Today: Intro to State Space (ss)

Lecture 8

ECE 147B

Digital Control

Reading: FPW 4.3.3, 7.5

or: FV 7.1, 7.2, 6.6

Topics:

- Root locus (Review from Lecture 7)
- Step response calculation: CT (ECE 147A Review)
- Direct design
- e^{sT} approximation (for small "sT")
- State space equations

Next Class: State space (ss).

- FPW: 6.1

or: - FV: 7.6, 7.7

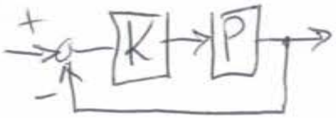
Root Locus

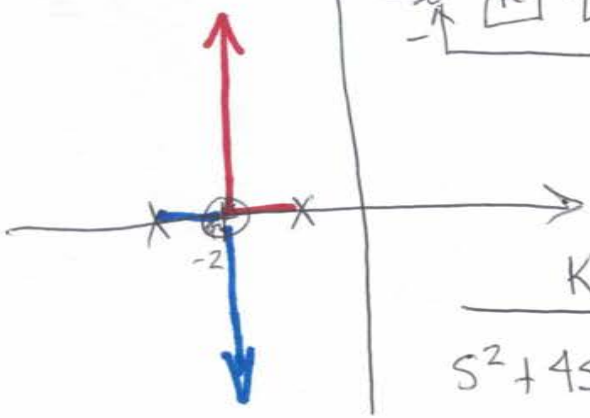
The root locus rules for z-plane and s-plane are identical!

What is different is the meaning of the resulting pole locations, i.e., stable poles in the left-hand plane (CT) versus within the unit circle (DT).

$$P(s) = \frac{1}{(s+1)(s+3)}$$

Poles: $s=-1$, $s=-3$

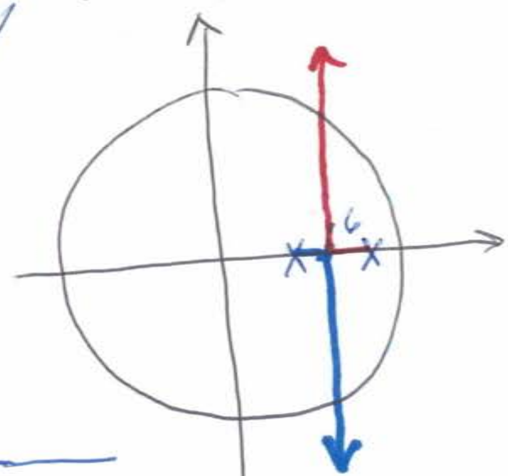




$$\frac{K}{s^2+4s+\boxed{3+K}}$$

Gets LARGER.

$$G(z) = \frac{1}{(z-.4)(z-.8)}$$



$$\frac{K}{z^2 \underbrace{-1.2z} + K + \text{stuff}}$$

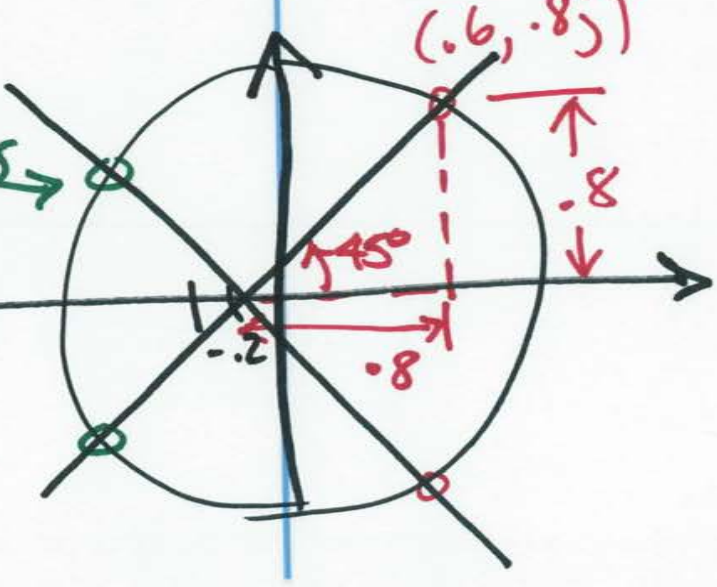
Root Locus

The root locus rules for z-plane and s-plane are identical!

$$P(s) = \frac{1}{(s+0.2)^4} \qquad G(z) = \frac{1}{(z+0.2)^4}$$

identical root locus

These poles actually go unstable first.



Root Locus

Root locus intuition...

$$G(z) = \frac{1}{(z^n + \underbrace{M}z^{n-1} + \ldots + \underbrace{K})}$$

Example, 6 poles,

$$(s-a)(s-b)\ldots(s-f) = 0$$

OR: z z z

$$s^6 + \underbrace{(-a-b-c-d-e-f)}s^5 + \ldots \underbrace{(a\cdot b\cdot c\cdot d\cdot e\cdot f)(-1)^6}_{K \text{ is like } "\omega_n^{\,n}" \text{ or } "R^n"}$$

$\frac{M}{n}$: "centroid" for all the asymptotes

CT Step Response: Analytic Solution

Last lecture, we discussed how to do this for a DT system. As review, here is a methodology for 2nd-order CT systems:

Review - How to calculate time response (e.g., step response) for CT system.

Example: $G(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \dfrac{Y(s)}{U(s)}$

① Form will be: $y(t) = K + C_1 e^{s_1 t} + C_2 e^{s_2 t}$, $t \geq 0$

...which can also be: $y(t) = K(1 + A\sin(\omega_d t + \phi)e^{-\sigma t})$

where: $\begin{cases} \omega_d = \omega_n\sqrt{1-\zeta^2} & \leftarrow \text{freq. of osc.} \\ \sigma = \zeta\omega_n \end{cases}$

← negative of the Real part

poles are: $\boxed{s = -\sigma \pm \omega_d j}$

K is the "final value", if you have decaying "e^{st}" terms.

CT Step Response: Analytic Solution

② @ $t=0$, $\dot{y}(t)=0$ (init. cond is zero. 2^{nd} order system)

I.C.

$\therefore\ 0 = A\omega_d\cos(\omega_d t+\phi)e^{-\sigma t} - A\sigma\sin(\omega_d t+\phi)e^{-\sigma t} = \dot{y}$

$t=0$ $t=0$ $t=0$ $t=0$

$0 = A\omega_d\cos(\phi) - A\sigma\sin(\phi)$

$\frac{\omega_d}{\sigma} = \frac{\sin(\phi)}{\cos(\phi)} \rightarrow \therefore \boxed{\phi = \text{atan}\left(\frac{\omega_d}{\sigma}\right)}$

③ @ $t=0$, $y=0 = K(1+A\sin(\phi))$

$\therefore A\sin(\phi) = -1 \rightarrow \boxed{A = \frac{-1}{\sin(\phi)}}$

A does not yet account for "K" scaling...

Example

$G = \frac{K\cdot 1}{s^2+.6s+1}$

$\omega_n = 1,\ \zeta = .3$

$y(t) = ?$

(rad

$\boxed{\phi = \text{atan}\left(\frac{\omega_d}{\sigma}\right) = 1.26}$

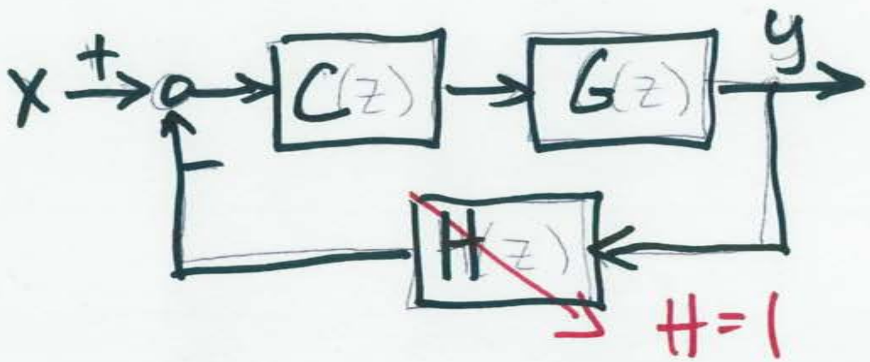
$\boxed{A = \frac{-1}{\sin\phi} = -1.04}$

$y(t) = K(1 - A\sin(\omega_d t + \phi$

$*e^{-\sigma t}$

$\omega_d \approx .95$

"Direct Design"



$H=1$ (no sensor dynamics) (unity f.b.)

Question: If we have some $P(z)$ in mind, can we pick $C(z)$ to EXACTLY achieve this?

$$P = \frac{Y(z)}{X(z)}$$

$(H=1)$, $P = \frac{CG}{1+CG}$, $P(1+CG) = CG$

$P = CG - PCG$

$P = C(G - PG)$

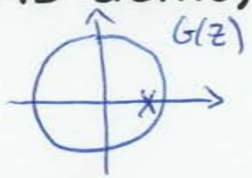
$$\therefore C = \frac{P}{G-PG} = \frac{1 \cdot P}{G(1-P)}$$

$$\boxed{C(z) = \frac{1}{G} \cdot \frac{P}{(1-P)}}$$

$$P = \frac{CG}{1+CGH}$$

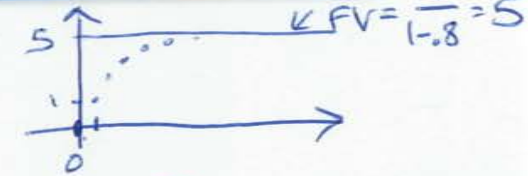
Example (MATLAB demo):

$$G(z) = \frac{1}{z-0.8}$$

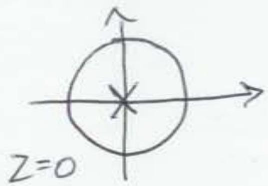


} open-loop plant

unit step for $G(z)$?



$$P(z) = \frac{1}{z}$$

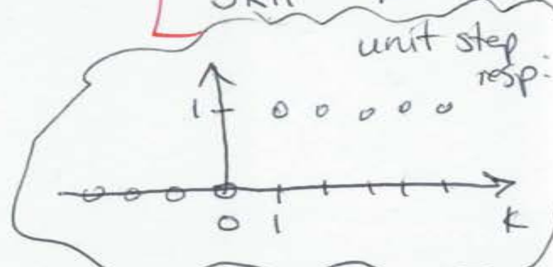


} desired closed-loop plant

"Deadbeat" system.

$\frac{Y}{X} = \frac{1}{z} = \frac{\text{"output"}}{\text{"input"}}$

$y_{k+1} = x_k$



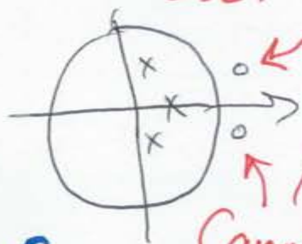
$$H(z) = 1$$

Using MATLAB:

$$C(z) = \frac{z^2 - 0.8z}{z^2 - z} = \frac{z - .8}{z - 1}$$

"Direct Design"

Example $G(z)$



Cannot invert!

Important Notes:

1. Controller "$C(z)$" cannot have unstable poles!

$\therefore$ We cannot "cancel out" zeros outside the unit circle in $G(z)$ (open-loop plant)

$\frac{1}{G(z)}$ ← "inverting" the plant

2. $C(z)$ must also be CAUSAL.

So, $P(z)=1$ Not possible w/ $G(z)=\frac{1}{z-.8}$

Output from $C(z)$ uses only current &/or previous inputs – NOT future info. So... if $G(z)$ does not respond until (say) "$k=2$" (to a unit step) then neither can $P(z)$. $C(z)=\frac{az^m+\ldots}{bz^n+\ldots}$, $\boxed{m \le n}$

e^{sT} Approximation (for small sT)

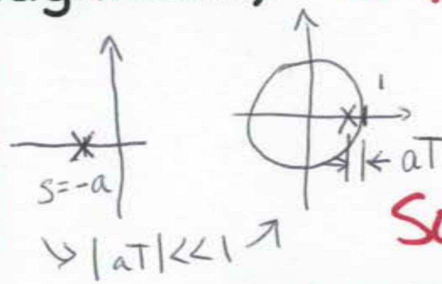
A hint when using "pole zero matching"...

$$z \leftrightarrow e^{sT} \qquad z = e^{sT}$$

When sT << 1 (small in magnitude):

Often, T is VERY "small", So e^{sT} is small, too...

$$\boxed{e^{sT} \approx 1 + sT \,\substack{=\\ z}}$$



$$\boxed{z \approx 1 + sT}$$

Analogously, where |z–1| << 1
("z–1" small in mag):

$$\boxed{\log(z) \approx z - 1} \qquad \boxed{s \approx (z-1)/T}$$

e^{sT} Approximation (for small sT)

$z \leftrightarrow e^{sT}$

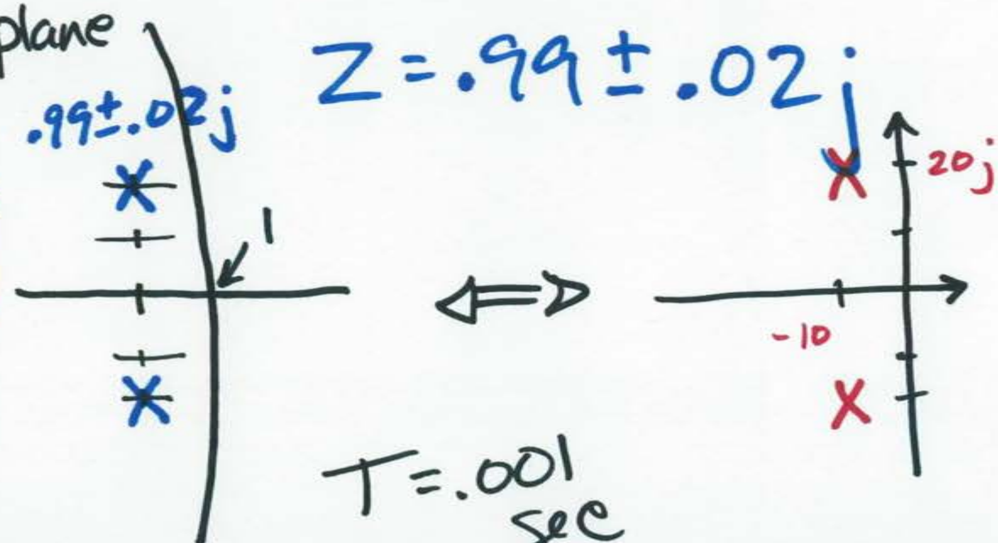
MATLAB examples...

$e^{sT} \approx 1 + sT \approx z$

$s = -2$	$s = -5 \pm 6j$
$T = .01$	$T = .001$
$z \approx .98$	$z \approx .995 \pm .006j$

$\log(z) \approx z - 1 \approx sT$

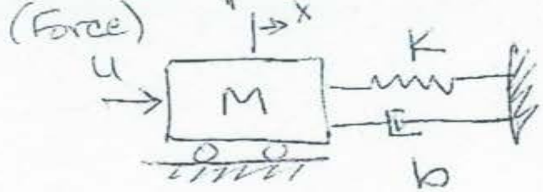
$\boxed{(z-1)/T \approx s}$



State Space: Introduction

2nd-order:
- 2 state variable (pos & vel, e.g.)
- 2nd-order char. polynomial
- 2 poles

CT Example: spring-mass-damper (2nd-order) system



$$m\ddot{x} + b\dot{x} + kx = u$$

(equation(s) of motion...)

Let $X = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$, $Y = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{C} X + \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{D} u$

$$\dot{X} = AX + Bu$$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}}_{A} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1/m \end{bmatrix}}_{B} u$$

State Space: Introduction

can put elements in $\underline{X}$ in any order.

vs Alternate form: $\underline{X} = \begin{bmatrix} \dot{x} \\ x \end{bmatrix}$, $Y = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$

$$Y = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{C} X + \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{D} u$$

$$\dot{\underline{X}} = \begin{bmatrix} \ddot{x} \\ \dot{x} \end{bmatrix} = \underbrace{\begin{bmatrix} -b/m & -k/m \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} \dot{x} \\ x \end{bmatrix} + \underbrace{\begin{bmatrix} 1/m \\ 0 \end{bmatrix}}_{B} u$$

Matrix form Equations of Motion

x : states
y : outputs

CT:

$$\boxed{\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}}$$

DT:

$$\boxed{\begin{aligned} x(k+1) &= A_d x(k) + B_d u(k) \\ y(k) &= C_d x(k) + D_d u(k) \end{aligned}}$$

matrix equations, for multi-input, multi-output systems ("MIMO")