Today: **Discrete-Time Transfer Functions**

Lecture 3
ECE 147B
Digital Control

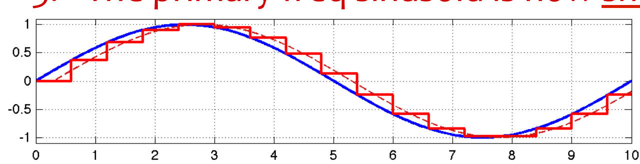
Reading: FPW 4.2, 4.3.1, 6.1 (p.189-195)
or: FV 3.3-3.7, 6.3.3

Topics:
- ZOH time delay effect (Review from Lecture 2)
- ZOH equivalent
- Definition of the Z Transform
- Discrete-Time Transfer Functions
- Impulse Response of G(z)
- Tustin with Pre-Warp

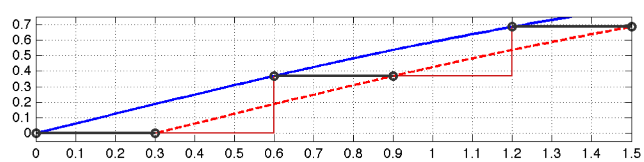
Next Class: Z Transform, DT TF, Tustin with pre-warp
- FPW: 6.3.1, 4.6.1, 4.2.1,
or: - FV: 2.3, 2.5, 2.8, 6.3.3

Sampling: ZOH Time Delay

1. Fourier: CT signal can be broken into sinusoids.
2. After ZOH, the CT waveform of a sinusoid has the original (primary) freq plus higher freqs, too.
3. The primary freq sinusoid is now shifted by T/2…

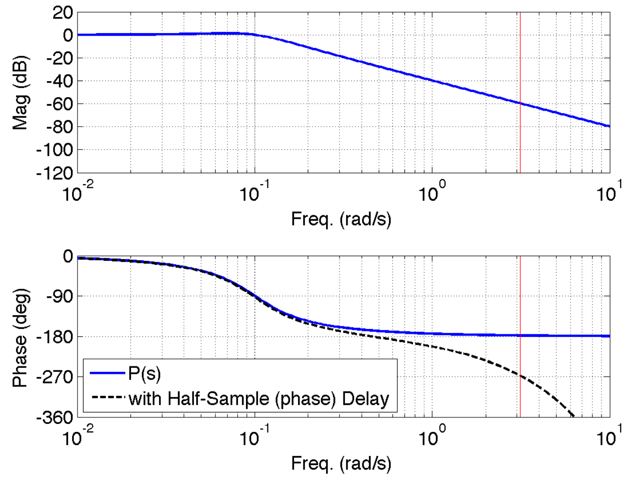


"Half-sample delay" approximation works well – until you get "very close" to Nyquist freq.



ZOH Consequences in FREQUENCY DOMAIN

- But wait, $G(z)$ at $z=-1$ (Nyquist) must be REAL-VALUED! i.e., phase of 0 or +/-180 deg.

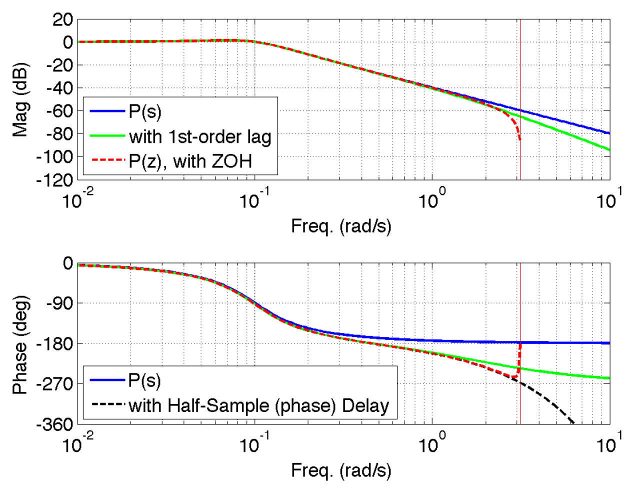


"Half-sample delay" approximation works well – until you get "very close" to Nyquist freq.

…Phase follow this approx until VERY close to Nyquist!!

ZOH Consequences in FREQUENCY DOMAIN

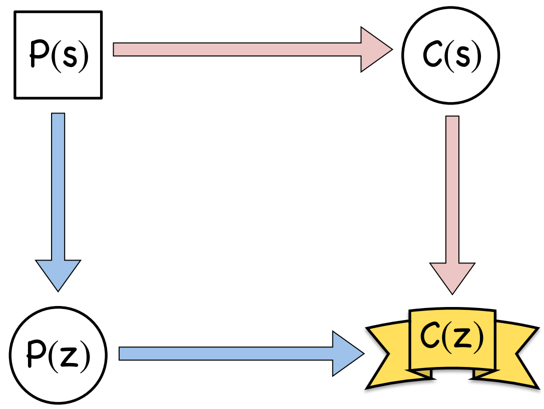
- Lab 1 using a 1^{st}-order lag approximation for the effects of the ZOH:



$$P_{lag}(s) = \frac{1}{\left[\frac{T}{2}\right]s+1}$$

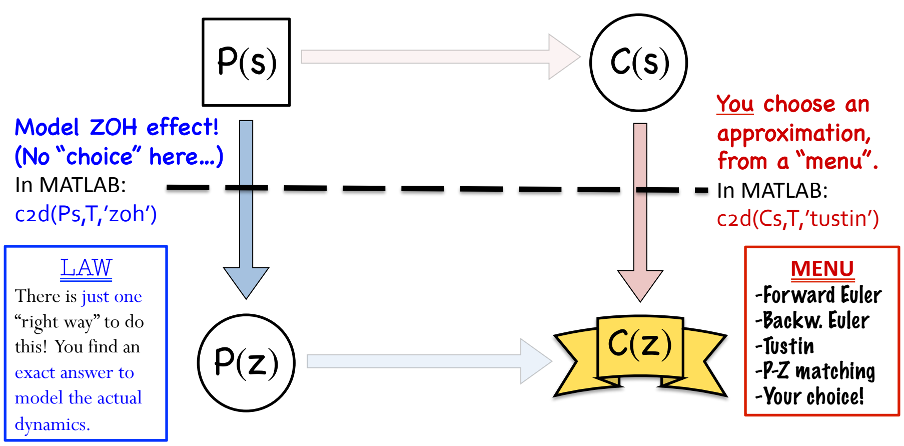
Digital Control Design Approaches

- Recall we have two design path options:



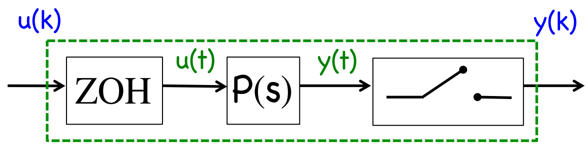
MATLAB c2d...

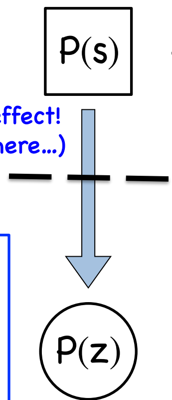
- There are two DIFFERENT ways we will convert a TF from CT to DT…



The "Direct Design" Path

- Converting P(s) to P(z) captures the ZOH.

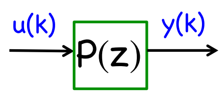




Model ZOH effect!
(No "choice" here...)
In MATLAB:
c2d(Ps,T,'zoh')

LAW
There is just one "right way" to do this! You find an exact answer to model the actual dynamics.

Given P(s), there is ONE SOLUTION for P(z).

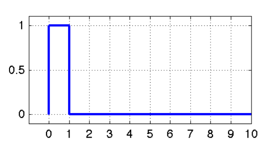


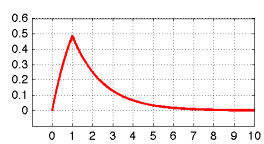
ZOH Equivalent, P(z)

$$P_{ZOH}(z) = (1 - z^{-1}) \cdot \mathcal{Z}\left\{\frac{P(s)}{s}\right\}$$

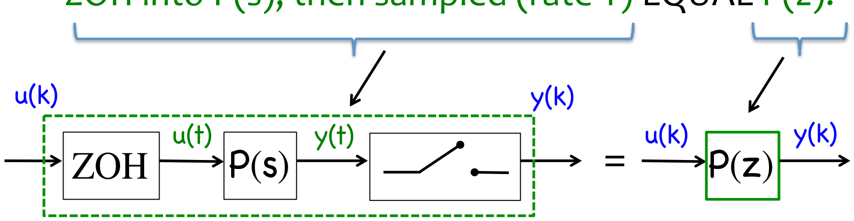
Zero-Order Hold Equivalent:

- u(k) gives a "pulse", to be integrated over "T".





- ZOH into P(s), then sampled (rate T) EQUAL P(z).



ZOH Equivalent: Math Definition

$$P_{ZOH}(z) = (1 - z^{-1}) \cdot \mathcal{Z}\left\{\frac{P(s)}{s}\right\}$$

Use Tables to find Z-Transform: $\mathcal{Z}\left\{\frac{P(s)}{s}\right\}$

Franklin, Power, and Workman (p.702)

Number	$\mathcal{F}(s)$	$f(kT)$	$F(z)$
1	—	$1,\ k=0;\ 0,\ k \neq 0$	1
2	—	$1,\ k=m;\ 0,\ k \neq m$	z^{-m}
3	$\frac{1}{s}$	$1(kT)$	$\frac{z}{z-1}$
4	$\frac{1}{s^2}$	kT	$\frac{Tz}{(z-1)^2}$
5	$\frac{1}{s^3}$	$\frac{1}{2!}(kT)^2$	$\frac{T^2}{2}\frac{z(z+1)}{(z-1)^3}$
6	$\frac{1}{s^4}$	$\frac{1}{3!}(kT)^3$	$\frac{T^3}{6}\frac{z(z^2+4z+1)}{(z-1)^4}$
7	$\frac{1}{s^m}$	$\lim_{a\to 0}\frac{(-1)^{m-1}}{(m-1)!}\frac{\partial^{m-1}}{\partial a^{m-1}}e^{-akT}$	$\lim_{a\to 0}\frac{(-1)^{m-1}}{(m-1)!}\frac{\partial^{m-1}}{\partial a^{m-1}}\frac{z}{z-e^{-aT}}$
8	$\frac{1}{s+a}$	e^{-akT}	$\frac{z}{z-e^{-aT}}$
9	$\frac{1}{(s+a)^2}$	kTe^{-akT}	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
10	$\frac{1}{(s+a)^3}$	$\frac{1}{2}(kT)^2e^{-akT}$	$\frac{T^2}{2}e^{-aT}\frac{z(z+e^{-aT})}{(z-e^{-aT})^3}$
11	$\frac{1}{(s+a)^m}$	$\frac{(-1)^{m-1}}{(m-1)!}\frac{\partial^{m-1}}{\partial a^{m-1}}(e^{-akT})$	$\frac{(-1)^{m-1}}{(m-1)!}\frac{\partial^{m-1}}{\partial a^{m-1}}\frac{z}{z-e^{-aT}}$
12	$\frac{a}{s(s+a)}$	$1-e^{-akT}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$

ZOH Equivalent: Math Definition

Fadali and Visioli (p.533)

$$P_{ZOH}(z) = (1 - z^{-1}) \cdot \mathcal{Z}\left\{\frac{P(s)}{s}\right\}$$

Example: $P(s) = \frac{1}{s^2}$

No.	Continuous Time	Laplace Transform	Discrete Time	z-Transform
1	$\delta(t)$	1	$\delta(k)$	1
2	$1(t)$	$\frac{1}{s}$	$1(k)$	$\frac{z}{z-1}$
3	t	$\frac{1}{s^2}$	kT^{**}	$\frac{zT}{(z-1)^2}$
4	t^2	$\frac{2!}{s^3}$	$(kT)^2$	$\frac{z(z+1)T^2}{(z-1)^3}$
5	t^3	$\frac{3!}{s^4}$	$(kT)^3$	$\frac{z(z^2+4z+1)T^3}{(z-1)^4}$
6	$e^{-\alpha t}$	$\frac{1}{s+\alpha}$	a^{k} ***	$\frac{z}{z-a}$
7	$1-e^{-\alpha t}$	$\frac{\alpha}{s(s+\alpha)}$	$1-a^k$	$\frac{(1-a)z}{(z-1)(z-a)}$
8	$e^{-\alpha t}-e^{-\beta t}$	$\frac{\beta-\alpha}{(s+\alpha)(s+\beta)}$	a^k-b^k	$\frac{(a-b)z}{(z-a)(z-b)}$
9	$te^{-\alpha t}$	$\frac{1}{(s+\alpha)^2}$	kTa^k	$\frac{azT}{(z-a)^2}$
10	$\sin(\omega_n t)$	$\frac{\omega_n}{s^2+\omega_n^2}$	$\sin(\omega_n kT)$	$\frac{\sin(\omega_n T)z}{z^2-2\cos(\omega_n T)z+1}$
11	$\cos(\omega_n t)$	$\frac{s}{s^2+\omega_n^2}$	$\cos(\omega_n kT)$	$\frac{z[z-\cos(\omega_n T)]}{z^2-2\cos(\omega_n T)z+1}$
12	$e^{-\zeta\omega_n t}\sin(\omega_d t)$	$\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}$	$e^{-\zeta\omega_n kT}\sin(\omega_d kT)$	$\frac{e^{-\zeta\omega_n T}\sin(\omega_d T)z}{z^2-2e^{-\zeta\omega_n T}\cos(\omega_d T)z+e^{-2\zeta\omega_n T}}$

ZOH Equivalent: "Intuitively Obvious!"

$$\cancel{P_{ZOH}(z) = (1 - z^{-1}) \cdot \mathcal{Z}\left\{\frac{P(s)}{s}\right\}}$$

Yikes! That was a bit mysterious (last page). We'll explain the derivation later. FOR NOW, let's think about a **MORE INTUITIVE** way to get the same solution…

Example: $P(s) = \dfrac{1}{s^2}$

ZOH Equivalent: "Intuitively Obvious!"

(Example continued…)

$$\cancel{P_{ZOH}(z) = (1 - z^{-1}) \cdot \mathcal{Z}\left\{\frac{P(s)}{s}\right\}}$$

Example: $P(s) = \dfrac{1}{s^2}$

Discrete-Time Transfer Function

DT TF's are analogous to CT TF's in many ways.
Ratio of polynomials:

$$P(z) = \frac{Y(z)}{X(z)} = \frac{b(z)}{a(z)} = \frac{b_0 z^0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_m z^{-m}}{1 + a_1 z^{-1} + \ldots + a_n z^{-n}}$$

$$= \frac{b_0 z^n + b_1 z^{n-1} + b_2 z^{n-2} + \ldots + b_m z^{n-m}}{z^n + a_1 z^{n-1} + \ldots + a_n z^0}$$

Example:

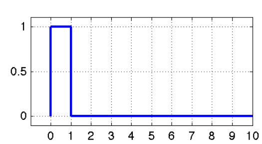
$$P(z) = \frac{Y(z)}{X(z)} = \frac{4z^2 + 3z}{z^2 + 0.5z + 0.25}$$

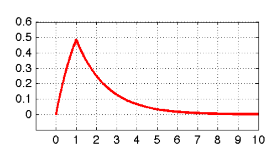
ZOH Equivalent: Back to the Definition...

$$P_{ZOH}(z) = (1 - z^{-1}) \cdot Z\left\{\frac{P(s)}{s}\right\} = Z\left\{\frac{P(s)}{s}\right\} - \frac{1}{z} Z\left\{\frac{P(s)}{s}\right\}$$

At left, a unit pulse is a step of +1 at t=0, plus a step of -1 at t=T.

At right, he response looks like a step response until t>=T. Then you subtract a time-shifted step response.





A closer look: what does $\frac{1}{z}$ mean?

$$P_{ZOH}(z) = (1 - z^{-1}) \cdot \mathcal{Z}\left\{\frac{P(s)}{s}\right\} = \underbrace{\mathcal{Z}\left\{\frac{P(s)}{s}\right\}}_{G_1(z)} + \underbrace{\frac{1}{z}\mathcal{Z}\left\{\frac{P(s)}{s}\right\}}_{G_2(z) = \frac{1}{z}G_1(z)}$$

Example:

$$G_1(z) = \frac{z}{z-1}$$

(Continued...)

$$P_{ZOH}(z) = (1 - z^{-1}) \cdot \mathcal{Z}\left\{\frac{P(s)}{s}\right\} = \underbrace{\mathcal{Z}\left\{\frac{P(s)}{s}\right\}}_{G_1(z)} + \underbrace{\frac{1}{z}\mathcal{Z}\left\{\frac{P(s)}{s}\right\}}_{G_2(z) = \frac{1}{z}G_1(z)}$$

Same Example:

$$G_2(z) = \frac{1}{z-1} = \frac{1}{z}\left[\frac{z}{z-1}\right]$$

G2 is just delayed in time by one sample.

Definition of Z Transform

Given a sequence of samples, x_k:

$$X(z) \equiv \mathcal{Z}\{x(k)\} \equiv \sum_{-\infty}^{\infty} x_k z^{-k}$$

Example:

$$G_1(z) = \frac{4z^3 + 2z^2 + 5z + 3}{z^3}$$

Impulse Response of G(z)

Same Example:

$$G_1(z) = \frac{4z^3 + 2z^2 + 5z + 3}{z^3}$$