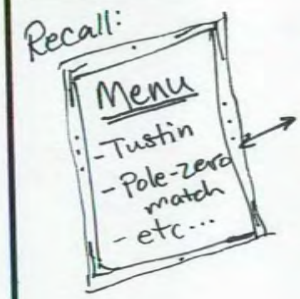
Today: "Matched" pole/zero method

Lecture 7
ECE 147B
Digital Control

Reading: FPW 5.5, 4.5 (7.4), 6.2, 7.3, 7.5
or: FV 6.7, 2.8 (6.5), 6.3.2, 6.1, 6.6

Topics:



- Time response: Ripple
- Frequency Response
- Pole-zero matching $C(s) \leftrightarrow C(z)$
- Root locus
- ~~direct design~~ (Next Time)

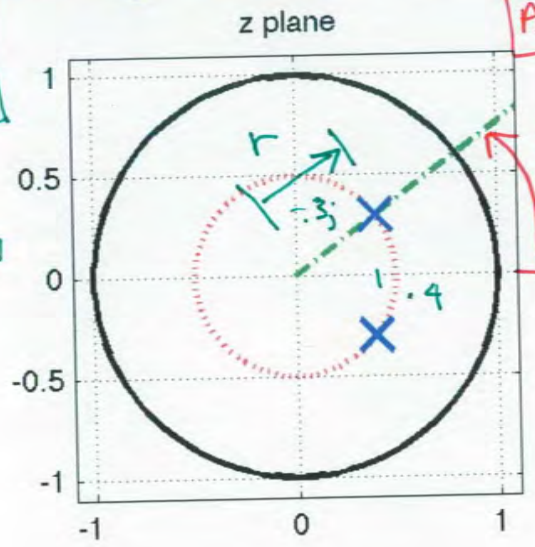
Next Class: Intro to state space (ss).
- FPW: 4.3.3
or: - FV: 7.1, 7.2

Good news:
NEW DOC Camera

Next time...

Time Response: "Ripple"

Radius gives decay rate...



Angle gives "ω_d"

$\theta = \text{atan}\left(\frac{.3}{.4}\right)$

$z \leftrightarrow e^{sT}$

USE MATLAB

"natural log" (ln) is just "log" in MATLAB

$S_1 = \log(z_1)/T$

$r = \sqrt{.3^2 + .4^2}$

$= \sqrt{.25}$

$= .5$

$z_1 = 0.4 + 0.3j$

$z_2 = 0.4 - 0.3j$

z_2

$$S_1 = \frac{\ln(r) + \theta j}{T} = \frac{\log(r)}{T} + \frac{\theta}{T}$$

$$S_2 = \frac{\ln(r) - \theta j}{T}$$

$$G(z) = \frac{1.8}{(z - (0.4 + 0.3j))(z - (0.4 - 0.3j))} = \frac{1.8}{z^2 - 0.8z + 0.25}$$

z_1 z_2

$\omega_d = \frac{\theta}{T}$

r = 0.5

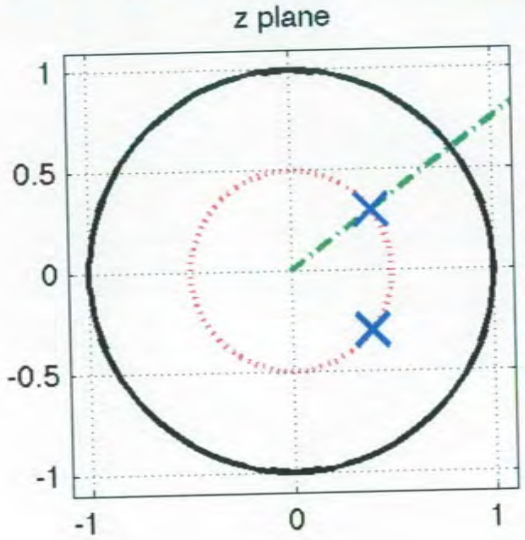
θ = atan(.75) = 0.6435

$$s = -6.9315 \pm 6.4350j$$

$-\zeta\omega_n = \frac{\ln(r)}{T}$ $\omega_d = \frac{\theta}{T}$

for T = 0.1

Time Response: "Ripple"



$$G(z) = \frac{1.8}{z^2 - 0.8z + 0.25}$$

(Final Value Thm...) $\lim_{z\to 1}\left((z-1)\left(\frac{z}{z-1}\right)G(z)\right) = K_{fv}$

$X(z) = \frac{z}{z-1}$

k	u(k)	y(k)
-1	0	0
0	1	0
1	1	0
2	1	1.8
n	1	$y(n) = K_{fv} + c_1 z_1^n + c_2 z_2^n$
∞	1	4

Unit step response

For stable poles, this part goes to zero (as $k \to \infty$)

↑ Final Value

$$K_{fv} = \frac{1.8}{.2+.25} = \frac{1.8}{.45} = 4$$

Eq.1: $y(0) - K_{fv} = (A_{11}c_1)z_1 + (A_{12}c_2)z_2$

$$y(k=0) = K_{fv} + c_1 z_1^0 + c_2 z_2^0$$

$$y(k=1) = K_{fv} + c_1 z_1^1 + c_2 z_2^1$$

with 2 eqn's, we can solve for the 2 (remaining) unknowns, c_1 & c_2

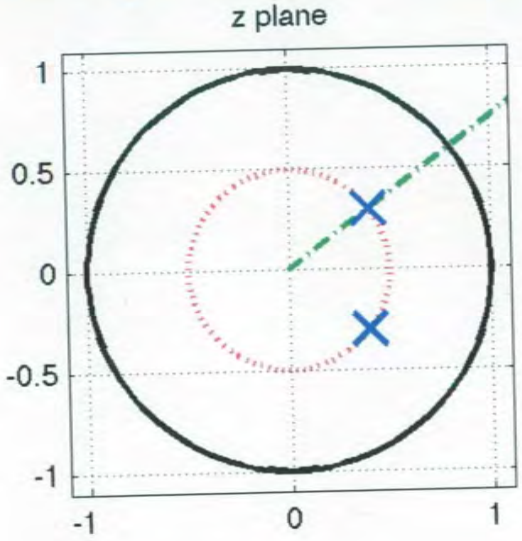
$$Ax = b$$

$$\begin{bmatrix} 1 & 1 \\ z_1 & z_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y(0) - K_{fv} \\ y(1) - K_{fv} \end{bmatrix} \leftarrow \text{Eq.1}$$

A · x = b

x = A\b

Time Response: "Ripple"



Use MATLAB at your desk!!

$Ax = b$

`x = A\b`

$A = \begin{bmatrix} 1 & 1 \\ .4+.3j & .4-.3j \end{bmatrix}$

$b = [-4\ ;\ -4]$

$x = -2 \pm 4j$

$\therefore C_1 = -2+4j\ ,\ C_2 = -2-4j$

$$C_1 = \frac{-2+4j}{\text{complex conj}}$$

$$C_2 = \frac{-2-4j}{}$$

$(K_{fv} = 4)$

$\therefore y(n) = \ldots$

$= 4 + (-2+4j)(.4+.3j)^n + (-2-4j)(.4-.3j)^n$

$$y(n) = \left[K_{fv} + c_1 z_1^n + c_2 z_2^n\right]$$

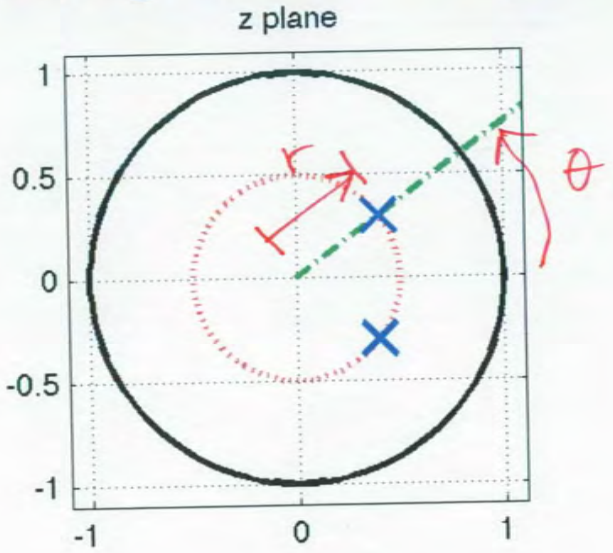
k	u(k)	y(k)
3	0	3.24
4	1	3.942
5	1	4.1436
6	1	4.1294
7	1	4.0676
∞	1	4 ← K_{fv}

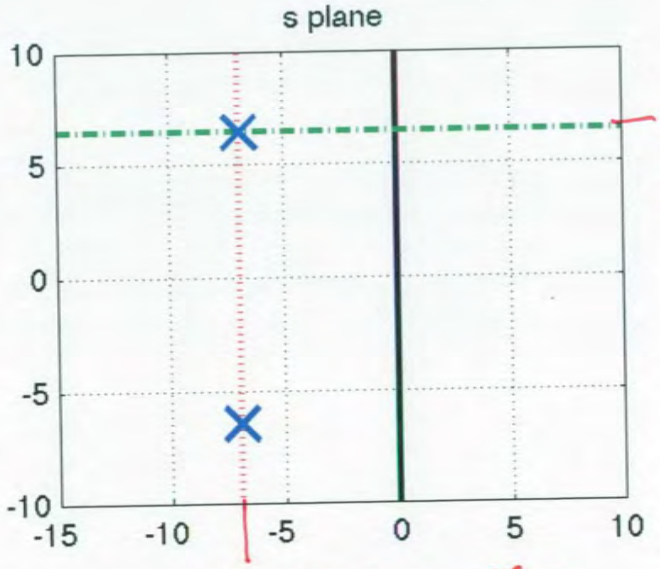
from Eqn or matlab step...

check:

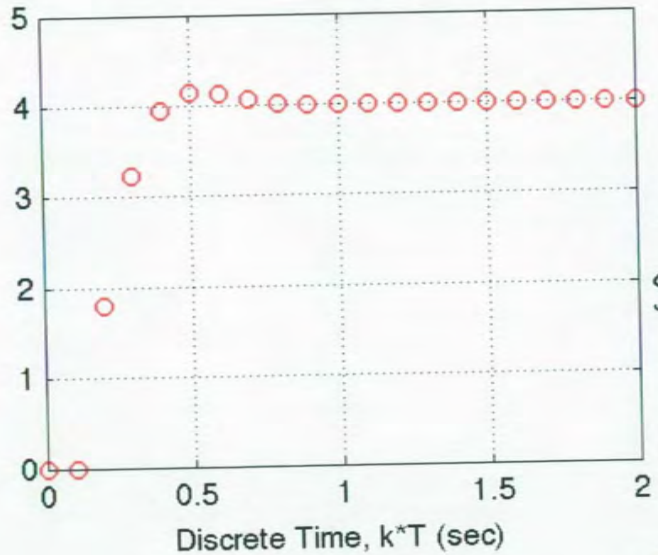
```
G = tf(1.8, [1, -.8, .25], .1)
[y, t] = step(G, 7)
```

Time Response: "Ripple"

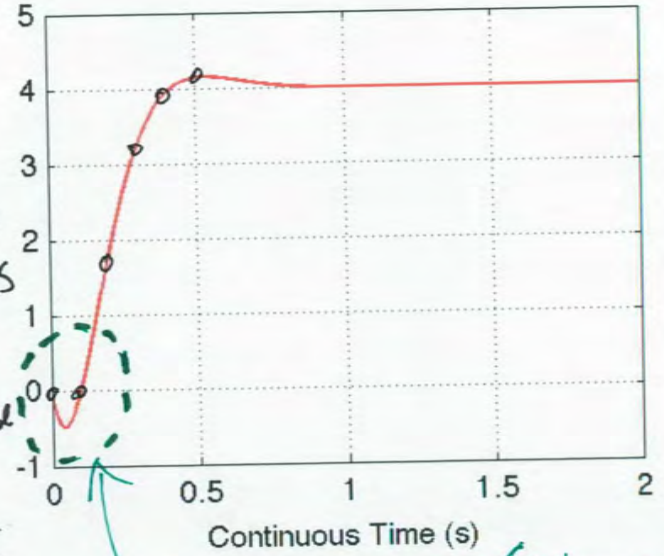




$$G(z) = \frac{1.8}{z^2 - 0.8z + .25}$$

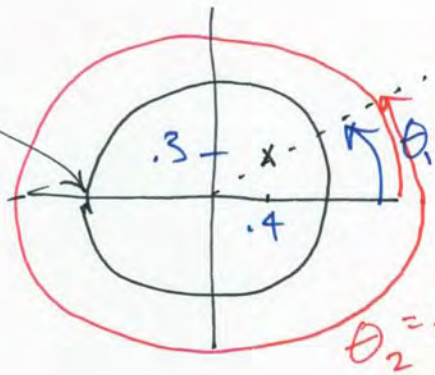


Samples of CT response @ $t = kT$



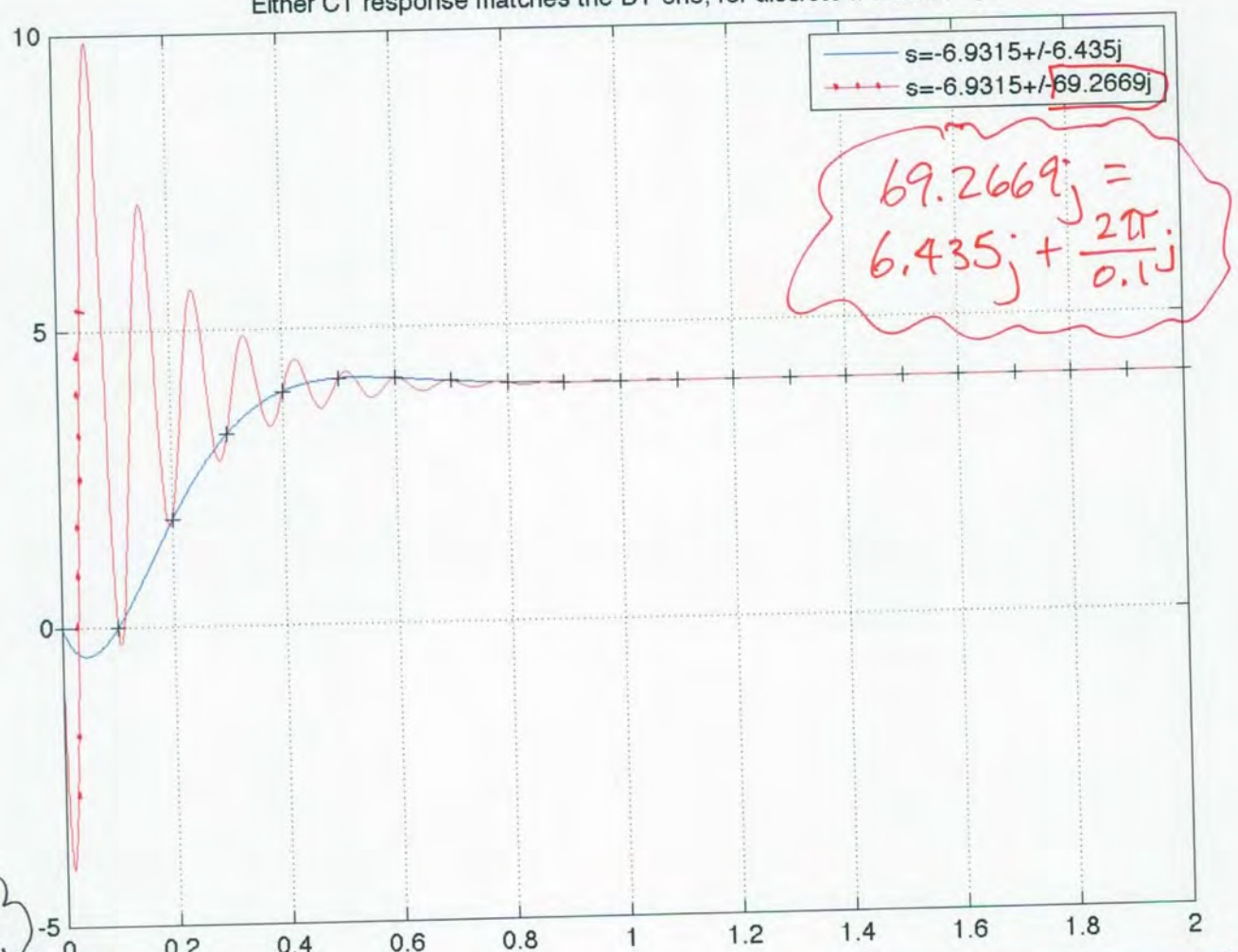
Looks odd?? (B/C no zeros, so $y(k) = 0$, $k < 2$)

*Nyquist freq: $\omega_{Nyq.} = \frac{\pi}{T}$ (@ $z = -1$)



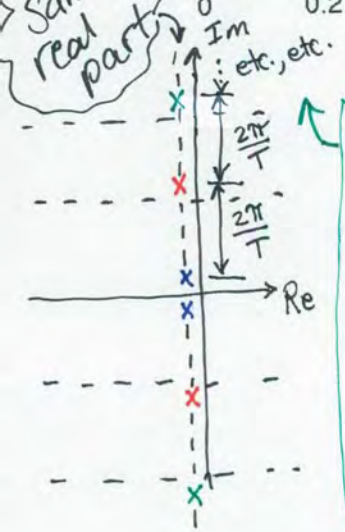
same real part $\left(\frac{\log(r)}{T}\right)$ (nat. log. "ln")

Either CT response matches the DT one, for discrete k values!



$69.2669j = 6.435j + \frac{2\pi}{0.1}j$

same real part

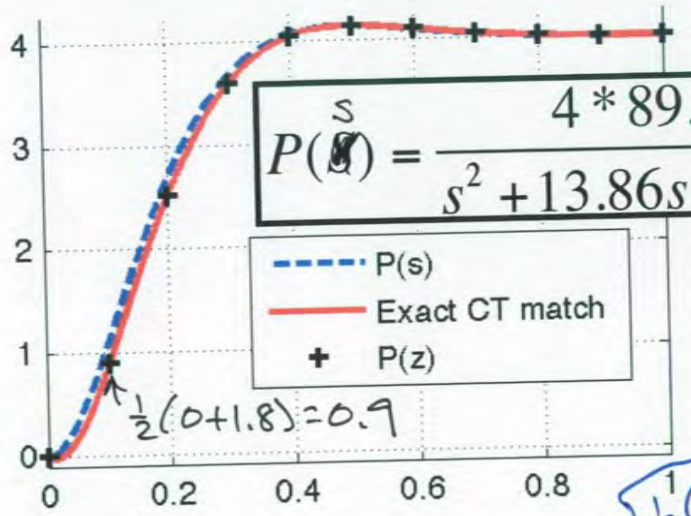


The DT values (@ $t = kT$) of the unit step response for "$G(z) = \frac{1.8}{z^2 - .8z + .25}$" have the same values as samples from a step response for any of a FAMILY of C.T. systems... (X's vs X's vs X's, at left ↑ corresponding s-plane poles)

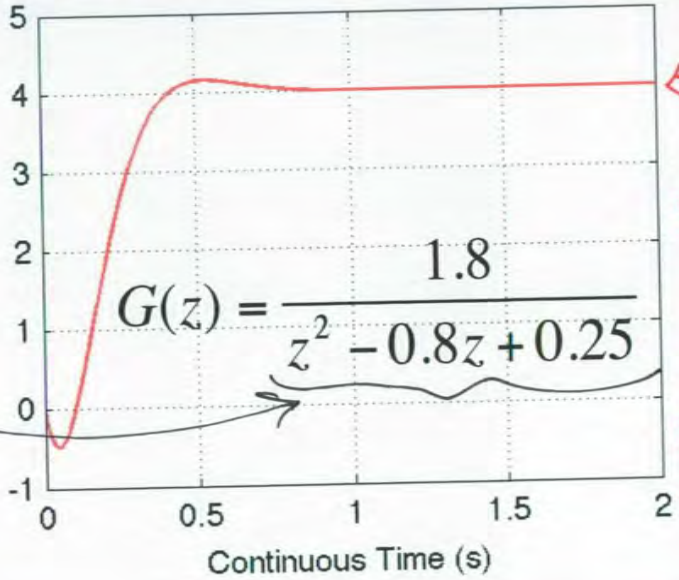
For an arbitrary CT TF w/ zeros, → ...Recall, $P(s) = \frac{2s+1}{s^2+2\zeta\omega_n s+\omega_n^2} = \left(\frac{\text{char. poly.}}{\text{char. poly.}} + \frac{\text{char. poly.}}{\text{char. poly.}}\right)$

→ (step response of P(s) is sum of responses of its two parts, at right)

Time Response: Same poles / Different zeros



$Z \leftrightarrow e^{sT}$ ($T=0.1$ sec)

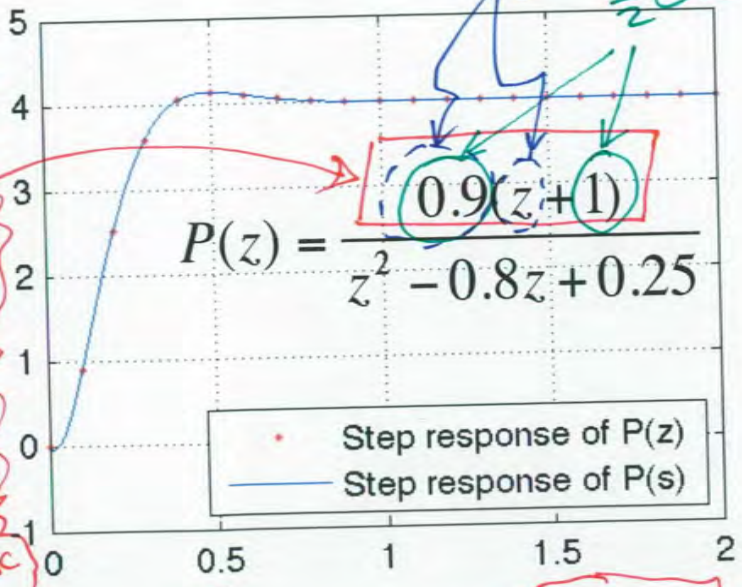


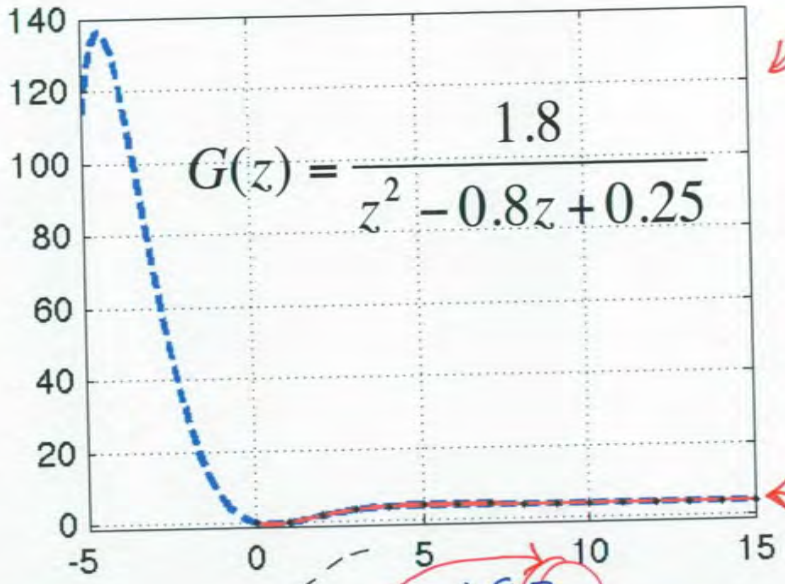
Same plot, w/ artificial "rewind" in

½G(z) shifted to the left

½G(z)

change in the zeros just shifts the "phase" of the response, NOT its characteristic SHAPE!



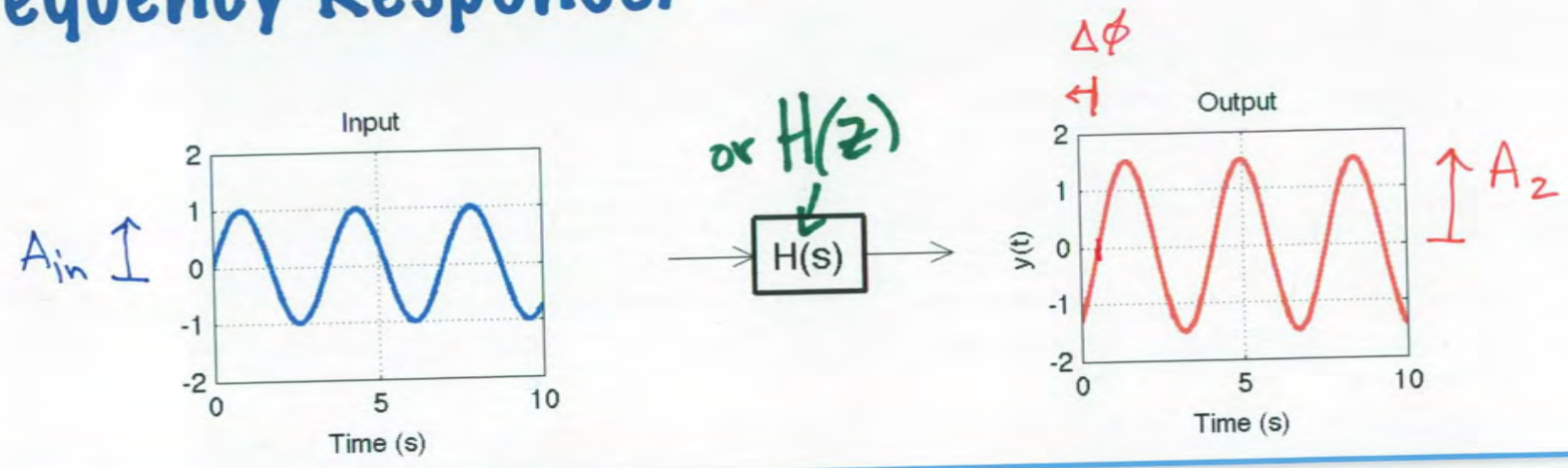


$\frac{1.8}{z^2-.8z+.25} \rightarrow y_k = 1.8x_{k-2} + .8y_{k-1} - .25y_{k-2}$

$y_k = 1.8x_{k-1} + .8y_{k-1} - .25y_{k-2}$

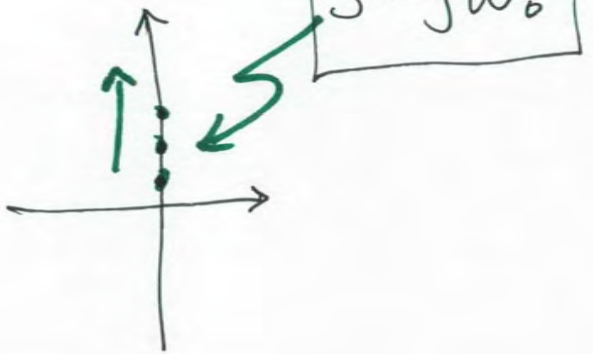
vs... $\frac{1.8z}{z^2-.8z+.25}$ ...just shifts step response "to the left" one step...!

Frequency Response:



Continuous Time (CT)

for some H(s), plug in $s = j\omega_0$ to evaluate mag & phase @ freq. ω_0

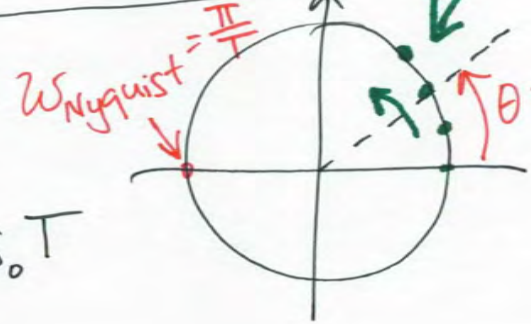


Discrete Time (DT)

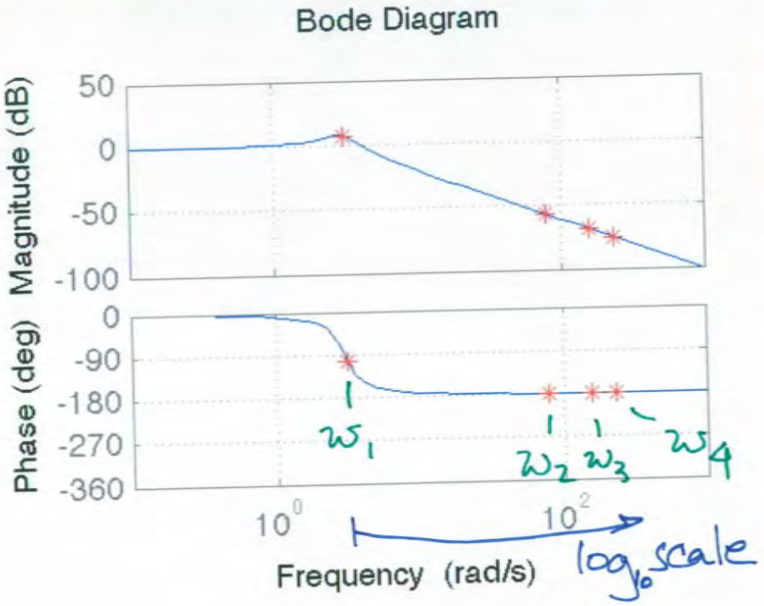
for H(z), $z = e^{j\omega_0 T}$

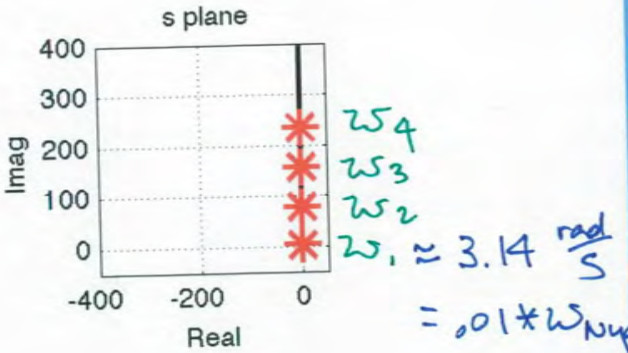
$H(e^{j\theta})$,

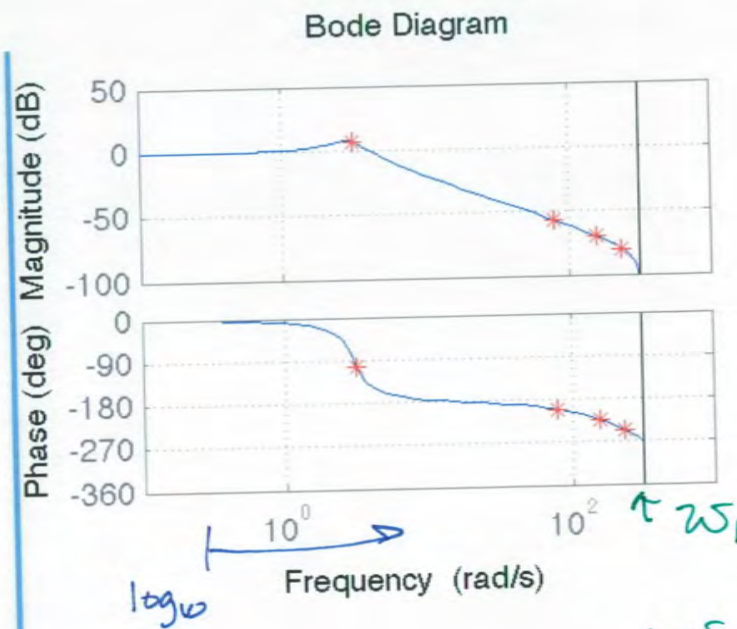
where $\theta = \omega_0 T$

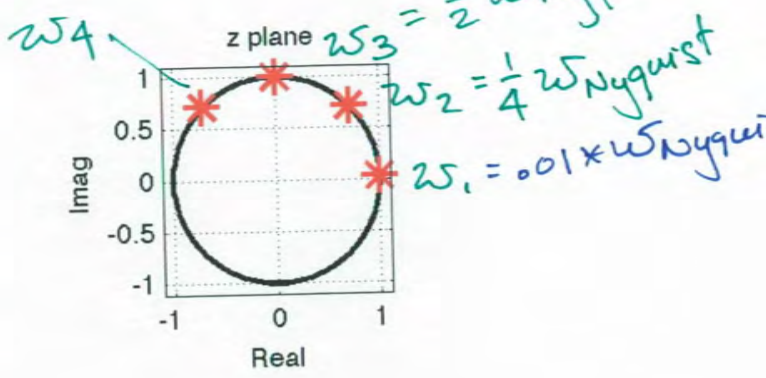


Frequency Response:









Continuous Time (CT)

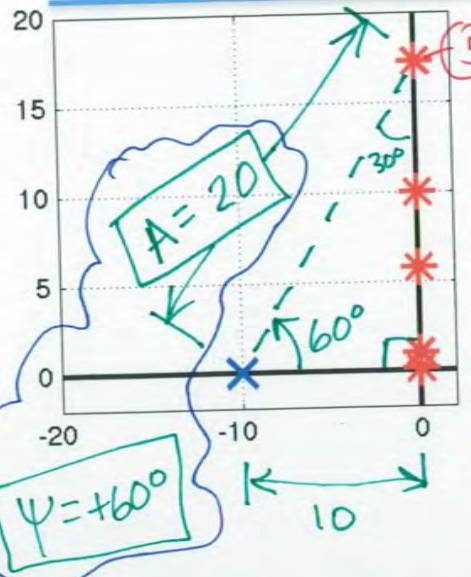
Discrete Time (DT)

Frequency Response: H(s)

C.T.T.F.

$$G(s) = s + 10 \qquad H(s) = \frac{10}{G(s)} = \frac{10}{s+10}$$

① $\omega = 0.1$	$A \approx 1$, $\psi \approx .57°$	$A \approx 1$, $\psi \approx -.57°$
② $\omega = 1$	$A = 10.05$, $\psi \approx \frac{1}{10}$	$A \approx .995$, $\psi \approx -5.7°$
③ $\omega = \frac{10}{\sqrt{3}}$	$A = 2\left(\frac{10}{\sqrt{3}}\right)$, $\psi = 30°$	$A \approx .866$, $\psi = -30°$
④ $\omega = 10$	$A = 10\sqrt{2}$, $\psi = 45°$	$A = \frac{10}{10\sqrt{2}} = \frac{1}{\sqrt{2}}$, $\psi = -45°$
⑤ $\omega = 10\sqrt{3}$		$\frac{10}{G(s)}$



$$|H(s)| = \frac{10}{|G(s)|} = \frac{10}{20} = \frac{1}{2}$$

$$\angle H(s) = -\angle G(s) = -60°$$

Use Geometric Intuition...

Consider some "comparable" H(z)...

$$H(s) = \frac{10}{s+10}$$

DC gain? $(s=0)$ $(\omega_0 = 0)$

$\frac{10}{10} = 1$

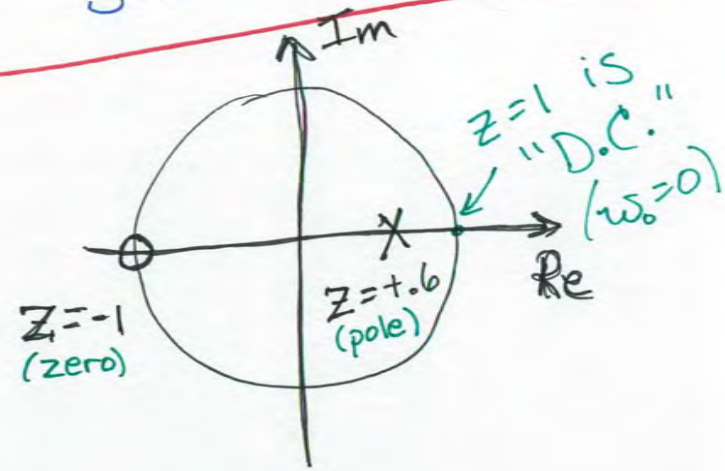
$$s \rightarrow \frac{2}{T}\frac{(z-1)}{(z+1)}$$

Matlab.

Tustin:

```
Hs = tf([10],[1, 10])
T = 0.05;
Hz = c2d(Hs,T,'tustin')
```

$$H(z) = \frac{10}{\left(\frac{2}{T}\left(\frac{z-1}{z+1}\right) + 10\right)} = K\frac{(z+1)}{\left(z-\frac{3}{5}\right)} = \frac{1}{5}\frac{(z+1)}{\left(z-\frac{3}{5}\right)}$$



DC gain? $(z=1) \Rightarrow K\frac{(1+1)}{\left(1-\frac{3}{5}\right)}$

$= K\frac{2}{(2/5)} = K*5$

$K = \frac{1}{5}$ would gives $|H(z=1)| = 1$ ✓

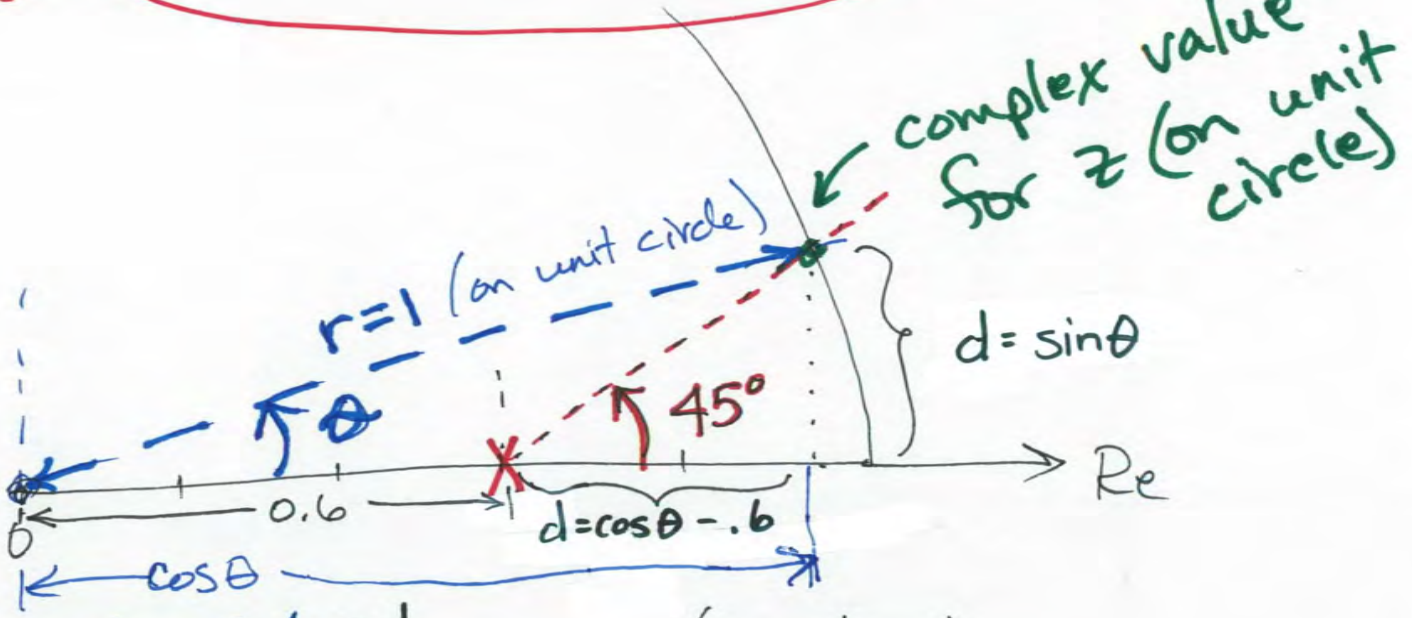
Consider some "comparable" H(z)...

$$H_2(z) = \frac{2}{5}\frac{1}{(z-0.6)}$$

pole, only.

Consider only the POLE first (same DC gain).

When is the phase -45°?



From geometry,

$$d = \sin\theta = \cos\theta - 0.6 = d$$

$$\theta = .347 \text{ (rad)}$$

$$\theta = \omega_0 T$$

$$\therefore \omega_0 = 6.945 \left(\frac{\text{rad}}{\text{sec}}\right)$$ ← on a Bode plot

Frequency Response: H(z)

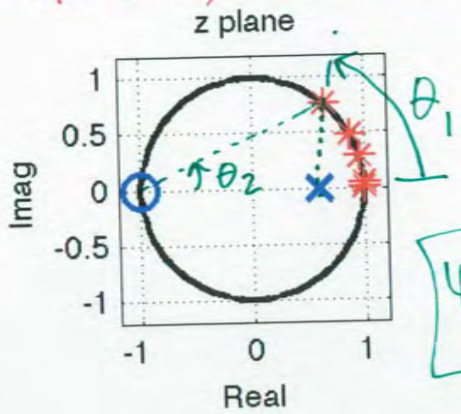
Right hand Side? Exactly as given 3 slides back

$$H(z) = \frac{1}{5}\left(\frac{z+1}{z-0.6}\right)$$

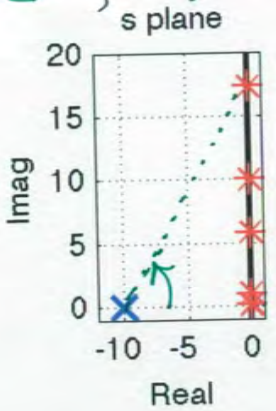
$$H(s) = \frac{10}{s+10}$$

$A \approx 1$, $\Psi \approx -.57°$ closer to H(s) @ lower freq.	$\omega = 0.1$	$A \approx 1$, $\Psi \approx -.57°$
$A = .995$, $\Psi = -5.7°$	$\omega = 1$	$A = .995$, $\Psi \approx -5.7°$
$A = .8645$, $\Psi = -30.2°$	$\omega = \frac{10}{\sqrt{3}}$	$A = .866$, $\Psi = -30°$
$A = .6996$ $\Psi = -45.6°$	$\omega = 10$	$A = .707$, $\Psi = -45°$
$A = .4757$, $\Psi = -61.6°$	$\omega = 10\sqrt{3}$	$A = .5$, $\Psi = -60°$

etc. (from prev. slides)



$\Psi = \theta_2 - \theta_1$



"Pole-Zero Matching" approximation

Heuristic method. This is another option on our "menu" to convert C(s) to C(z)... The idea is to use the same mapping between s and z planes for BOTH poles AND zeros:

$$z \leftrightarrow e^{sT}$$

Rules to convert some CTTF G(s) to a DTTF G(z):

1. Map poles to: $z_p = e^{s_p T}$
2. Map zeroes to: $z_z = e^{s_z T}$

same mapping (between s-plane & z-plane zeroes (as used to map between poles))

3. Keep adding zeros @ "z=-1" to G(z) until the order of the numerator (of G(z)) is one less than the order of the denominator.
4. Match DC gains (for G(z), to match G(s)).

Usage in MATLAB (can go either way):

```
Gz = c2d(Gs,T,'matched')
```

```
Gs = d2c(Gz,'matched')
```

"Pole-Zero Matching" approximation $z \leftrightarrow e^{sT}$

```
T=1;
Gz = c2d(Gs,T,'matched')
```

```
Gs = d2c(Gz,'matched')
```

$z \leftrightarrow e^{sT}$

$\frac{\ln(z)}{T} \leftrightarrow s$

$$G(z) = \frac{(z+1)}{(z^2 - z + .5)} \leftrightarrow G(s) = \frac{4 * 0.737}{(s^2 + 0.6931s + 0.737)}$$

$$z_1 = 0.5 + 0.5j$$

$$s_1 = \ln(0.5 + 0.5j)/T = -0.3466 + 0.7854j$$

$$z_2 = 0.5 - 0.5j$$

$$s_1 = \ln(0.5 - 0.5j)/T = -0.3466 - 0.7854j$$

```
>> conv([1 -s1],[1 -s2])
ans =
    1.000      0.6931     0.737
```

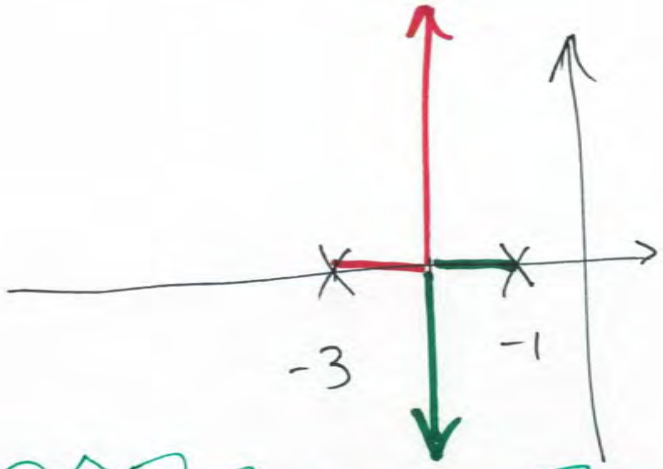
← to get the characteristic polynomial...

Root Locus

The root locus rules for z-plane and s-plane are identical!

What is different is the meaning of the resulting pole locations, i.e., stable poles in the left-hand plane (CT) versus within the unit circle (DT).

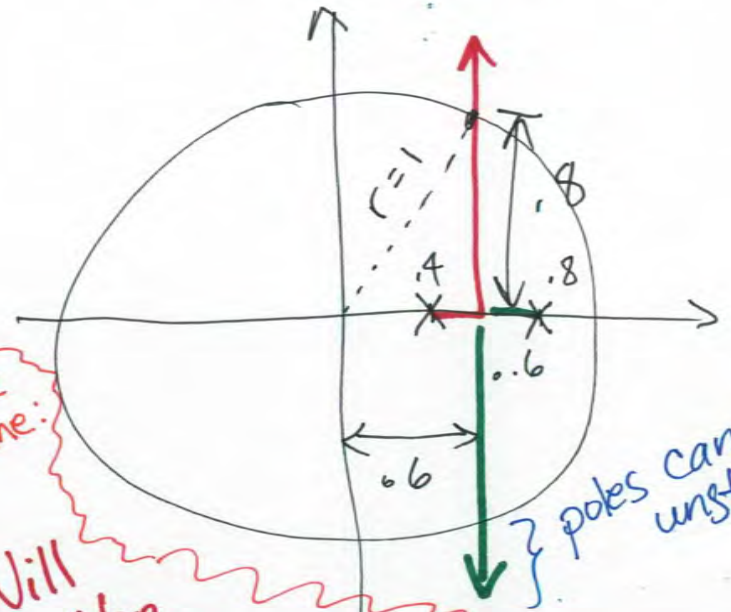
$$P(s) = \frac{1}{(s+1)(s+3)}$$



on the s-plane: Never goes unstable.

VS

$$G(z) = \frac{1}{(z-.4)(z-.8)}$$



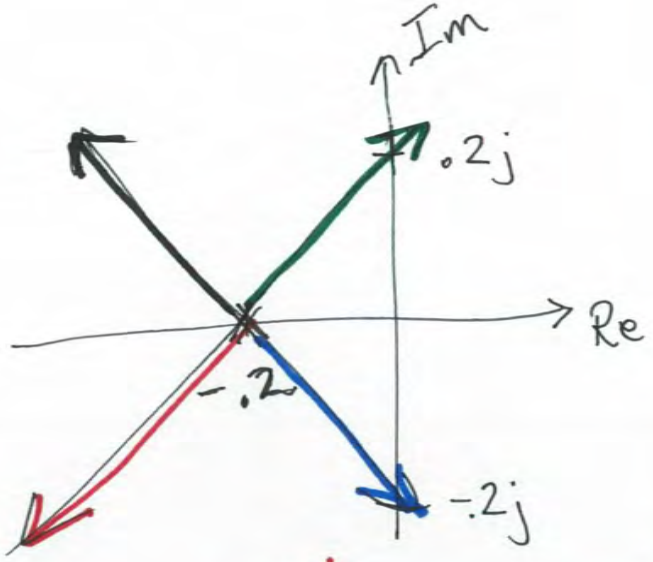
poles can go unstable

on the z-plane: Will exit the unit circle (go unstable) for "large" K...

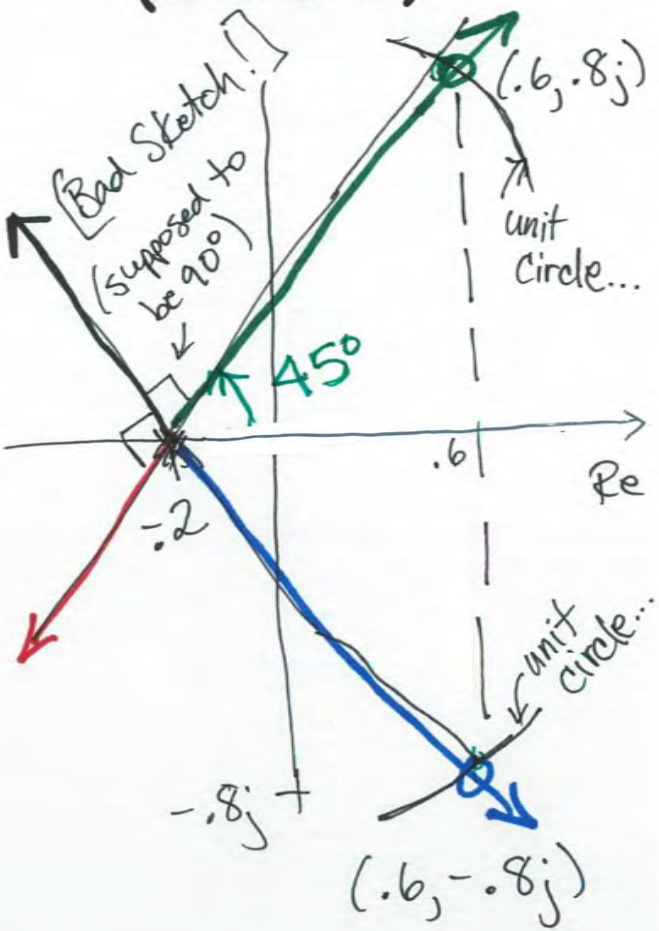
Root Locus

The root locus rules for z-plane and s-plane are identical!

$$P(s) = \frac{1}{(s+0.2)^4}$$



$$G(z) = \frac{1}{(z+0.2)^4}$$



* Easy to calculate pole locations where system is marginally stable...