:

```
A=1; B=1; C=1; D=0; T=1;
ss1=ss(A,B,C,D,T)
Q=1; R=1;
[Kest,Lss,P] = kalman(ss1,Q,R);
```

Transfer functions: Control Canonical Form systems

A quick summary…

Control canonical form example, with one input (u) and output (y), i.e., "SISO":

$$z\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$X \equiv \begin{bmatrix} z^2 x \\ zx \\ x \end{bmatrix}$$

$$y = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} X$$