**Today:** # Time Response: s-plane vs z-plane

**Lecture 5**
ECE 147B
Digital Control

**Reading:** FPW 4.4, 7.3.1
*or*: FV 2.6

**Topics:**
- Classic examples (Review from Lecture 4)
- Properties of the Z Transform
- Tustin with Pre-Warp
- Time response calculation
- HW 2, HW1 Sol's. Prelab TYPO. (3.78)

**Next Class:** Frequency response. Direct design; deadbeat control.
- FPW: 4.5, 7.5

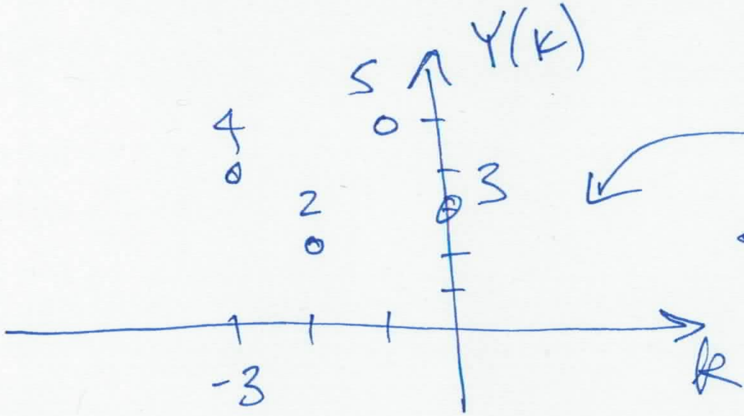
or: - FV: 2.8, 4.1-4.6, 6.7

$C_6(z)$...
See TA's...

# Definition of Z Transform

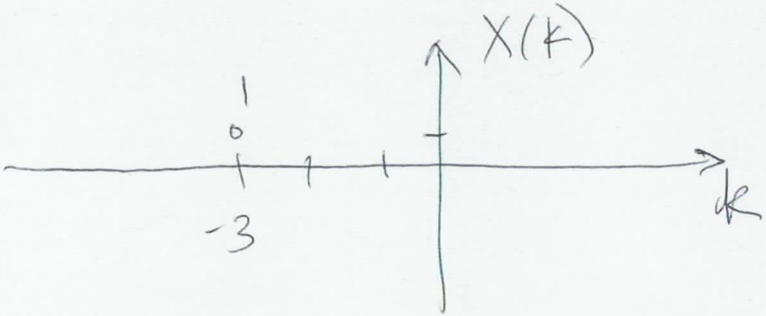
Recall: A given sequence of samples, $x_k$, has a corresponding Z Transform, defined as:

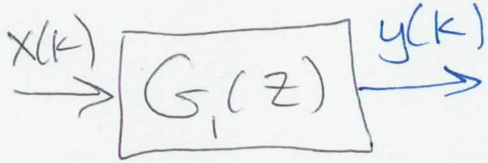
$$X(z) \equiv \mathcal{Z}\{x(k)\} \equiv \sum_{-\infty}^{\infty} x_k z^{-k}$$




Example:

$$G_1(z) = \frac{Y(z)}{X(z)} = \frac{4z^3 + 2z^2 + 5z + 3}{z^3}$$



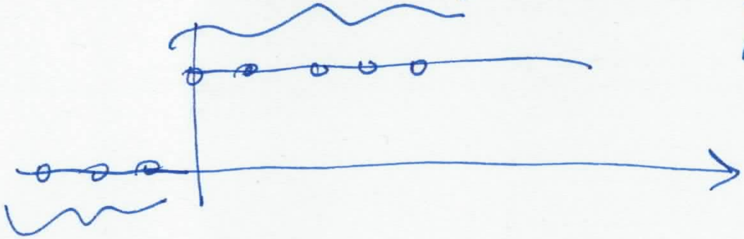



# Example: impulse response is a UNIT STEP

$$X(z) \equiv \mathcal{Z}\{x(k)\} \equiv \sum_{k=-\infty}^{\infty} x_k z^{-k}$$

> Output y(k) are SAMPLES of the IMPULSE response for: $G(s) = \dfrac{1}{s}$

$x(k) = 1,\ k \geq 0$



$x(k) = 0,\ k < 0$

$$X(z) = 1 \cdot z^0 + 1 \cdot z^{-1} + 1 \cdot z^{-2} \ldots$$

$$= \frac{z}{z-1} \quad \text{(last time...)}$$

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Another perspective: • What if we have a ~~step~~ + impulse @ $k=0$

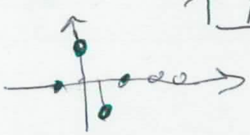
And

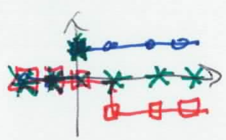
• a ~~step~~ − impulse (unit) @ $k=1$ ?

input: $\boxed{1 \cdot z^0 - 1 z^{-1}}$

output: $\boxed{1 \cdot z^0}$

$$u(k) \to \boxed{X(z)} \to v(k)$$





$$\frac{\text{Output}}{\text{Input}} = \frac{1}{1 - \frac{1}{z}} = \frac{z}{z-1}$$

✓ "unit step"

# Ex.: imp resp is a DECAYING EXPONENTIAL

$$\boxed{X(z) \equiv \mathcal{Z}\{x(k)\} \equiv \sum_{-\infty}^{\infty} x_k z^{-k}}$$

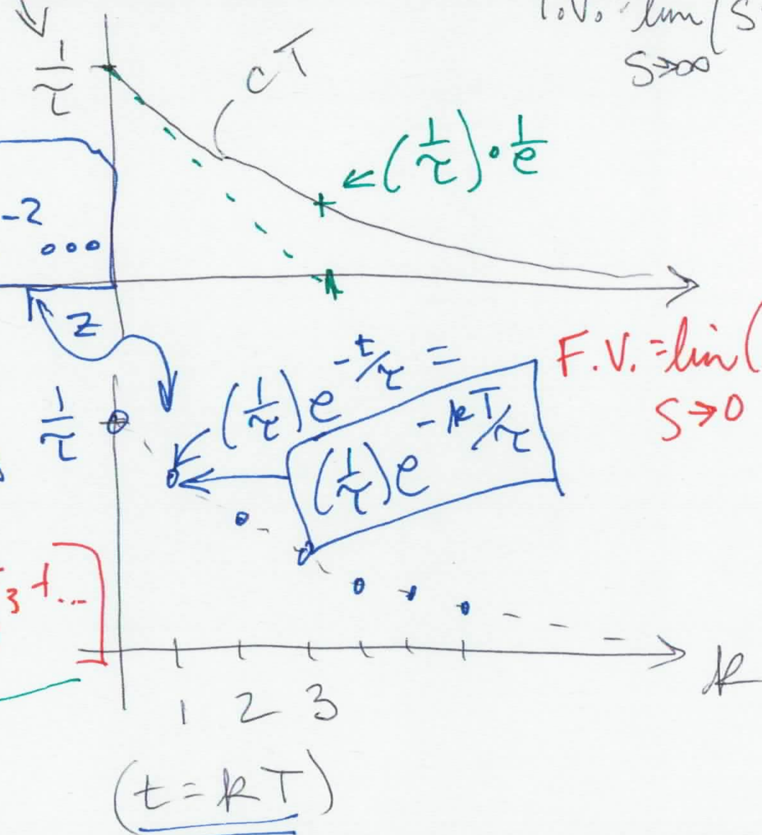
Output y(k) are SAMPLES of the IMPULSE response for: $G(s) = \dfrac{1}{\tau s + 1}$

I.V. = $\lim_{s \to \infty}\left(s \cdot \frac{1}{\tau s + 1}\right)$

$x(k) = 0, \quad k < 0$

$$\boxed{\left(\tfrac{1}{\tau}\right) z^0 + \left(\tfrac{1}{\tau}\right) e^{-T/\tau} z^{-1} + \left(\tfrac{1}{\tau}\right) e^{-2T/\tau} z^{-2} \dots}$$

$$\boxed{= a \sum_{k=0}^{\infty} e^{-aTk} z^{-k}}, \text{ where } \boxed{a = \tfrac{1}{\tau}}$$




F.V. = $\lim_{s \to 0}\left(s \cdot \frac{1}{\tau s + 1}\right)$

$$= a \underbrace{\left[1 + \frac{1}{e^{aT} z} + \frac{1}{(e^{aT} z)^2} + \frac{1}{(e^{aT} z)^3} + \dots\right]}_{``\Delta"}$$

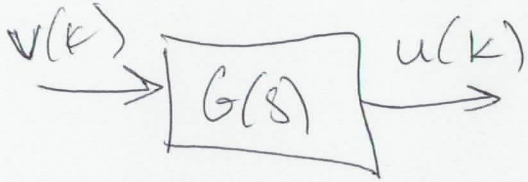
$$= a \frac{\Delta}{\Delta\left(1 - \frac{1}{e^{aT} z}\right)}$$

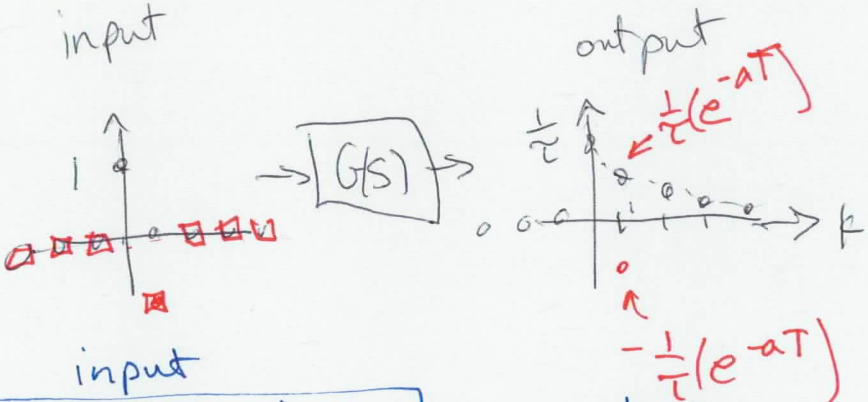
$$\boxed{= a \frac{z}{z - e^{-aT}}}$$ ← for a = 0, this is just the step response.

# Ex.: imp resp is a DECAYING EXPONENTIAL

<u>OR</u>, use a TRICK to find (intuitively) some input, $v(k)$, s.t. the input & output are non-zero only over a <u>limited</u> range.

(<u>NOT</u> an infinite sum...)




input

$$\boxed{1 \cdot z^0 - (\cancel{\tfrac{1}{\tau}})(e^{-aT})\,z^{-1}}$$

output

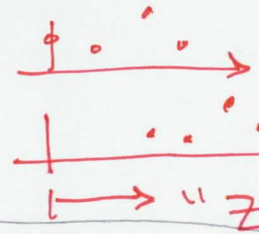
$$\boxed{\left(\tfrac{1}{\tau}\right) z^0}$$

# Properties of the Z Transform

1. Linearity $Z\{a\cdot x(k)+b\cdot y(k)\} = \sum_{-\infty}^{\infty}\{ax_k+by_k\}z^{-k}$
   $= aX(z)+bY(z)$

2. Convolution of time sequences <-> Multiplication of xforms
   $\hookrightarrow Z\{\sum_{m=-\infty}^{\infty} x(m)\cdot y(k-m)\} = X(z)Y(z)$

3. Time Shift: $Z\{f(k+n)\}=z^nF(z)$
   $Z(x(k+n)) = z^n F(z)$



"$z^2$" (shift by +2)

$z \to$ "$rz$"

$\frac{z}{z-1}$

4. Scaling on the z-plane: $Z\{r^{-k}f(k)\}=F(rz)$
   e.g. step: 1 vs scaled step r $\to k$
   $\frac{rz}{rz-1} = \frac{z}{z-\frac{1}{r}}$

5. Final Value Theorem:
   See HW 2! $\boxed{\lim_{k\to\infty}(y(k)) = \lim_{z\to 1}((z-1)Y(z))}$

6. and…(drumroll)…Inversion, $X(z) \Leftrightarrow \sum_{-\infty}^{\infty} x_k z^{-k}$

Can use partial frac exp. (for example), but this is PAINFUL! So, intuition is useful (more practical).

So, for F. V. to a UNIT STEP $\leftarrow$ X, input

$Y(z) = \left(\frac{z}{z-1}\right)G(z)$

$\lim_{z\to 1}\left(\frac{z}{\cancel{z-1}}\right)(\cancel{z-1})G$

$(X \to \boxed{G} \to Y)$

# Tustin with Pre-warp

$C(s) = \frac{s+2}{s+40}$

Recall the standard Tustin transformation (aka "trapezoid rule").

$$s \leftrightarrow \boxed{\frac{2}{T}} \frac{(z-1)}{(z+1)}$$

← Commonly used in the "EMULATION": $\boxed{C(s) \rightarrow C(z)}$

(147A) (147B)

"Tustin with pre-warp" is similar, except is it <u>scaled</u> to preserve MAGNITUDE and PHASE of C(z) at a "pre-warp" frequency, $\omega_o$.

$$s \leftrightarrow \boxed{\alpha} \frac{(z-1)}{(z+1)}$$

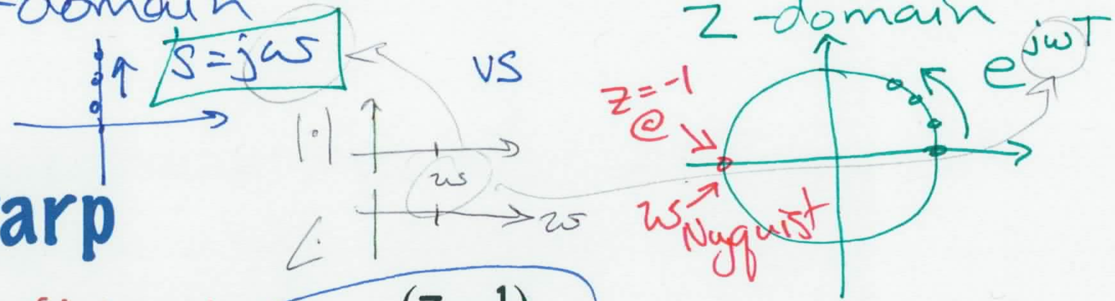
$$\alpha = \frac{\omega_0}{\tan\left(\frac{\omega_0 T}{2}\right)}$$

"$\frac{2}{T}$" becomes "$\alpha$"

Basically, the point is (usually) that "$\omega_o$" represents the desired cross-over freq: $\boxed{\omega_{co} = \omega_o}$

(Gain & Phase of loop transm. (L.T.) stay the same

Recall, for freq. Resp.




# Tustin with Pre-warp

1. Let $z = e^{j\omega_o T}$ for some $\omega_o$ of interest, and evaluate Gain and Phase of:

$$S = \alpha \frac{(z-1)}{(z+1)}$$

$C(s) = \frac{s+2}{s+40}$, if we can have mag. & phase of the expression for S be the SAME as for s-domain freq. resp., the mag & phase of C(z) will match C(s) @ this freq. ($\omega_o$)

2. Goal: If we match gain and phase for "s" (in the expression above),

we will match gain and phase for ANY: $H(s) = \frac{b(s)}{a(s)}$

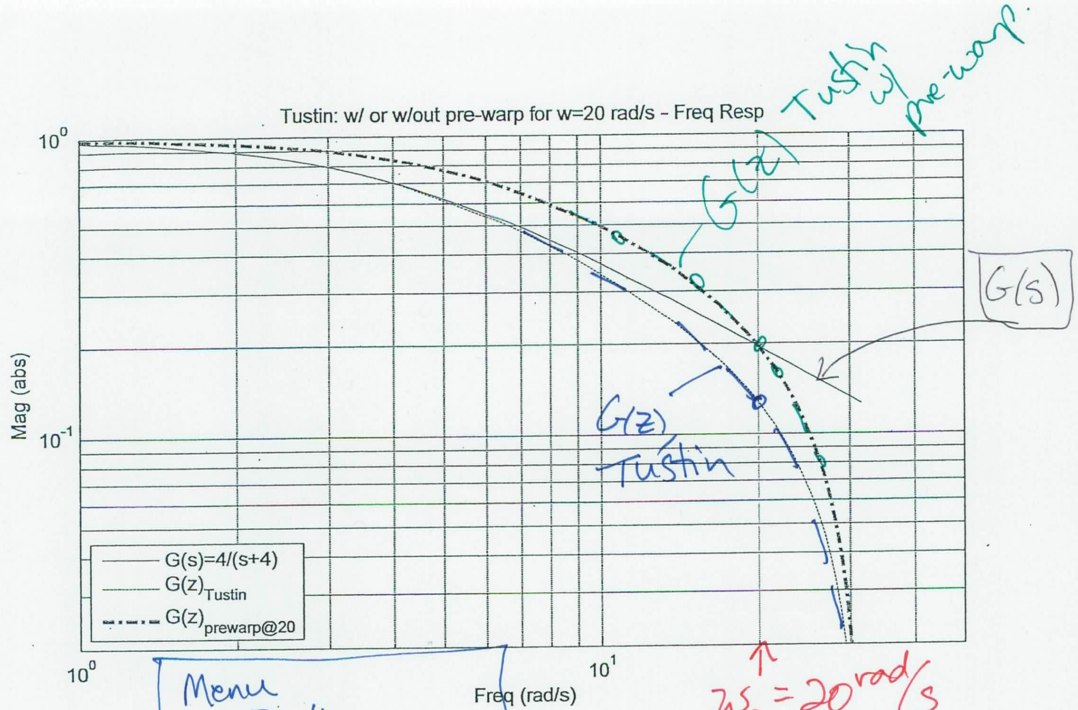
$$S = \alpha \frac{\left(e^{j\omega_o T} - 1\right)}{\left(e^{j\omega_o T} + 1\right)} = \alpha \frac{\left(e^{j\omega_o \frac{T}{2}} - e^{-j\omega_o T/2}\right)\left(e^{j\frac{\omega_o T}{2}}\right)}{\left(e^{j\omega_o T/2} + e^{-j\omega_o T/2}\right)\left(e^{j\omega_o \frac{T}{2}}\right)}$$

$$= \alpha \frac{2\sin\left(\frac{\omega_o T}{2}\right) j}{2\cos\left(\frac{\omega_o T}{2}\right)} = \alpha \tan\left(\frac{\omega_o T}{2}\right) j$$

@ $\omega = \omega_o$, we <u>want</u> $s = j\omega_o$,

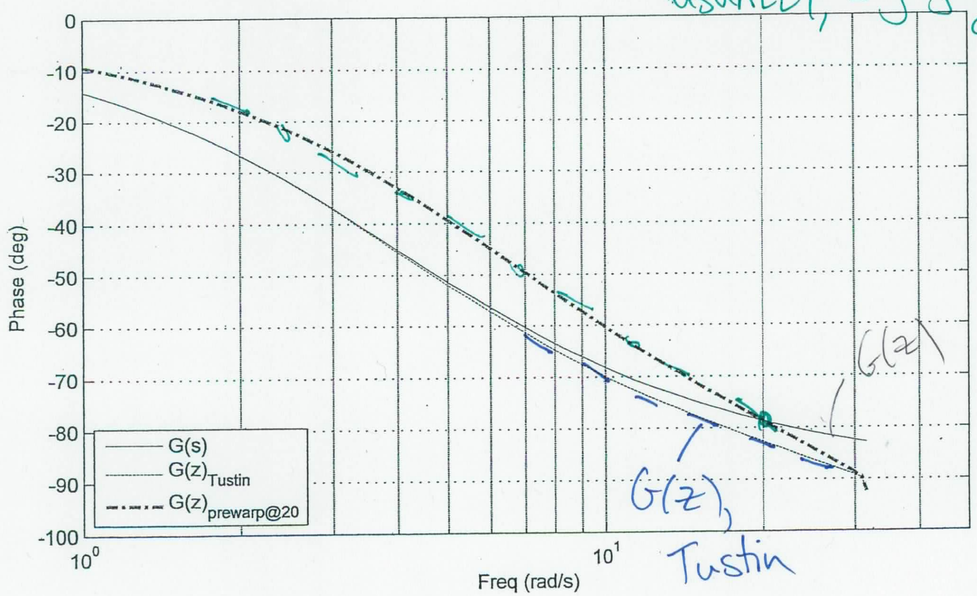
$\therefore \cancel{j}\omega_o = \alpha \tan\left(\frac{\omega_o T}{2}\right)\cancel{j}$

$$\boxed{\alpha = \frac{\omega_0}{\tan\left(\frac{\omega_0 T}{2}\right)}}$$



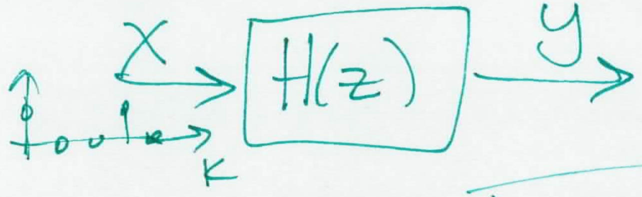

Menu
- Tustin
- Tustin pre-warp

(Again, $w_0 = w_{co}$, USUALLY, by your design.)



# Solving for TIME RESPONSE, y(k) <-> Y(z)

Given some H(z) = Y(z)/X(z),

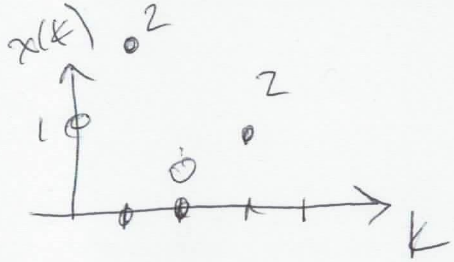



Steps:

1. Compute plant TF, H(z).
2. Compute input x(k)'s xform, X(z).

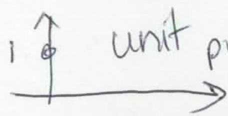
use definition $\longrightarrow$ $\boxed{X(z) \Leftrightarrow \sum_{-\infty}^{\infty} x_k z^{-k}}$

a)



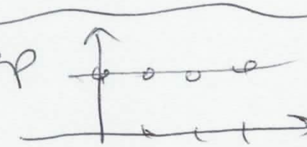

$\longleftrightarrow 1\cdot z^{0} + 2\cdot z^{-1} + 0\cdot z^{-2} + 2\cdot z^{-3}$

b)




unit pulse $\longleftrightarrow 1$

c) step



$\longrightarrow 1+\frac{1}{z}+\frac{1}{z^2}\cdots+\frac{1}{z^{\infty}} = \frac{1}{1-\frac{1}{z}} = \frac{z}{z-1}$

3. Form product: Y(z)=X(z)H(z)
4. Invert Y(z) to get y(kT)

# Solving for TIME RESPONSE, y(k) <-> Y(z)

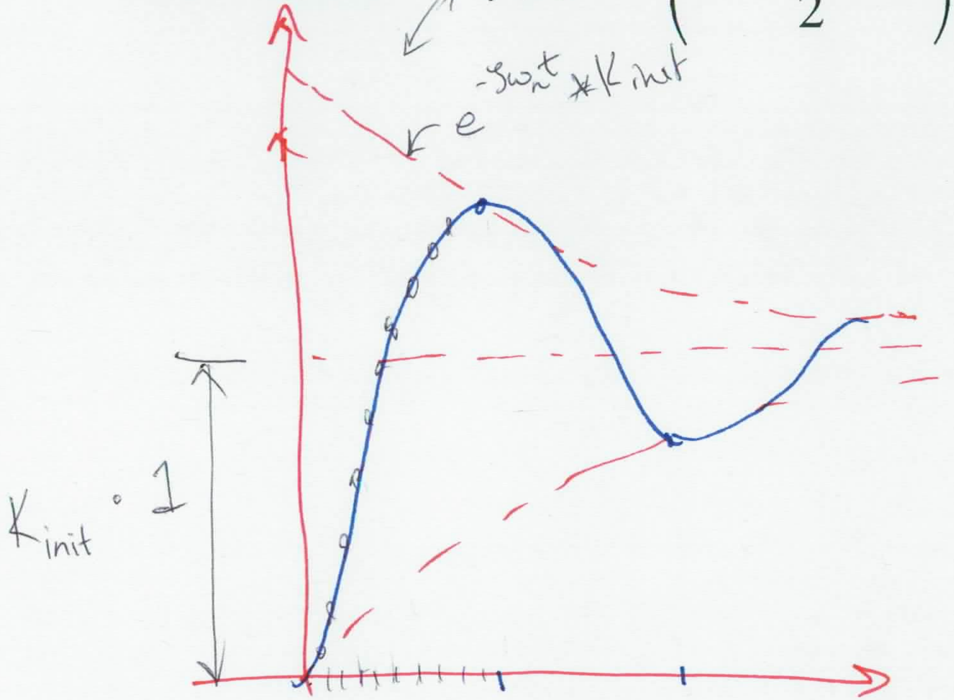
Approaches:

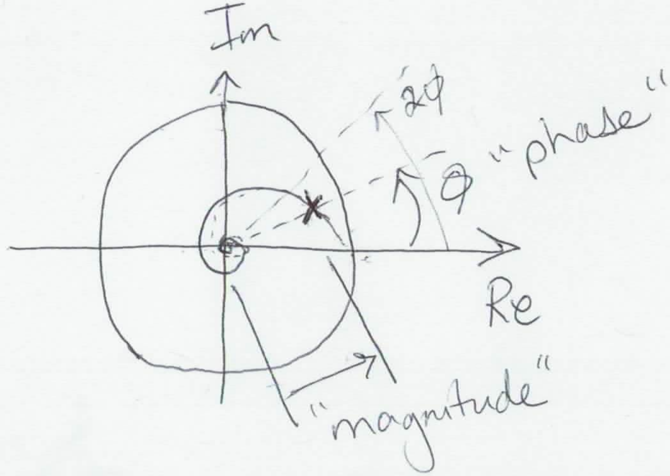
1. By hand ("inversion")... $X(z) \Leftrightarrow \sum_{-\infty}^{\infty} x_k z^{-k}$

2. Learn known "classic" patterns!
   (better for intuition AND more practical)

(We did an exponential already...)

**Sinusoid:** $y(k) = r^k\left(\dfrac{e^{jk\theta} - e^{-jk\theta}}{2}\right)$




T.B.C.