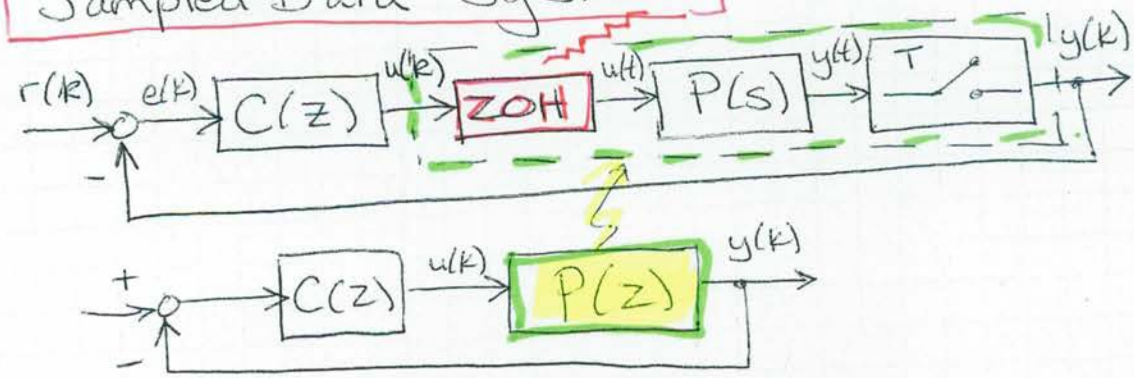
Digital Control {ECE 147B, UCSB}

Lecture 19 Review

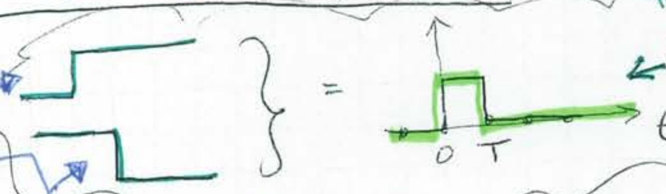
- "Sampled-Data System"



① – Understanding the dynamic system $P(z)$, given some $P(s)$:
- preceded by ZOH (rate $\frac{2\pi}{T}$)
- sampled every T sec @ output

$$\boxed{P(z) = \left(1 - \frac{1}{z}\right) \mathcal{Z}\left\{\frac{P(s)}{s}\right\}} \leftarrow \text{ZOH}$$

* To find eqn for... (Response of P(s) to a unit step (and to) a time-shifted negative step)



to find the "pulse response"

- $P(z)$ describes a DIFFERENCE EQUATION.

e.g., "$P(z) = \frac{az+b}{z^2+cz+d} = \frac{Y(z)}{U(z)}$" corresponds to:

$y(k+2) + cy(k+1) + d(k) = au(k+1) + bu(k)$

or: $y(k) + cy(k-1) + d(k-2) = au(k-1) + bu(k-2)$

* only **RELATIVE** k values matter! (for one term, compared with another in the equation.

z-Transform definition:

$$X(z) = \mathcal{Z}\{x(k)\} = \sum_{k=-\infty}^{\infty} x_k z^{-k}, \quad r_0 < |z| < R_0 \text{ (region of convergence)}$$

often, $x_k = 0,\ \forall\ k<0$ ... so sum is from $k=0$ to $k=\infty$, effectively

* IMPORTANT example:

1^{ST}-order Response

$y_k = Me^{-aTk}$

samples of impulse of $G(s) = \frac{1}{\tau s+1} = \frac{a}{s+a}$

$a = \frac{1}{\tau}$

Response is at $Me^{-1} = \frac{M}{e}$ at ONE time constant, τ

*Know this! → infinite series solution (Do NOT write out individual terms...)

$$G(z) = M\left(\frac{z}{z-e^{-aT}}\right) = \frac{Y(z)}{X(z)}$$

$M\left(\frac{z}{z-r}\right)$ ← alternate form

diff. eqn → $y(k) - (e^{-aT})\,y(k-1) = M\,x(k)$

Special 1^{st}-order case:

e.g., "unit step" has "$\tau = \infty$"

$\frac{1}{s} \longleftrightarrow \frac{z}{z-1}$

CT (unit step) DT

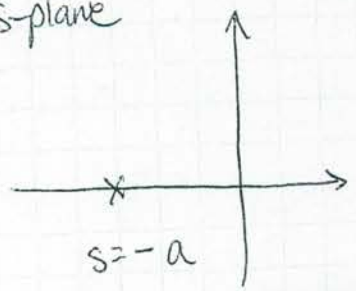
"Linearity" (scaled by a constant)

$$G(z) = 17\left(\frac{z}{z-1}\right)$$

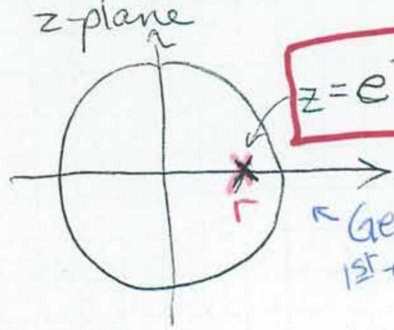
$e^{-\frac{1}{\infty}} = e^0 = 1$

1^{st}-order pole locations...

s-plane



z-plane



$z = e^{-aT} = r$

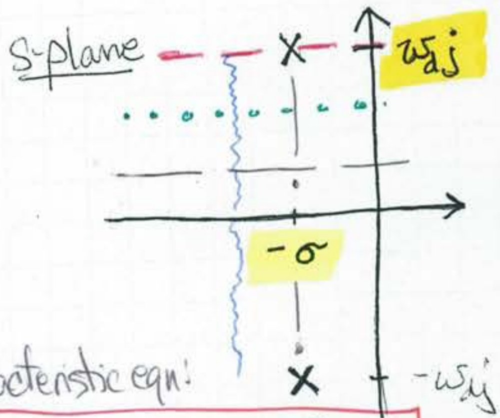
← General 1^{st}-order case (stable pole)

• s-plane vs z-plane correspondence

$$\boxed{z = e^{sT}}$$

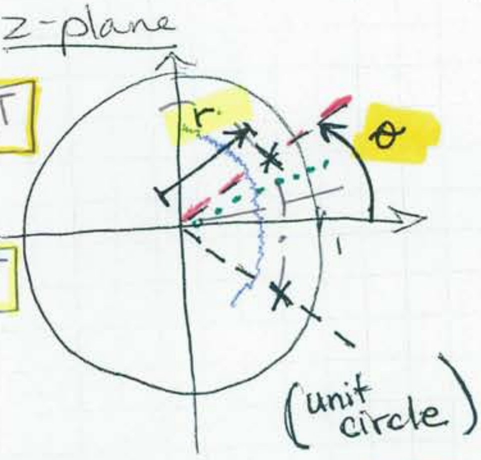
← (G(z) pulse response are the samples of G(s) impulse response.)

2nd-order systems (2 poles)



$\omega_d \rightarrow \boxed{\theta = \omega_d T}$

$\sigma \rightarrow \boxed{r = e^{-\sigma T}}$



characteristic eqn:

$$\boxed{s^2 + 2\zeta\omega_n s + \omega_n^2 = 0}$$

poles @ $s_1 = -\sigma + \omega_d j$

$s_2 = -\sigma - \omega_d j$

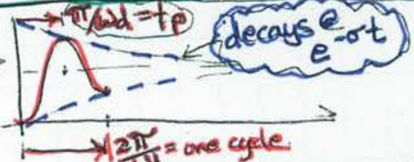
$$\boxed{\sigma = \zeta\omega_n, \ \omega_n = \sqrt{\sigma^2 + \omega_d^2}}$$

poles @

$z_1 = re^{j\theta}$
$= r\cos\theta + jr\sin\theta$

$z_2 = re^{-j\theta}$
$= r\cos\theta - jr\sin\theta$

Recall



$$\boxed{r = e^{-\sigma T}} \qquad \boxed{\theta = \omega_d T}$$

sampling time, T

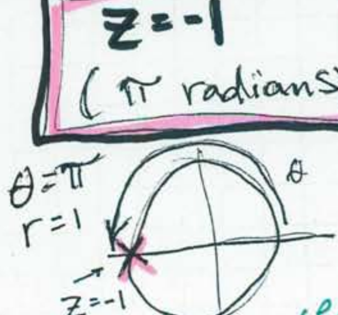
One time step corresponds to:

$T\omega_d$ (radians) of phase.

Nyquist freq. is $\boxed{\omega_{Nyq} = \frac{\pi}{T}}$

and corresponds to

$\boxed{z = -1}$ (π radians)



"½-sample delay" would result approx. in -90° phase @ $\omega_{Nyquist}$ ...

$$\Delta\phi = \left(\frac{\omega}{\omega_{Nyq}}\right)\left(\frac{-\pi}{2}\right)(\text{rad})$$

$$= \omega \cdot \frac{T}{\pi} \cdot \frac{\pi}{2} = \frac{-T\omega}{2}$$

$$\boxed{\Delta\phi = \frac{-T\omega}{2} \text{ @ freq. } \omega}$$

To approx. the ZOH effect. Only use the "half-sample" delay approx BELOW ω_{Ny}.

Use $z = -1$ for exact ω

Problem: estimate z-plane pole, given a time response.

"r"

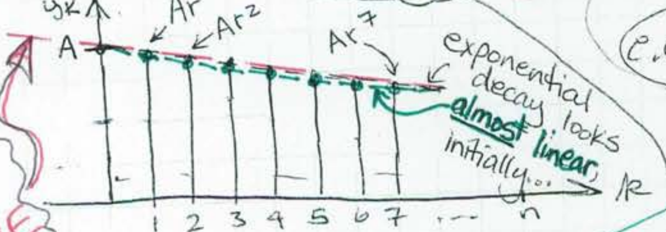
A trick:

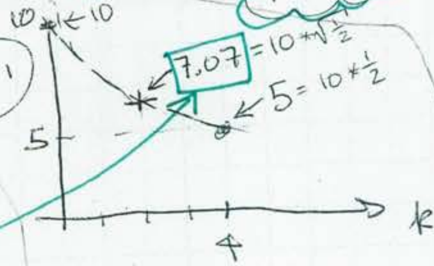
Estimating r when $r = 1 - \Delta$ $(\Delta << 1)$

$$\boxed{(1-\Delta)^n \approx 1 - n\Delta}, \ \Delta << 1$$

$\sqrt{\frac{1}{2}} \approx .707$

... in other words, approximate the decay envelope LOCALLY as a LINE



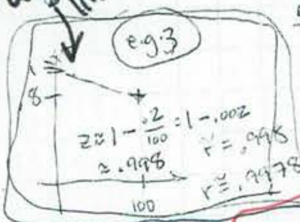


$$r^4 = \frac{5}{10} = \frac{1}{2}$$

$$\boxed{r = \left(\frac{1}{2}\right)^{1/4}}$$

$$(z = r)$$

@ $t = .8$ sec, $y(t) = 2.5 = \frac{1}{4} 10$ (factor of $\frac{1}{2}$ every 400 samples)



$\frac{5}{10} = \frac{1}{2}$

$$\boxed{r^{400} = \frac{1}{2}}$$

$(1-\Delta)^{400} = \frac{1}{2} \approx 1 - 400\Delta$

$400\Delta \approx 1 - \frac{1}{2}$

$$r = \left(\frac{1}{2}\right)^{1/400}$$

$$r = \left(\frac{5}{10}\right)^{1/400} \approx 1 - \left(\frac{1}{400}\right)\left(\frac{1}{2}\right)$$

$= 1 - \frac{1}{800}$

$= 1 - .00125$

$$\boxed{r \approx .99875}$$

$r^{400} = .722$

OR

$\frac{7.07}{10} = .707$

$r^{200} = \left(\frac{1}{2}\right)^{\frac{1}{2}} = .707$

$(1-\Delta)^{200} = .707 \approx 1 - 200\Delta$

$200\Delta \approx 1 - .707$

$\Delta \approx \frac{.3}{200} = .0015$

$$\boxed{r = 1 - \Delta \approx .9985}$$

r is actually $r = .9983$

$z = .99875$

$s \approx \left(\frac{1}{T}\right)[z - 1]$

$s \approx \frac{-.00125}{T}$

for $T = .001$ sec,

$s \approx -1.25$

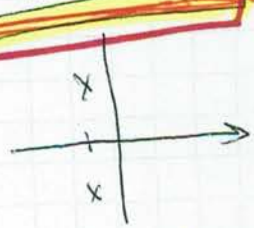
$\tau \approx \frac{1}{1.25}$

$= .8$ sec

$$z = e^{sT} \quad \text{for } \textbf{small} \text{ "}sT\text{" } (|sT| << 1),$$

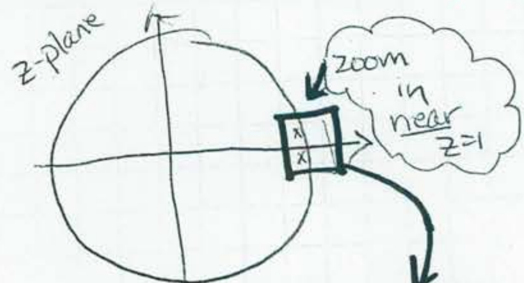
← i.e., $|z - 1| << 1$

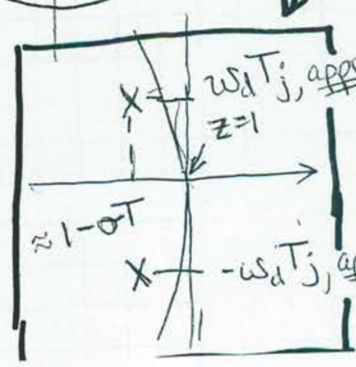
$$\boxed{z \approx 1 + sT}$$



$$\boxed{e^{\Delta} \approx 1 + \Delta}, \ \Delta << 1$$

$$\Delta \approx \log(1 + \Delta)$$





② Design of Controller $C(z)$

$$\begin{tikzcd}
P(s) \arrow[r, "\text{CT design (ece 147a)}"] \arrow[d, "\text{model ZOH}"'] & C(s) \arrow[d, "\text{A) ``Emulation'' of } C(s)"] \\
P(z) \arrow[r, "\text{B) ``Direct Design''}"'] & C(z)
\end{tikzcd}$$

SISO or SS → Ⓐ

"Emulation" of $C(s)$ w/ some $C(z)$ (Tustin, etc.) "Menu"

SISO or SS → Ⓑ

"Direct Design" ← place poles directly: via state space!

count on $C(s)$ & $C(z)$ "matching @ w_{co}"

Two Methods:

So, for "loop shaping", use $\omega_0 = \omega_{co}$ ← crossover freq.

Ⓐ Emulation

[Ⓑ See page 10! Direct Design...]

– Tustin w/ **pre-warp**

prewarp frequency ω_0

$$\boxed{s \leftrightarrow \alpha \frac{(z-1)}{(z+1)}, \quad \alpha = \frac{\omega_0}{\tan\left(\frac{\omega_0 T}{2}\right)}}$$

Goal for $C(z)$? → match Gain & Phase of $C(s)$ @ pre-warp freq. ω_0

list on p. 192 (FPW)

usually "good enough" (w/out "prewarp")

– **Tustin**, or limit as $w_0 \ll w_{Ny}$ ("small" T)

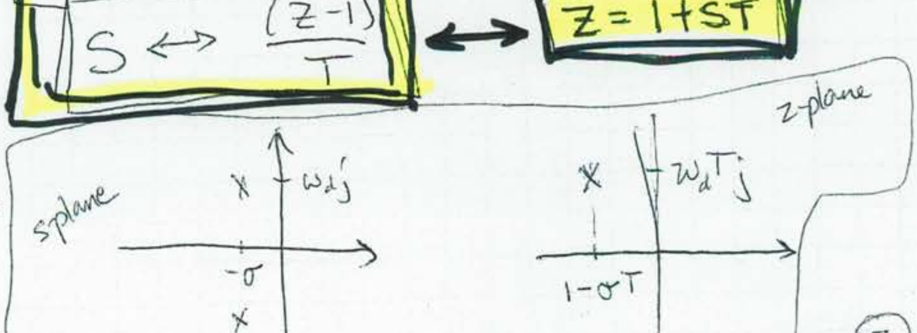
$$\boxed{s \leftrightarrow \frac{2}{T}\frac{(z-1)}{(z+1)}} \qquad z = \frac{1 + Ts/2}{1 - Ts/2}$$

– **Forward Euler**

$$\boxed{s \leftrightarrow \frac{(z-1)}{T}} \longleftrightarrow \boxed{z = 1 + sT}$$

← FW Euler

Forw. Euler is exactly the approx for "small" sT from last page! ⟷



So $P(s)$ & $P(z)$ have similar TIME RESPONSES.

(Not as useful for "$C(z)$" via emulation...)

"Pole-zero matching"

1. $z = e^{sT}$ for poles, $z_p = e^{s_p T}$
2. " for zeros, $z_z = e^{s_z T}$
3. $\boxed{\text{Keep adding zeros @ } z=-1}$ until order of $G(z)$ numerator is one less than denominator: $G(z) = \dfrac{(z+1)*K}{z^2+az+b}$

$(z+1)^{(n_p - n_z - 1)}$ in numerator

4. Match DC gains "FVT"

Only use FVT if STABLE!!

Final Value thm → **FVT:** $\lim_{z\to 1} \left((z-1)\cdot F(z)\right)$ ← IFF poles are all inside unit circle! (R.O.C.)

some total Z x-form... e.g. INPUT AND PLANT

- For unit step response of $G(z)$,

DT → $\lim_{z\to 1}\left(\cancel{(z-1)}\cdot\left(\dfrac{z}{\cancel{z-1}}\cdot \underbrace{G(z)}_{\text{plant}}\right)\right)$ ← unit step, $\dfrac{z}{z-1}$

CT → vs $\lim_{s\to 0}\left(s\cdot\left(\frac{1}{s}\cdot G(s)\right)\right)$

Initial Value Thm → **IVT:** $\lim_{z\to\infty}\left(F(z)\right)$

Example →

e.g. $G(z) = \dfrac{8z}{z-.5}$

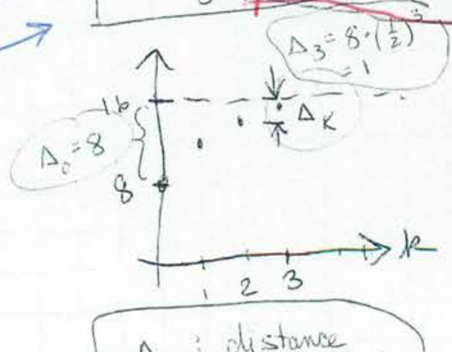
step resp: $F_f(z) = \left(\dfrac{z}{z-1}\right)\cdot\dfrac{8z}{(z-.5)}$ ← unit step is "$\frac{z}{z-1}$"

Stability check!, $\boxed{z=.5}$ stable!

F.V. → $\cancel{(z-1)}\cdot\left(\dfrac{z}{\cancel{z-1}}\right)\cdot\dfrac{8z}{(z-.5)}$

$\therefore$ $\boxed{\text{F.V.} = \dfrac{8}{.5} = 16}$

$\boxed{\text{I.V.} \rightarrow \left(\dfrac{\infty}{\infty-1}\right)\left(\dfrac{8\infty}{\infty-.5}\right) = 8}$



Δ_K: distance from F.V.

$\Delta_K = (\Delta_0)\cdot r^k$

closed-loop

Deadbeat control

$z = 0$ ← ALL pole(s) @ **origin**

∴ impulse resp. Output goes to zero in finite time

Freq. Resp.

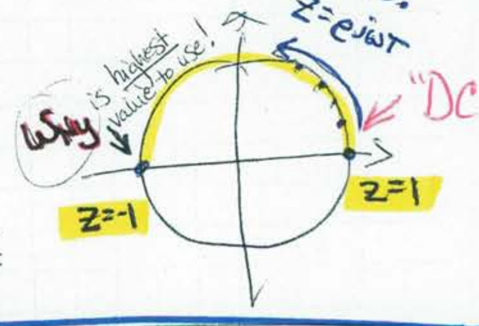
Bode plots and Nyquist plots →

Eval $G(z)$ for

$$z = e^{j\omega T}$$

$$0 \le \omega \le \omega_{Ny} = \frac{\pi}{T}$$

↙ Values are on unit circle!



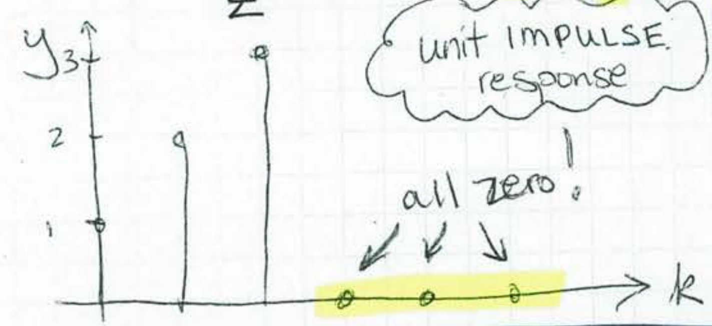
Impulse Response for Deadbeat system:

$$G(z) = \frac{z^2 + 2z + 3}{z^2} = 1 + \frac{2}{z} + \frac{3}{z^2}$$

All poles @ $z = 0$. ∴ Can expand into a finite number terms.

∴ FINITE IMPULSE RESPONSE

unit IMPULSE response



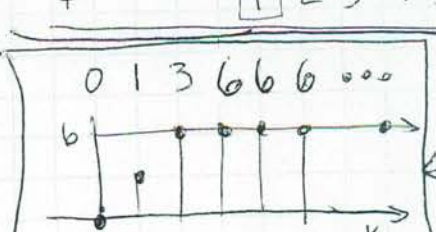
① How to: Derive the unit step response, Given a unit impulse response

add an infinite # of time-shifted copies!

← Add up time-shifted impulses!

max of 3 terms

e.g. :	0	1	2	3	0	0		impulse
+		0	1	2	3	0	0	+ shifted by 1
+			0	1	2	3	0 0	+ shifted by 2
+					1	2	3	+ shifted by 3
								∴ + shifted by ∞
	0	1	3	6	6	6	...	



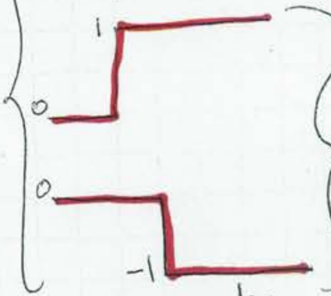
← $G(z) = \frac{z^2+2z+3}{z^2}$,

← unit STEP response

2 (inverse of the previous problem...)

Derive the Impulse Response ...given a unit step response

pos step @ $k=0$

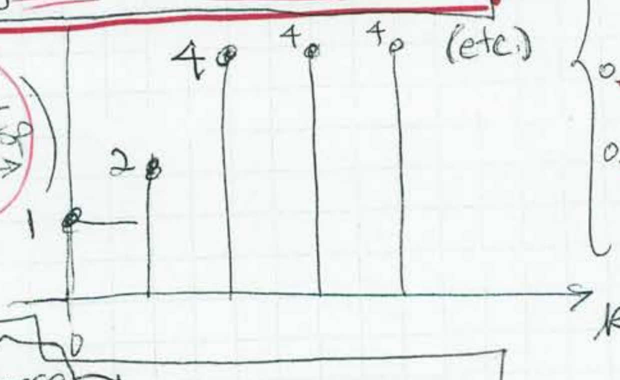


neg-step @ $k=1$

We want "response to a pulse"

pulse

(Example step resp @ right →)



Impulse Response is "a unit step @ $k=0$" minus "a unit step @ $k=1$"

1	2	4	4	4	...
	-1	-2	4	4	
1	1	2	0	0...	

Resulting impulse response.

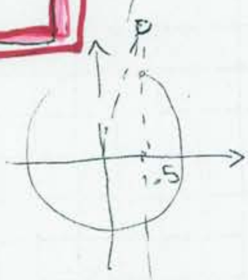
$$\therefore G(z) = 1 + \frac{1}{z} + \frac{2}{z^2} = \boxed{\frac{z^2 + z + 2}{z^2}}$$

Definition of Z transform!

zeros @ $r = \sqrt{2}$

$\theta = \text{atan}\left(\frac{\sqrt{1.75}}{.5}\right)$

$r = \sqrt{2}$



$x = .5$

$y = \sqrt{2 - .5^2}$

$y = \sqrt{1.75}$

$z = .5 \pm \sqrt{1.75}\, j$

Write out finite terms IFF it is a finite impulse response.

(otw, write as 1st- or 2nd-order form...)

S.S. (State Space)

CT

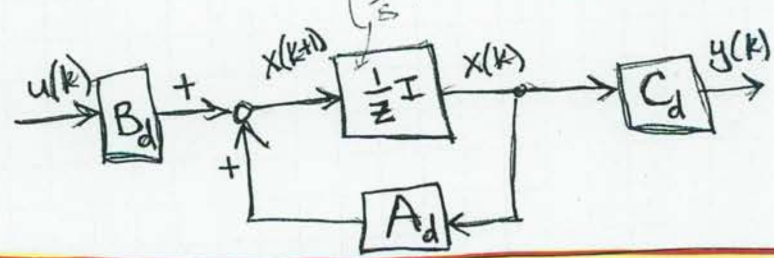
$$\boxed{\begin{array}{c} sX = \dot{X} = AX + Bu \\ y = Cx \\ (\text{or: } y = Cx + Du) \end{array}}$$

DT

i.e., $\boxed{X_{k+1} = A_d X_k + B_d u_k}$

$$\boxed{\begin{array}{c} zX = A_d X + B_d u \\ y = C_d X, \\ (\text{or: } y = C_d X + D_d u) \\ \text{usually, } C_d = C \end{array}}$$

Block Diag:



* **Stability**: $\boxed{\text{eigenvals of } A_d}$ inside unit circle, for STABILITY!

" $\boxed{\det[zI - A_d] = 0}$ " ← these are the eigenvalues of A_d

(Poles, open-loop system.)

S.S: Forward Euler Approx:

(for the ZOH acting on some P(s)...)

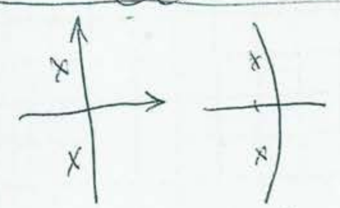
$\left(e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k\right)$

$$\boxed{\begin{array}{c} A_d \approx I + AT \\ B_d \approx BT \end{array}}$$

$C_d = C \quad (D_d = D)$

for small sT ($|sT| << 1$)

$e^{sT} \approx 1 + sT$



↑ Again, know how to map between S & Z planes...

Really

$$\boxed{\begin{array}{l} A_d = e^{AT} \\ B_d = \int_0^T e^{A\tau} d\tau\, B = A^{-1}(A_d - I)B \end{array}}$$

n states
m inputs
l outputs

"Regulator"

State Feedback

$u = -Kx$

No ref.

Drive all states to ZERO

CL poles from

$\mathrm{eig}(A_d - B_d K)$ ← DT

or $\mathrm{eig}(A - BK)$ ← CT

→ size of K?

$u = -K\,x$

(m×1) (m×n) (n×1)

(m×1) (n×1)

Ⓑ "Direct Design" → Can place poles directly, anywhere in stable z plane. "Pole Placement"

→ Fast poles typically require higher gains in K (may saturate u)

Optimal Control

→ **LQR** can set K to minimize a quadratic

cost: $J = \sum_{k=1}^{\infty} x^T Q x + u^T R u$ — R, Q

(R ↑, control actuation needed ↓)

* **LQR is an alternative to "pole placement".** It is Optimal Control

✓ "Regulator" → Reference is zero.

✓ Linear → System dynamics have linear eqns $\left\{ \begin{array}{l} \text{differential: CT} \\ \text{difference: DT} \end{array} \right\}$.

LQR: **Q** penalizes the STATES, x.

R penalizes the INPUTS, u.

• Set closed-loop poles w/ MATLAB: K = place(A, B, poles)

(direct) pole placement

Estimators : Control based on $u = -K\hat{x}$

$\uparrow$ estimate $\hat{x}$ of true states

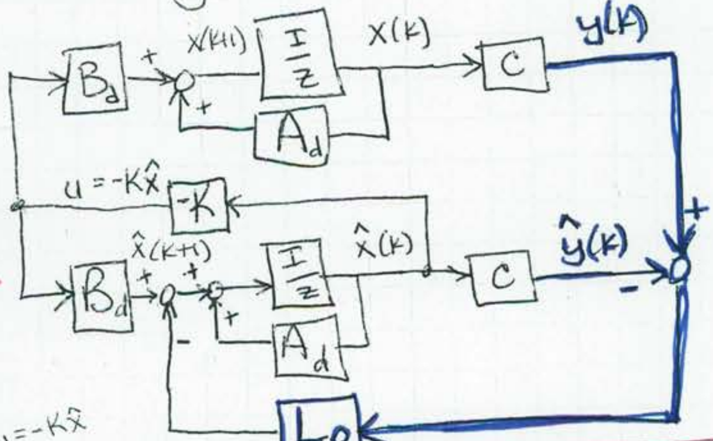
* Below is a diagram of control w/ estimation:

Both

1) Control: $x(k+1) = A_d x(k) - \underbrace{B_d K \hat{x}(k)}_{u=-K\hat{x}}$

AND

2) Estimator: $\hat{x}(k+1) = A_d\hat{x}(k) - \underbrace{B_d K \hat{x}(k)}_{u=-K\hat{x}} + L_p(y(k) - \hat{y}(k)) = (A_d - B_d K - L_p C_d)\hat{x}(k) + L_p y(k)$



Design gains in L_p

* Estimator Poles @ $\boxed{eig(A_d - L_p C_d)}$

Can do pole placement to set K & L_p as SEPARATE problems! (Separation principle)

Can place: $\boxed{L_p = place(A', C', poles)'}$

can use place or acker.

$\uparrow$ use of transpose 3 places!

(You should know size of matrix L_p, & of matrix K...)

(For Final Exam know what SIZE both K & L_p must be...)

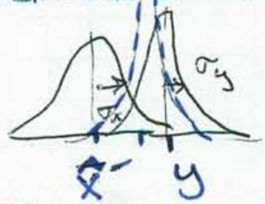
* Rule of thumb, make estimator poles 2x to 4x Faster that CL plant poles.

-> If too fast, L_p gets large & NOISE may be magnified...

* Blending 2 noisy estimates ($\hat{x}^-$ & y) can be done OPTIMALLY:

(KALMAN filter...)

$\uparrow$ optimal estimation.



$\hat{x}^+ = \left(\frac{\sigma_y^2}{\sigma_x^2+\sigma_y^2}\right)\hat{x}^- + \left(\frac{\sigma_x^2}{\sigma_x^2+\sigma_y^2}\right)y$

$\boxed{K = \mathrm{lqr}(A, B, Q, R)}$ ← MATLAB

Optimal Control $\boxed{u = -K_{LQR}x}$

Linear Quadratic Regulator : LQR

Minimize a quadratic cost over time:

DT: $\boxed{J = \sum_{k=0}^{\infty} (x^T Q x + u^T R u)_k} \Rightarrow \underbrace{\sum_{k=0}^{\infty} \left[\sum_{i=1}^{n} (Q_{ii} x_i^2) + \sum_{j=1}^{m} (R_{jj} u_j^2) \right]}_{\text{when } Q \text{ \& } R \text{ are diagonal.}}$

CT: $J = \int_{t=0}^{\infty} (x^T Q x + u^T R u)\, dt$

[typically/usually the case...]

→ Just the RELATIVE values in Q & R (wrt one another) matter.

check with MATLAB!

$Q\uparrow$ (or $R\downarrow$) penalizes STATE errors, ∴ $K\uparrow$, faster

$Q\downarrow$ (or $R\uparrow$) penalizes EFFORT, u, ∴ $K\downarrow$, slower

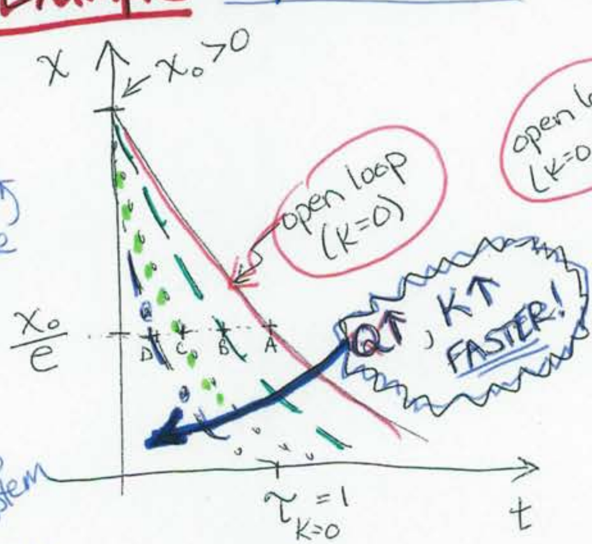
---

LQR Example w/ one state ($n=1$), & one input ($m=1$):

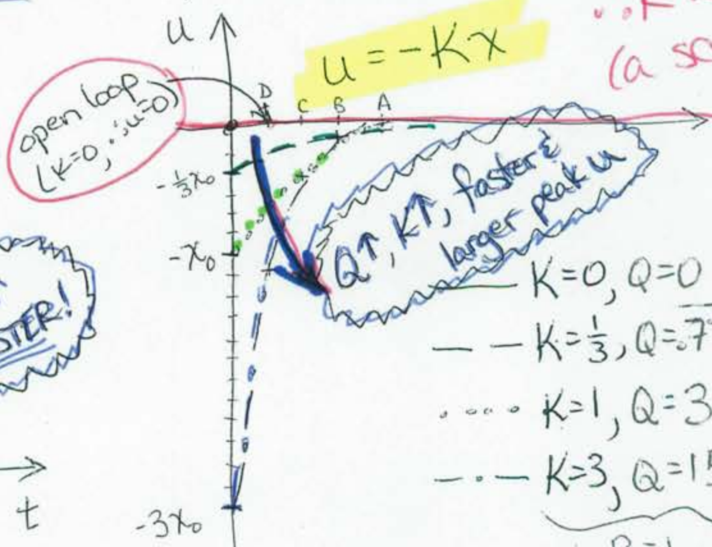
∴ K is (1x1) (a scalar)

$A = -1$
$B = 1$

state space form



$u = -Kx$



—— $K=0$, $Q=0$

– – $K=\frac{1}{3}$, $Q=.77$

····· $K=1$, $Q=3$

–·– $K=3$, $Q=15$

$R=1$

$R=1$ for all case...

SISO system

$\dot{x} = x + u$

$s = \frac{-1}{\tau}$

$(s+1)x = u$

$\boxed{\frac{x}{u} = \frac{1}{s+1}}$ open loop pole @ $s=-1$ .... closed loop → $\boxed{\frac{x}{u} = \frac{K}{s+1+K}}$

Kalman estimator gains, L_K

MATLAB → $[K_{est}, L_K, P] = \text{kalman}(sys, Q, R)$

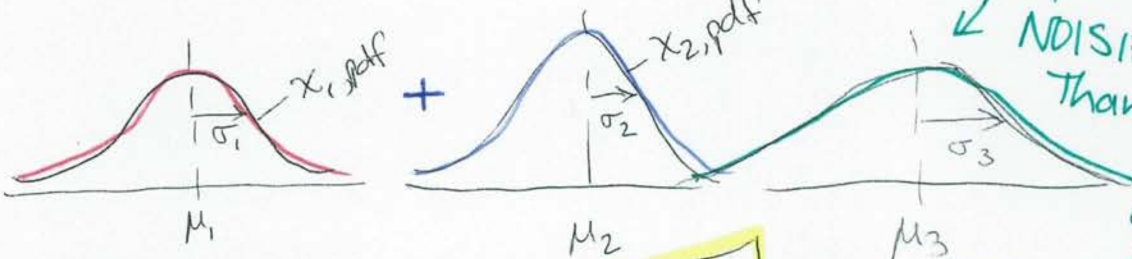
Optimal Estimation

Kalman filter

- Assumes PROCESS & MEASUREMENT noise are BOTH
  1. Gaussian : μ & σ
  2. White (zero mean)

Two Things about why Gaussians are "Nice" to use...

① When Gaussian noise adds to another Gaussian:

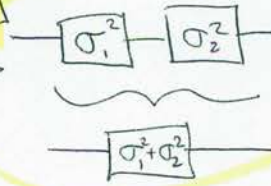


even NOISIER! Than either x_1 or x_2

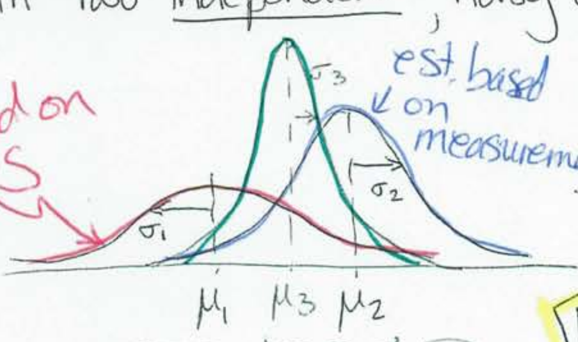
$$\mu_3 = \mu_1 + \mu_2$$

$$\sigma_3^2 = \sigma_1^2 + \sigma_2^2$$

Like impedances in SERIES...



② With two independent, noisy (Gaussian) estimates:



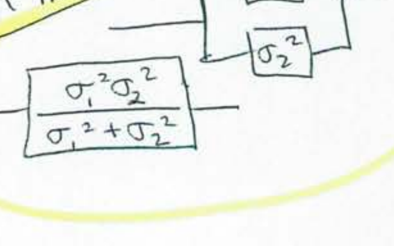
Est. based on dynamics

est. based on measurement

$$\mu_3 = \frac{\sigma_2^2\mu_1 + \sigma_1^2\mu_2}{\sigma_1^2 + \sigma_2^2}$$

$$\frac{1}{\sigma_3^2} = \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}$$

Like resistors in PARALLEL



μ_3 is closer to the less noisy estimate!

* KALMAN (ESTIMATION) FILTER:

5 STEPS...

Model Update:

1) $\boxed{\bar{x}_k^- = A\bar{x}_{k-1}^+ + Bu_{k-1}}$ (a prior est, $\bar{x}^-$)

~~Extra Credit likely on SCALAR case...~~ No! Not on exam!!!...

Not on Final in 2013...

2) $P_k^- = AP_{k-1}^+A^T + Q$

$$\boxed{P_k^- = a^2 P_{k-1}^+ + \sigma_Q^2}$$

($A = [a]$ in scalar case.)

for $C=1$ & **scalar** state x

Kalman Estimator 3) $L_k = P_k^- C^T (CP_k^- C^T + R)^{-1}$

("Kalman Gain")

$$= \frac{P_k^-}{P_k^- + R}$$

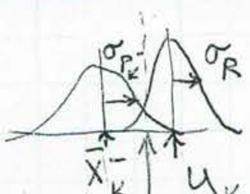
Boxed eqn's are for scalar case, with $c=1$.

$$\boxed{L_k = \frac{\sigma_{P_k^-}^2}{\sigma_{P_k^-}^2 + \sigma_R^2}}$$

Measurement Update:

4) $\bar{x}_k^+ = \bar{x}_k^- + L_k(y_k - C\bar{x}_k^-)$

$$\boxed{\bar{x}_k^+ = (1-L_k)\bar{x}_k^- + L_k y_k}$$



i.e.) New estimate is weighted as $\left(\frac{\sigma_R^2}{\sigma_P^2+\sigma_R^2}\right)\bar{x}_k^- + \left(\frac{\sigma_P^2}{\sigma_P^2+\sigma_R^2}\right) y$

5) $P_k^+ = (I - L_k C)P_k^-$

$$\boxed{P_k^+ = \frac{RP_k^-}{R+P_k^-} = \frac{\sigma_R^2\,\sigma_{P_k^-}^2}{(\sigma_R^2 + \sigma_{P_k^-}^2)}}$$ for $C=1$ & **scalar x**!

Combined Est. & Control:

$$\boxed{u = -K\hat{x}}$$

2n total poles...

Total control-estimator poles @ $\{eig(A_d - B_d K), eig(A_d - L_p C)\}$

- poles of CL plant (controller & Estr) =

$$\left\{ eig(A_d - L_p C), eig(A_d - B_d K) \right\}$$

"Separation Principle"

2n total "states"

x & $e_x = x - \hat{x}$
$(n \times 1)$ $(n \times 1)$

2n total poles

$$\begin{bmatrix} x(k+1) \\ e_x(k+1) \end{bmatrix} = \begin{bmatrix} A-BK, & BK \\ 0, & A-LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

- $\underline{\underline{C(z)}}$ poles are $eig(A - BK - LC)$

$$\frac{U(s)}{Y(s)} = -K(sI - \overbrace{A + BK + L_p C}) ^{-1} L_p$$

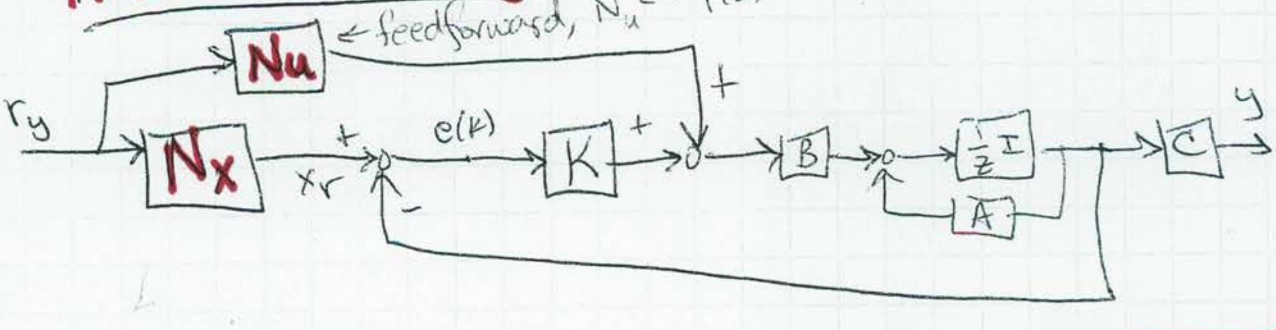
Nyquist: $Z = N_{cw} + P$ ← # unstable OL poles

Stability Criterion

Z: # unstable CL poles

N_{cw}: # CW encirclements of −1

Reference Tracking:

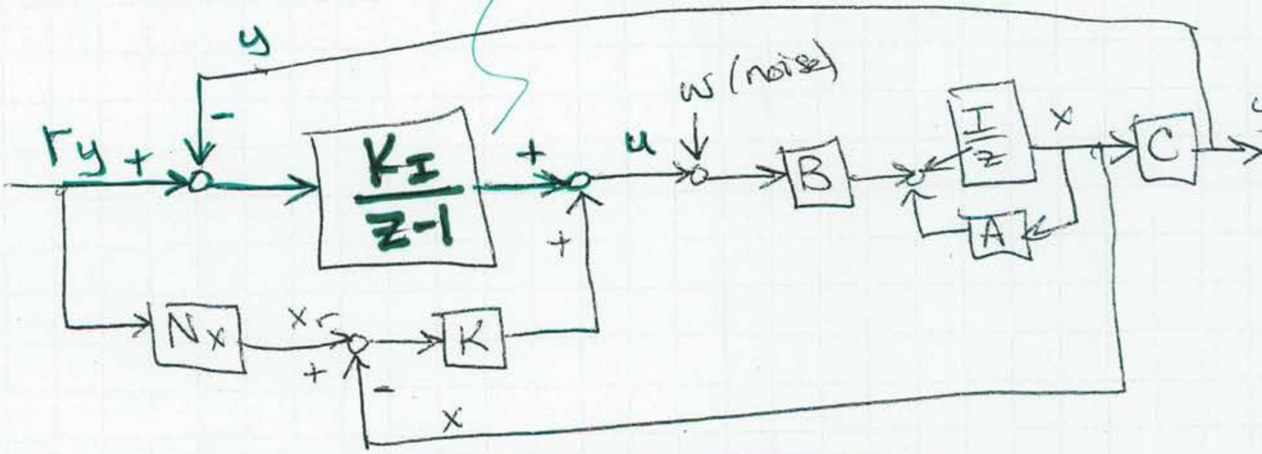


Integral Control

$$\boxed{\frac{K_I}{z-1}} \quad \text{vs} \quad \boxed{\frac{K_I}{s}}$$

DT CT

integral feedback term is added into u.



$$\boxed{u = K(x_r - x) + \left(\frac{K_I}{z-1}\right)(r_y - y)}$$

integral action

Good Luck!