Today: Discrete-Time Transfer Functions

Lecture 3

ECE 147B
Digital Control

Reading: FPW 4.2, 4.3.1, 6.1 (p.189-195)

or: FV 3.3-3.7, 6.3.3

Topics:

- ZOH time delay effect (Review from Lecture 2)
- ZOH equivalent
- Definition of the Z Transform
- Discrete-Time Transfer Functions
- Impulse Response of G(z) $= \frac{Y(z)}{X(z)}$
- ~~Tustin with Pre-Warp~~ Next time!

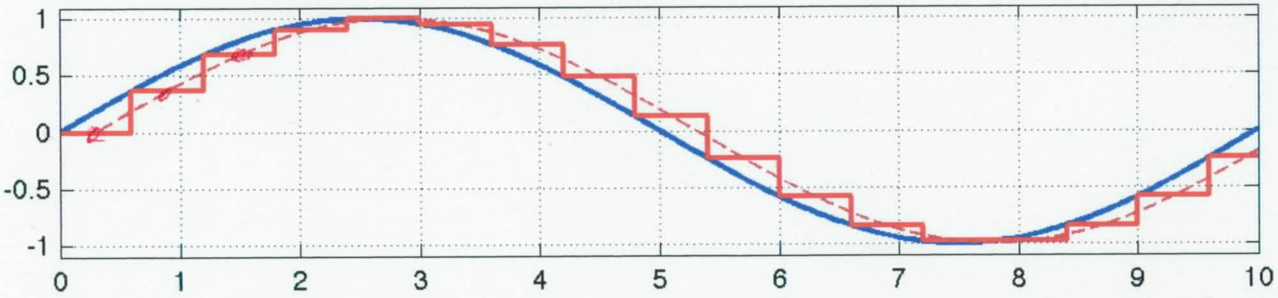
Next Class: Z Transform, DT TF, Tustin with pre-warp

- FPW: 6.3.1, 4.6.1, 4.2.1,

or: - FV: 2.3, 2.5, 2.8, 6.3.3

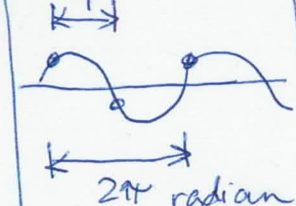
Sampling: ZOH Time Delay

1. Fourier: CT signal can be broken into sinusoids.
2. After ZOH, the CT waveform of a sinusoid has the original (primary) freq plus higher freqs, too.
3. The primary freq sinusoid is now shifted by T/2…

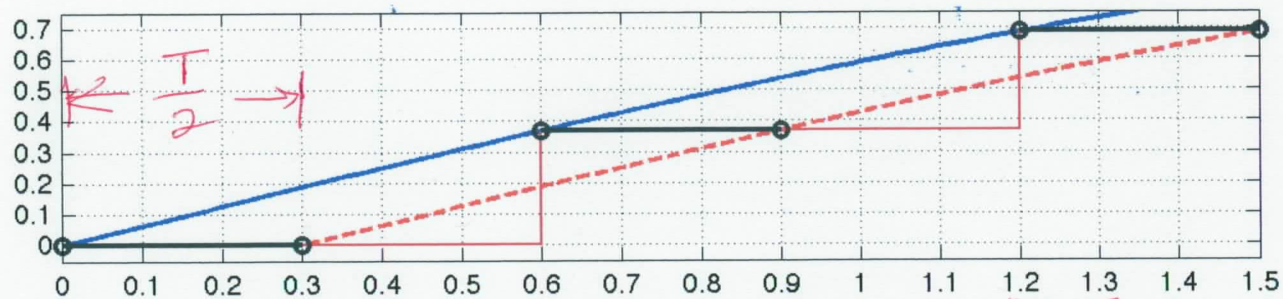


"Half-sample delay" approximation works well – until you get "very close" to Nyquist freq.

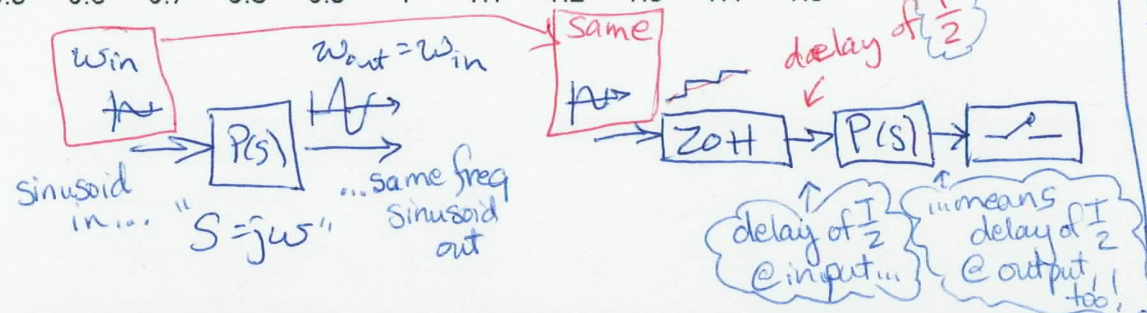
$$\boxed{\omega_{Nyquist} = \frac{\pi}{T}}$$



2 samples per cycle @ $\omega_{Nyquist}$



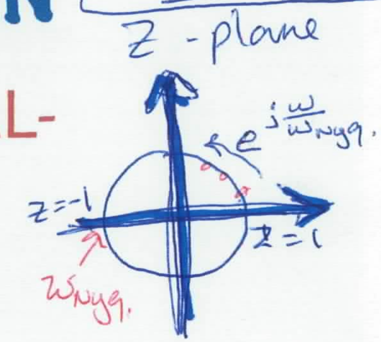
Freq. Resp:

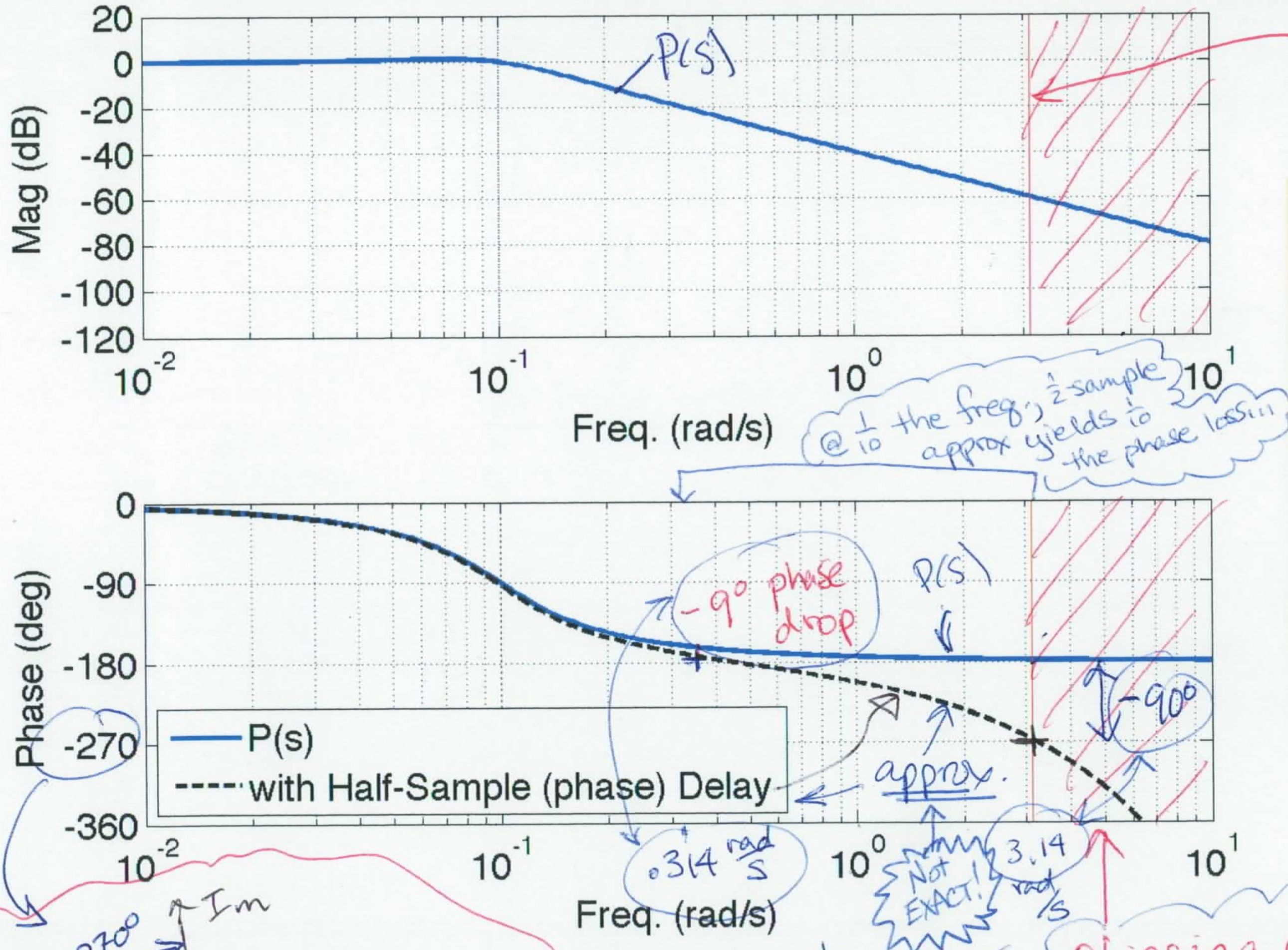


ZOH Consequences in FREQUENCY DOMAIN

- But wait, G(z) at z=-1 (Nyquist) must be REAL-VALUED! i.e., phase of 0 or +/-180 deg.

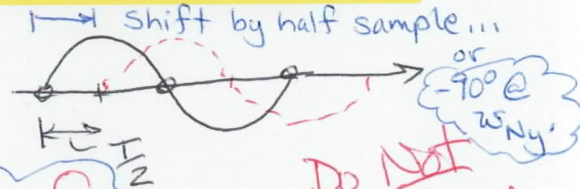
G(z)... is Real for z=-1

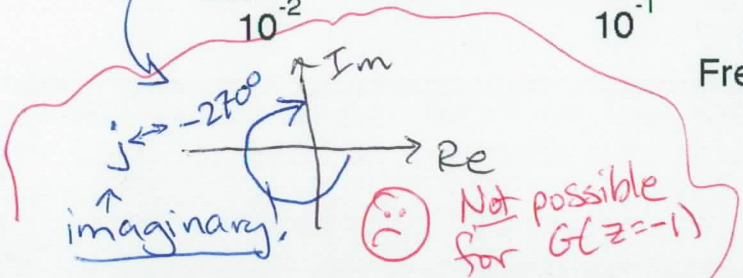




"Half-sample delay" approximation works well – until you get "very close" to Nyquist freq.

...Phase follow this approx until VERY close to Nyquist!!



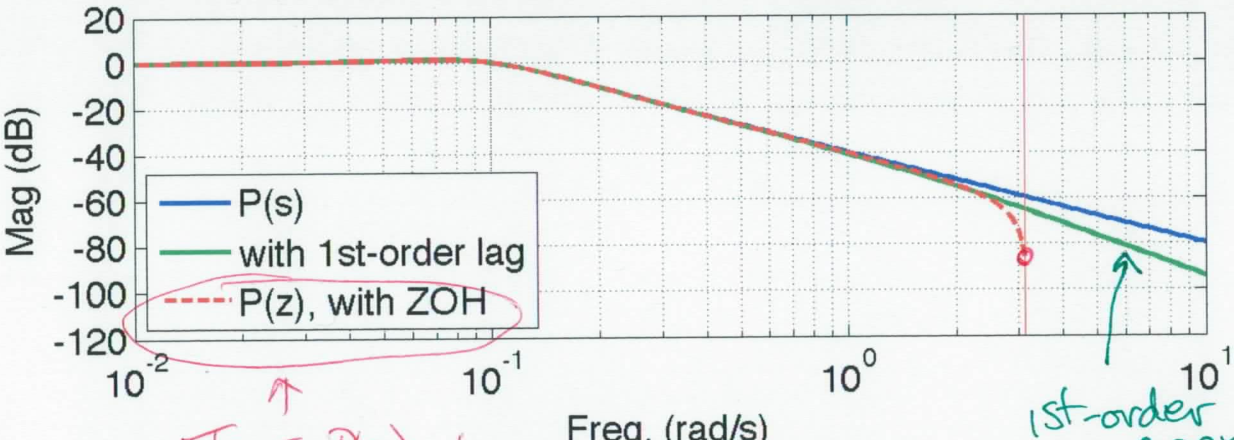


aliasing @ freqs above ω_{Nyq} ⟺ Do NOT try to control!

ZOH Consequences in FREQUENCY DOMAIN

- Lab 1 using a 1st-order lag approximation for the effects of the ZOH:

$$P_{lag}(s) = \frac{1}{\left[\frac{T}{2}\right]s + 1}$$

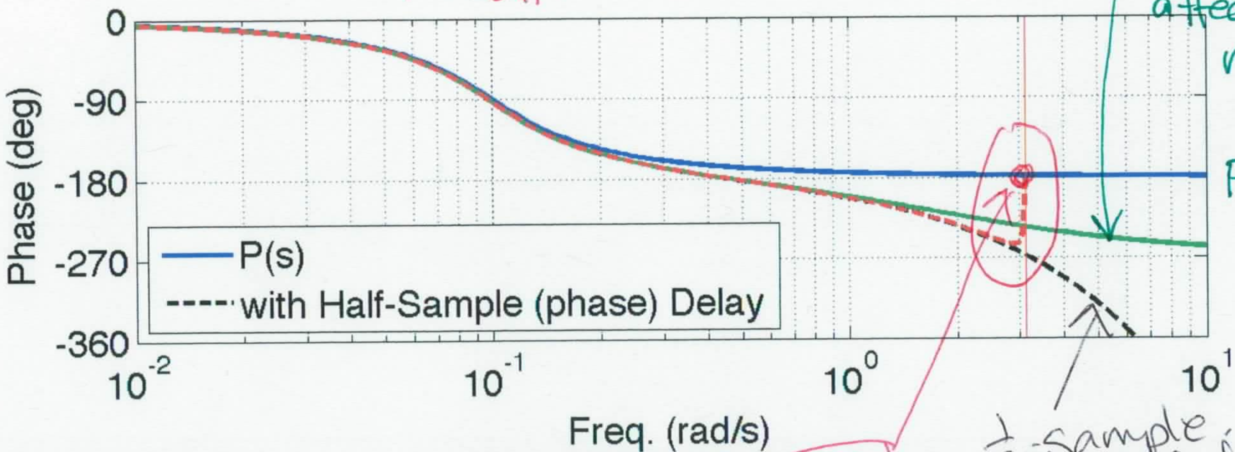


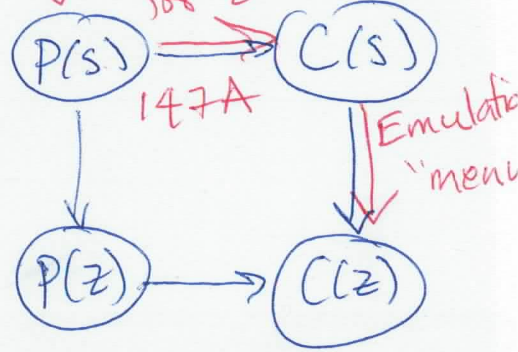
TRUE P(z), w/ ZOH.

1st-order approx, affects magn & phase.

Prelab 1 "1st-order" linear approx.

partly compensate for ZOH.





½-sample only affects phase.

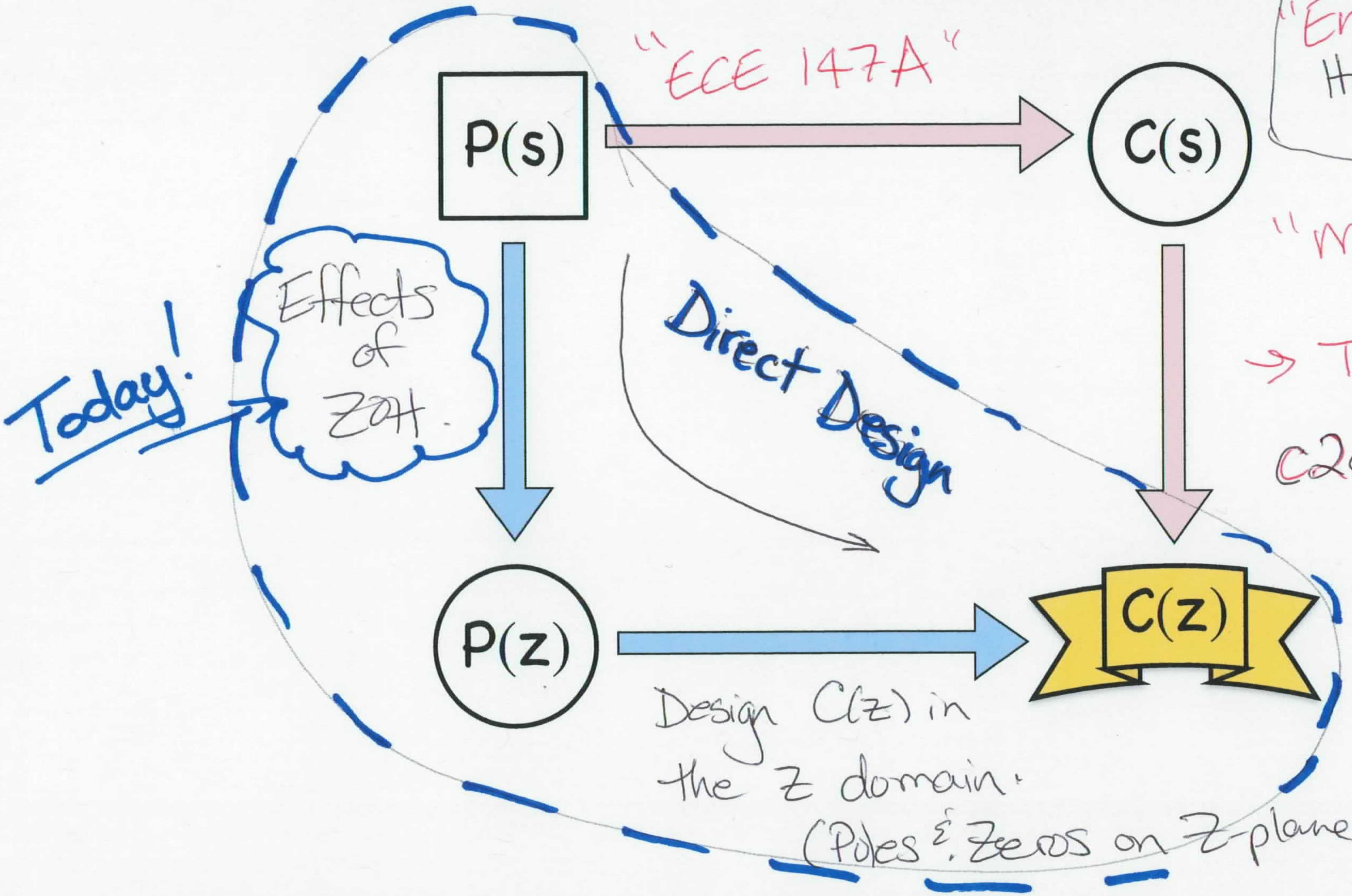
"-180°" is REAL-VALUED.

P(z): Actual phase is -180°

NOT -270°, as ½-sample approx est's...

Digital Control Design Approaches

- Recall we have two design path options:



"Emulation": Hope that "T" is small enough you "ignore" DT effects.

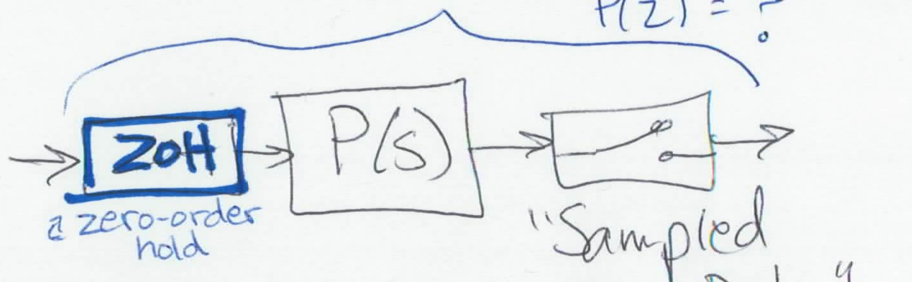
"menu":

→ Tustin

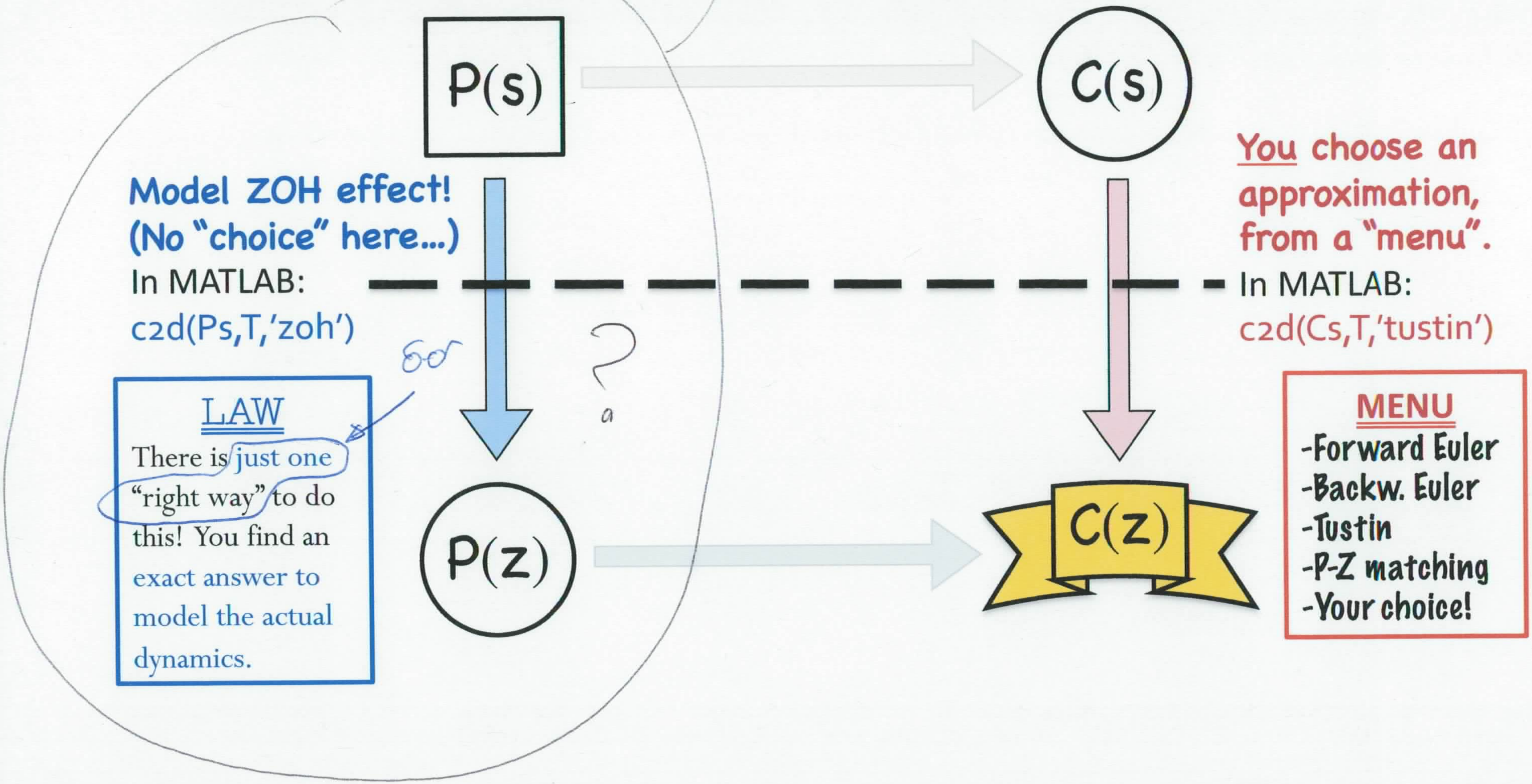
c2d(Cs, T, 'tustin')

C(s)

MATLAB c2d...

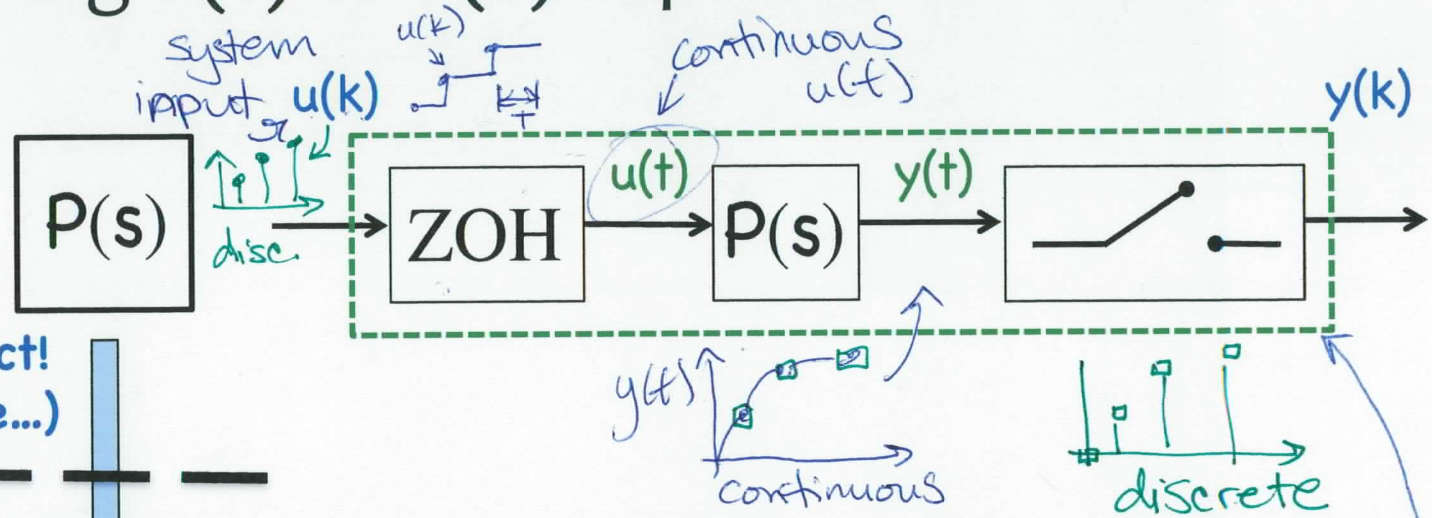


- There are two DIFFERENT ways we will convert a TF from CT to DT...

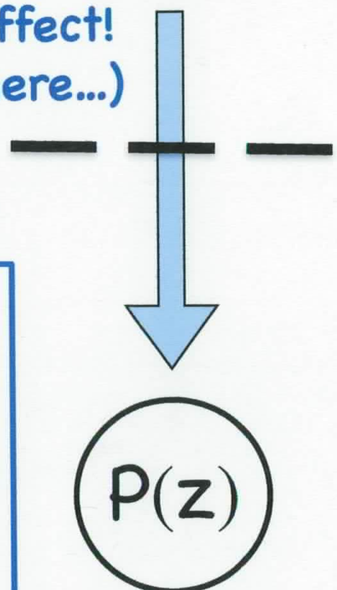


The "Direct Design" Path

- Converting P(s) to P(z) captures the ZOH.



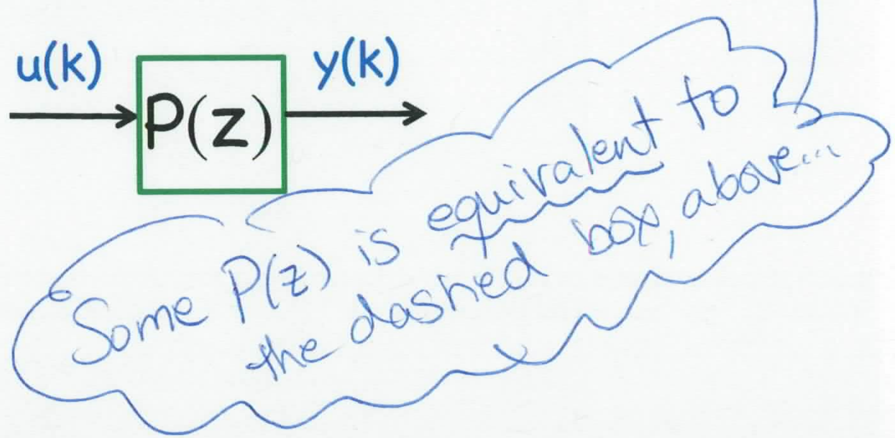
Model ZOH effect!
(No "choice" here…)
In MATLAB:
c2d(Ps,T,'zoh')



LAW

There is just one "right way" to do this! You find an exact answer to model the actual dynamics.

Given P(s), there is ONE SOLUTION for P(z).



ZOH Equivalent, P(z)

$$\left(1-\frac{1}{z}\right) = \left(\frac{z-1}{z}\right)$$

$$P_{ZOH}(z) = (1 - z^{-1}) \cdot \mathcal{Z}\left\{\frac{P(s)}{s}\right\}$$

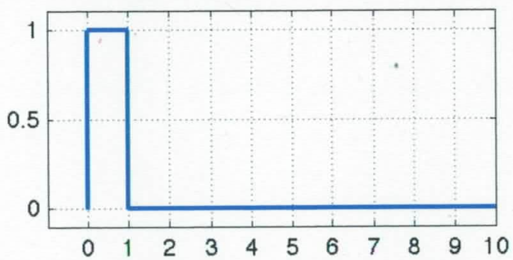
$P(s) \cdot \frac{1}{s}$

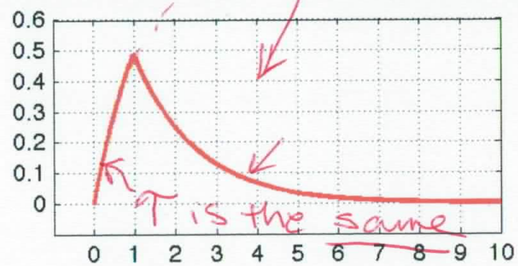
unit step

Zero-Order Hold Equivalent:

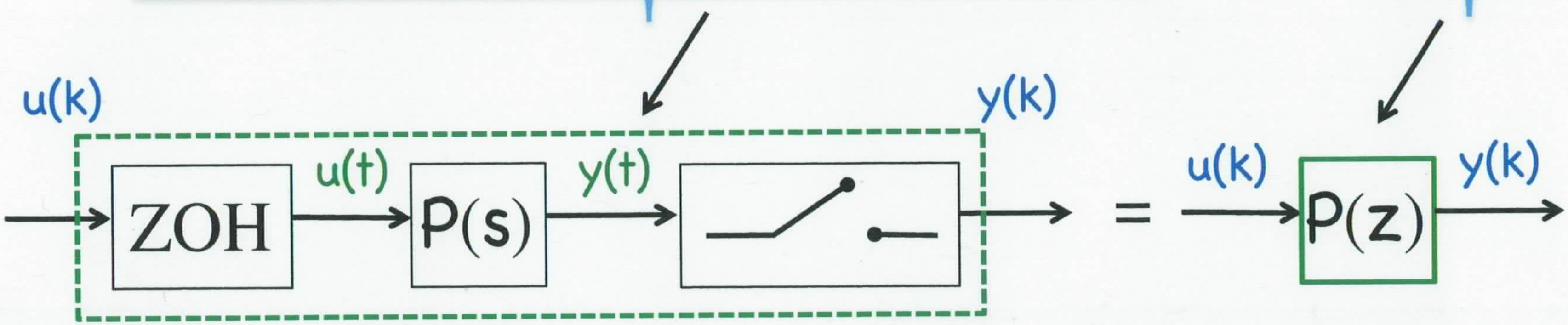
- u(k) gives a "pulse", to be integrated over "T".

1st-order example





- ZOH into P(s), then sampled (rate T) EQUAL P(z).



ZOH Equivalent: Math Definition

$$P_{ZOH}(z) = (1 - z^{-1}) \cdot \mathcal{Z}\left\{\frac{P(s)}{s}\right\}$$

You can Use Tables to find Z-Transform: $\mathcal{Z}\left\{\frac{P(s)}{s}\right\}$

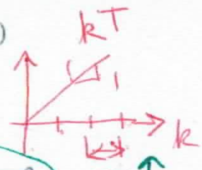
Franklin, Powell, and Workman (p.702)

Number	$\mathcal{F}(s)$	$f(kT)$	$F(z)$
1	—	$1,\ k=0;\ 0,\ k \neq 0$	1
2	—	$1,\ k=m;\ 0,\ k \neq m$	z^{-m}
3	$\frac{1}{s}$	$1(kT)$	$\frac{z}{z-1}$
4	$\frac{1}{s^2}$	kT	$\frac{Tz}{(z-1)^2}$
5	$\frac{1}{s^3}$	$\frac{1}{2!}(kT)^2$	$\frac{T^2}{2}\frac{z(z+1)}{(z-1)^3}$
6	$\frac{1}{s^4}$	$\frac{1}{3!}(kT)^3$	$\frac{T^3}{6}\frac{z(z^2+4z+1)}{(z-1)^4}$
7	$\frac{1}{s^m}$	$\lim_{a\to 0}\frac{(-1)^{m-1}}{(m-1)!}\frac{\partial^{m-1}}{\partial a^{m-1}}e^{-akT}$	$\lim_{a\to 0}\frac{(-1)^{m-1}}{(m-1)!}\frac{\partial^{m-1}}{\partial a^{m-1}}\frac{z}{z-e^{-aT}}$
8	$\frac{1}{s+a}$	e^{-akT}	$\frac{z}{z-e^{-aT}}$
9	$\frac{1}{(s+a)^2}$	kTe^{-akT}	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
10	$\frac{1}{(s+a)^3}$	$\frac{1}{2}(kT)^2e^{-akT}$	$\frac{T^2}{2}e^{-aT}\frac{z(z+e^{-aT})}{(z-e^{-aT})^3}$
11	$\frac{1}{(s+a)^m}$	$\frac{(-1)^{m-1}}{(m-1)!}\frac{\partial^{m-1}}{\partial a^{m-1}}(e^{-akT})$	$\frac{(-1)^{m-1}}{(m-1)!}\frac{\partial^{m-1}}{\partial a^{m-1}}\frac{z}{z-e^{-aT}}$
12	$\frac{a}{s(s+a)}$	$1-e^{-akT}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$

Example: $P(s) = \frac{1}{s^2}$

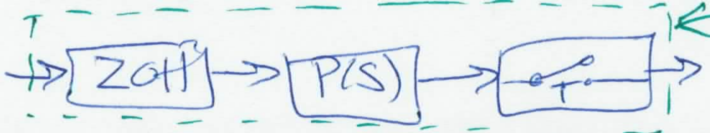
$P(s) = \frac{1}{s^2}$

$\frac{P(s)}{s} = \frac{1}{s^3}$



$Z(P(s))$

$Z\left(\frac{P(s)}{s}\right)$



$\leftarrow P(z) = ?$

ZOH Equivalent: Math Definition

Fadali and Visioli (p.533)

$$(1-\tfrac{1}{z})$$

$$P_{ZOH}(z) = (1 - z^{-1}) \cdot \mathcal{Z}\left\{\frac{P(s)}{s}\right\}$$

$\frac{1}{s}$

Example: $\boxed{P(s) = \frac{1}{s^2}}$

Using Table

$$\frac{P(s)}{s} = \frac{1}{s^3} \xrightarrow{Z} \left(\frac{z(z+1)T^2}{2(z-1)^3}\right)$$

$$\left(1 - \frac{1}{z}\right) = \left(\frac{z-1}{z}\right)$$

$$\left(\frac{\cancel{z-1}}{\cancel{z}}\right) * \left(\frac{\cancel{z}(z+1)T^2}{2(z-1)^{\cancel{3}\,2}}\right)$$

$$\boxed{P(z) = \frac{(z+1)T^2}{2(z-1)^2} = \left(\frac{T^2}{2}\right)\left(\frac{z+1}{z^2-2z+1}\right)}$$

No.	Continuous Time	Laplace Transform	Discrete Time	z-Transform
1	$\delta(t)$	1	$\delta(k)$	1
2	$1(t)$	$\frac{1}{s}$	$1(k)$	$\frac{z}{z-1}$
3	t	$\frac{1}{s^2}$	kT^{**}	$\frac{zT}{(z-1)^2}$
4	t^2	$\frac{2!}{s^3}$	$(kT)^2$	$\frac{z(z+1)T^2}{(z-1)^3}$
5	t^3	$\frac{3!}{s^4}$	$(kT)^3$	$\frac{z(z^2+4z+1)T^3}{(z-1)^4}$
6	$e^{-\alpha t}$	$\frac{1}{s+\alpha}$	$a^{k\,***}$	$\frac{z}{z-a}$
7	$1-e^{-\alpha t}$	$\frac{\alpha}{s(s+\alpha)}$	$1-a^k$	$\frac{(1-a)z}{(z-1)(z-a)}$
8	$e^{-\alpha t}-e^{-\beta t}$	$\frac{\beta-\alpha}{(s+\alpha)(s+\beta)}$	a^k-b^k	$\frac{(a-b)z}{(z-a)(z-b)}$
9	$te^{-\alpha t}$	$\frac{1}{(s+\alpha)^2}$	kTa^k	$\frac{azT}{(z-a)^2}$
10	$\sin(\omega_n t)$	$\frac{\omega_n}{s^2+\omega_n^2}$	$\sin(\omega_n kT)$	$\frac{\sin(\omega_n T)z}{z^2-2\cos(\omega_n T)z+1}$
11	$\cos(\omega_n t)$	$\frac{s}{s^2+\omega_n^2}$	$\cos(\omega_n kT)$	$\frac{z[z-\cos(\omega_n T)]}{z^2-2\cos(\omega_n T)z+1}$
12	$e^{-\zeta\omega_n t}\sin(\omega_d t)$	$\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}$	$e^{-\zeta\omega_n kT}\sin(\omega_d kT)$	$\frac{e^{-\zeta\omega_n T}\sin(\omega_d T)z}{z^2-2e^{-\zeta\omega_n T}\cos(\omega_d T)z+e^{-2\zeta\omega_n T}}$

$$\frac{2!}{s^3} \xrightarrow{Z} \frac{z(z+1)T^2}{(z-1)^3}$$

ZOH Equivalent: "Intuitively Obvious!"

$$\cancel{P_{ZOH}(z) = (1 - z^{-1}) \cdot Z\left\{\frac{P(s)}{s}\right\}}$$

Yikes! That was a bit mysterious (last page). We'll explain the derivation later. FOR NOW, let's think about a **MORE INTUITIVE** way to get the same solution…

Example: $P(s) = \frac{1}{s^2} = \frac{Y(s)}{U(s)}$ ← output ← input

Differential Eqn: $S^2 Y(s)$ $1 \cdot U(s)$

① $\boxed{u(t) = \ddot{y}(t)}$ Acceleration, C.T.

? Hmm… For a DT TF, we need:

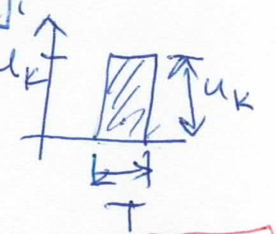
a) Expressions @ discrete values, $t = kT$ (k an integer.)

b) No derivative terms. (i.e., no $\dot{y}$ or $\ddot{y}$…)

a) $\boxed{\ddot{y}_k = u_k}$ ← ACCEL.

② What is the VELOCITY?

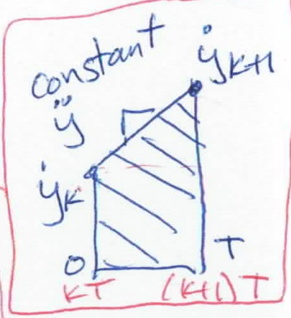
$$\dot{y}_{k+1} = \dot{y}_k + \int_{kT}^{(k+1)T} u_k \, dt$$



$$\boxed{\dot{y}_{k+1} = \dot{y}_k + u_k T}$$

③ And Position?

$$y_{k+1} = y_k + \int_{kT}^{(k+1)T} \dot{y} \, dt$$



$$= y_k + \int_0^T (\dot{y}_k + u_k t) \, dt$$

$$\boxed{y_{k+1} = y_k + \dot{y}_k T + \frac{1}{2} u_k T^2}$$

$$y_{k+2} = y_{k+1} + \dot{y}_{k+1} T + \frac{1}{2} u_{k+1} T^2$$

ZOH Equivalent: "Intuitively Obvious!"

(Example continued...)

$$\cancel{P_{ZOH}(z) = (1 - z^{-1}) \cdot \mathcal{Z}\left\{\frac{P(s)}{s}\right\}}$$

Example: $P(s) = \dfrac{1}{s^2}$

$$\boxed{\dot{y}_k = \frac{(y_{k+1} - y_k)}{T} - \frac{T}{2} u_k}$$

Recall $\boxed{\dot{y}_{k+1} = \dot{y}_k + u_k T}$ plug in for velocity...

$$\left(\frac{y_{k+2} - y_{k+1}}{T}\right) - \frac{T}{2} u_{k+1} = \left(\frac{y_{k+1} - y_k}{T}\right) - \frac{T}{2} u_k + u_k T$$

a) DT eqn? Yes!

b) No vel. or acc? Yes!!

Y terms | u terms

$$y_{k+2} - y_{k+1} - y_{k+1} + y_k = \frac{T^2}{2} u_{k+1} - \frac{T^2}{2} u_k + T^2 u_k$$

$$y_{k+2} - 2y_{k+1} + y_k = \frac{T^2}{2} u_{k+1} + \frac{T^2}{2} u_k$$

(difference eqn...) $\Downarrow$ Z xform

$$\boxed{(z^2 - 2z + 1)Y(z) = \frac{T^2}{2}(z+1)U(z)}$$

$$\frac{Y(z)}{U(z)} = \boxed{P(z) = \frac{T^2 (z+1)}{2(z^2 - 2z + 1)}}$$

Intuitively, P(z) matches result when using Tables...

Discrete-Time Transfer Function

DT TF's are analogous to CT TF's in many ways.

Ratio of polynomials:

$$\frac{b_1}{z} \qquad \frac{b_2}{z^2} \qquad \text{etc...}$$

$$P(z) = \frac{Y(z)}{X(z)} = \frac{b(z)}{a(z)} = \frac{b_0 z^0 + \underbrace{b_1 z^{-1}} + \underbrace{b_2 z^{-2}} + \ldots + b_m z^{-m}}{1 + a_1 z^{-1} + \ldots + a_n z^{-n}}$$

$$= \frac{b_0 z^n + b_1 z^{n-1} + b_2 z^{n-2} + \ldots + b_m z^{n-m}}{z^n + a_1 z^{n-1} + \ldots + a_n z^0}$$

Example:

$$P(z) = \frac{Y(z)}{X(z)} = \frac{4z^2 + 3z}{z^2 + 0.5z + 0.25}$$

$z^0 = 1$

$$Y \cdot (z^2 + .5z^1 + .25) = (4z^2 + 3z)X$$

Diff Eqn:

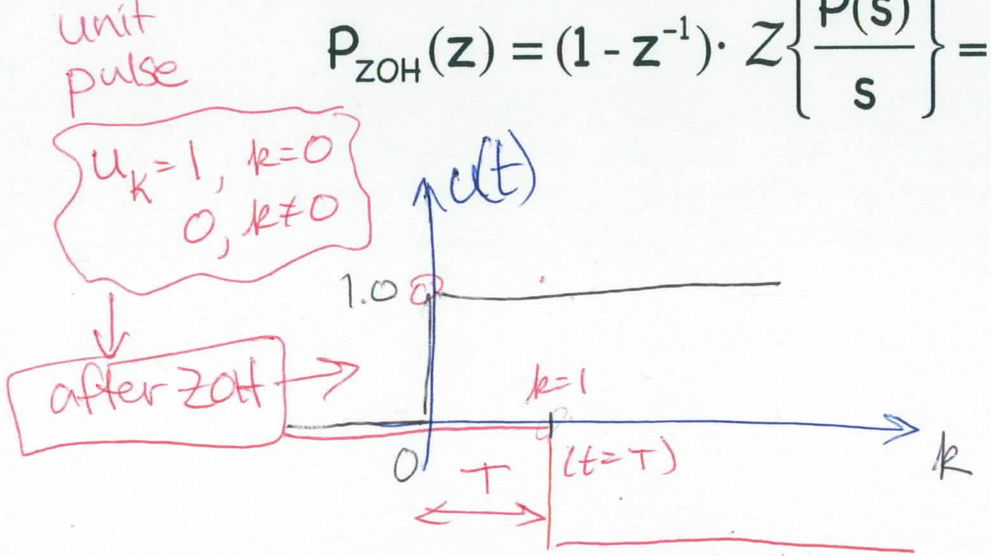
$$y_{k+2} + 0.5y_{k+1} + .25y_{k+0} = 4x_{k+2} + 3x_{k+1}$$

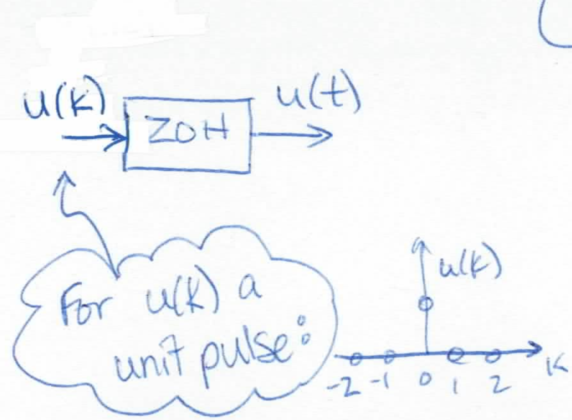
$$y_k = -.5y_{k-1} - .25y_{k-2} + 4x_k + 3x_{k-1}$$

So, P(z) (a DT TF) represents the DIFFERENCE EQUATION above...

ZOH Equivalent: Back to the Definition...

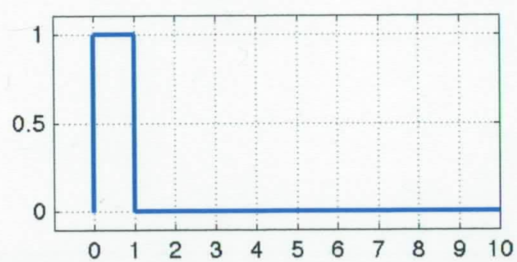
$$P_{ZOH}(z) = (1 - z^{-1}) \cdot \mathcal{Z}\left\{\frac{P(s)}{s}\right\} = \mathcal{Z}\left\{\frac{P(s)}{s}\right\} - \frac{1}{z}\mathcal{Z}\left\{\frac{P(s)}{s}\right\}$$

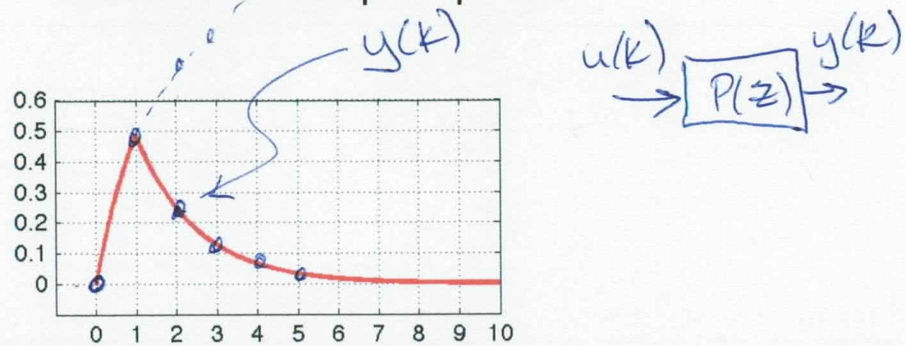




At left, a unit pulse is a step of +1 at t=0, plus a step of -1 at t=T.

At right, he response looks like a step response until t>=T. Then you subtract a time-shifted step response.





A closer look: what does "$\frac{1}{z}$" mean?

can write as... $\left(1-\frac{1}{z}\right)$

$$P_{ZOH}(z) = (1 - z^{-1}) \cdot \mathcal{Z}\left\{\frac{P(s)}{s}\right\} = \underbrace{\mathcal{Z}\left\{\frac{P(s)}{s}\right\}}_{G_1(z)} - \underbrace{\frac{1}{z}\mathcal{Z}\left\{\frac{P(s)}{s}\right\}}_{G_2(z) = \frac{1}{z}G_1(z)}$$

Example:

$$G_1(z) = \frac{z}{z-1} = \frac{Y(z)}{X(z)}$$

$$z \cdot X(z) = (z-1) \cdot Y(z)$$

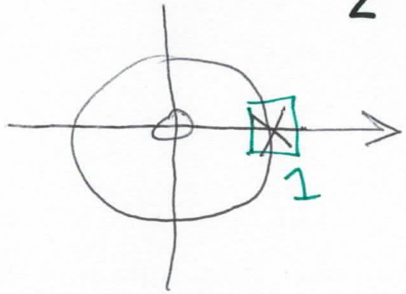
$$x_{k+1} = y_{k+1} - y_k$$

$$x_k = y_k - y_{k-1}$$

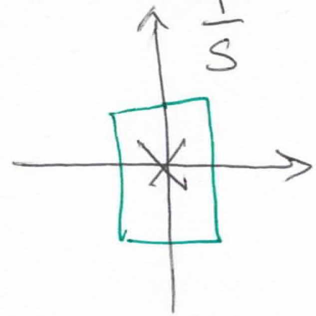
$$\boxed{y_k = y_{k-1} + x_k}$$

For an impulse,

input: $\cancel{x}(k) = \begin{cases} 1, & k=0 \\ 0, & k \neq 0 \end{cases}$

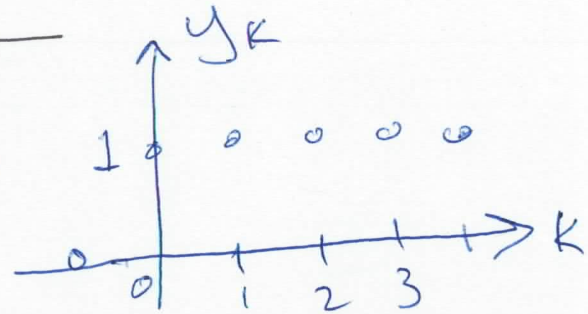


vs



unit pulse

k	x (in)	y (out)
-1	0	0
0	1	1
1	0	1
2	0	1
3	0	1
4	0	1



(Not defined @ non-integer k values...)

(Continued...)

$$P_{ZOH}(z) = (1 - z^{-1}) \cdot \mathcal{Z}\left\{\frac{P(s)}{s}\right\} = \underbrace{\mathcal{Z}\left\{\frac{P(s)}{s}\right\}}_{G_1(z)} - \underbrace{\frac{1}{z}\mathcal{Z}\left\{\frac{P(s)}{s}\right\}}_{G_2(z) = \frac{1}{z}G_1(z)}$$

G2 is just delayed in time by one sample.

compared w/ $G_1(z)$

Same Example:

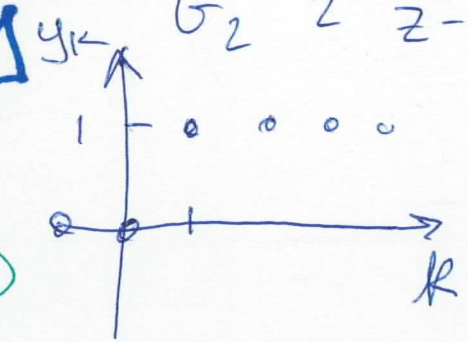
$$\boxed{G_2(z) = \frac{1}{z-1}} = \frac{1}{z}\left[\frac{z}{z-1}\right] = \frac{Y}{X}$$

$$G_2 = \frac{1}{z} \cdot \frac{z}{z-1}$$

$$x_k = y_{k+1} - y_k$$

$$\boxed{y_k = y_{k-1} + x_{k-1}}$$

k	x	y
-1	0	0
0	1	0
1	0	1
2	0	1
3	0	1
4	0	1



Definition of Z Transform

Given a sequence of samples, x_k:

$$X(z) \equiv \mathcal{Z}\{x(k)\} \equiv \sum_{-\infty}^{\infty} x_k z^{-k}$$

Example:

$$G_1(z) = \frac{4z^3 + 2z^2 + 5z + 3}{z^3}$$

$$\rightarrow G_1(z) = \frac{Y}{X}$$

$$4\underbrace{x_{k+3}}_{k} + 2\underbrace{x_{k+2}}_{k-1} + 5\underbrace{x_{k+1}}_{k-2} + 3\underbrace{x_{k}}_{k-3} = \underbrace{y_{k+3}}_{k}$$

Hmm! I see a PATTERN here...

k	x	y	=	"$4x_k$"	+	$2x_{k-1}$	+	$5x_{k-2}$	+	$3x_{k-3}$
-1	0	0								
0	1	4	=	4	+	0	+	0	+	0
1	0	2	=	0		2		0		0
2	0	5		0		0		5		0
3	0	3	=	0		0		0		3
4	0	0								

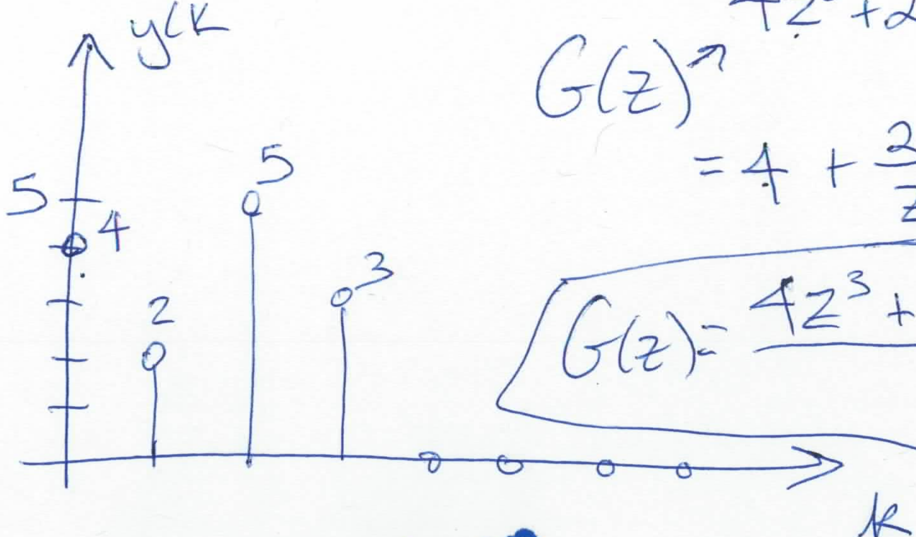
Impulse Response of G(z)

Same Example:

$$G_1(z) = \frac{4z^3 + 2z^2 + 5z + 3}{z^3}$$

$$X(z) \equiv \mathcal{Z}\{x(k)\} = \sum_{-\infty}^{\infty} x_k z^{-k}$$

impulse response



$$G(z) \rightarrow 4z^0 + 2z^{-1} + 5z^{-2} + 3z^{-3}$$

$$= 4 + \frac{2}{z} + \frac{5}{z^2} + \frac{3}{z^3}$$

$$G(z) = \frac{4z^3 + 2z^2 + 5z + 3}{z^3}$$

Note: "Deadbeat control" is one where closed-loop sys has... ...poles @ origin on z-plane

z-plane

(poles @ origin on z-plane are at $s = -\infty$ on s-plane, analogously...)