Example: Two-cart spring system.

Absolute Coord's

$$T^* = \tfrac{1}{2} m_1 \dot{x}_1^2 + \tfrac{1}{2} m_2 \dot{x}_{2a}^2$$

$$V = \tfrac{1}{2} k (x_1 - x_{2a})^2$$

$$\mathcal{L} = T^* - V$$
$$= \tfrac{1}{2} m_1 \dot{x}_1^2 + \tfrac{1}{2} m_2 \dot{x}_{2a}^2 + \ldots$$
$$- \tfrac{1}{2} k x_1^2 - \tfrac{1}{2} k x_{2a}^2 + k x_1 x_{2a}$$

$$\boxed{\Xi_{1a} = -F \qquad \Xi_{2a} = +F}$$

Note the difference

① $$-F = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right) - \frac{\partial \mathcal{L}}{\partial x_1}$$

$$-F = \frac{d}{dt}(m_1 \dot{x}_1) - (-k x_1 + k x_{2a})$$

$\boxed{1a}$ $$\boxed{-F = m_1 \ddot{x}_1 + k x_1 - k x_{2a}}$$

② $$F = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_{2a}}\right) - \frac{\partial \mathcal{L}}{\partial x_{2a}}$$

$$F = \frac{d}{dt}(m_2 \dot{x}_{2a}) - (-k x_{2a} + k x_1)$$

$\boxed{2a}$ $$\boxed{F = m_2 \ddot{x}_{2a} + k x_{2a} - k x_1}$$

Relative Coord's

$$T^* = \tfrac{1}{2} m_1 \dot{x}_1^2 + \tfrac{1}{2} m_2 (\dot{x}_1 + \dot{x}_2)^2$$

$$V = \tfrac{1}{2} k x_{2r}^2$$

$$\mathcal{L} = T^* - V$$
$$= \tfrac{1}{2} m_1 \dot{x}_1^2 + \ldots$$
$$\tfrac{1}{2} m_2 (\dot{x}_1^2 + 2 \dot{x}_1 \dot{x}_{2r} + \dot{x}_{2r}^2) + \ldots$$
$$- \tfrac{1}{2} k x_{2r}^2$$

$$\boxed{\Xi_{1r} = 0 \qquad \Xi_{2r} = +F}$$

① $$0 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right) - \frac{\partial \mathcal{L}}{\partial x_1}$$

$$0 = \frac{d}{dt}(m_1 \dot{x}_1 + m_2 \dot{x}_1 + m_2 \dot{x}_{2r})$$

$\boxed{1r}$ $$\boxed{0 = (m_1 + m_2) \ddot{x}_1 + m_2 \ddot{x}_{2r}}$$

② $$F = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_{2r}}\right) - \frac{\partial \mathcal{L}}{\partial x_{2r}}$$

$$F = \frac{d}{dt}(m_2 \dot{x}_1 + m_2 \dot{x}_{2r}) + \ldots$$
$$- (-k x_{2r})$$

$\boxed{2r}$ $$\boxed{F = m_2 \ddot{x}_1 + m_2 \ddot{x}_{2r} + k x_{2r}}$$

Let's compare!!!

- Write $\boxed{1a}$ in terms of x_{1r} & x_{2r}:

$$-F = m_1 \ddot{x}_1 + k x_1 - k(x_1 + x_{2r})$$

$$\therefore \boxed{-F = m_1 \ddot{x}_1 - k x_{2r}} \quad \boxed{B\text{-}1}$$

- Write $\boxed{2a}$ in terms of x_{1r} & x_{2r}:

$$F = m_2 \ddot{x}_1 + m_2 \ddot{x}_{2r} + (k x_1 + k x_{2r}) - k x_1$$

$$\therefore \boxed{F = m_2 \ddot{x}_1 + m_2 \ddot{x}_{2r} + k x_{2r}} \quad \boxed{B\text{-}2}$$

- Equation $\boxed{B\text{-}2}$ matches $\boxed{2r}$.

→ Now, add $\boxed{B\text{-}1}$ and $\boxed{B\text{-}2}$,

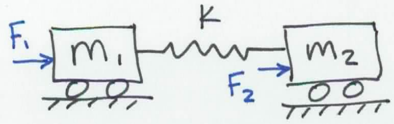
$$\boxed{0 = (m_1 + m_2) \ddot{x}_1 + m_2 \ddot{x}_{2r}}$$

Matches $\boxed{1r}$!

[1a & 2a can be written as 1r & 2r!

Another Perspective:

Ⓐ Absolute coordinates require Ξ_i includes forces (or torques) wrt the inertial reference frame.



$$\boxed{\begin{aligned}\Xi_{1a} = F_1 &= m_1\ddot{x}_1 + k(x_1 - x_{2a})\\ \Xi_{2a} = F_2 &= m_2\ddot{x}_2 + k(x_{2a} - x_1)\end{aligned}}$$

Same derivation as before, but $F_1 \leftrightarrow -F$, $F_2 \leftrightarrow +F$ $\boxed{1a}$ & $\boxed{2a}$

Ⓡ To match when using x_{2r} (relative coord) requires:

$$\boxed{\begin{aligned}\Xi_{1r} &= (m_1 + m_2)\ddot{x}_1 + m_2\ddot{x}_{2r}\\ \Xi_{2r} &= m_2(\ddot{x}_1 + \ddot{x}_{2r}) + kx_{2r}\end{aligned}}$$

See eqns $\boxed{1r}$ & $\boxed{2r}$

$$\therefore \boxed{\Xi_{1a} + \Xi_{2a} = \Xi_{1r}} = (m_1\ddot{x}_1 + k(x_1 - x_{2a})) + \ldots (m_2\ddot{x}_2 + k(x_{2a} - x_1))$$
$$= (m_1 + m_2)\ddot{x}_1 + m_2\ddot{x}_{2r}$$

And:

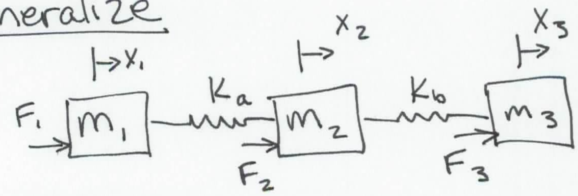
$$\boxed{\Xi_{2a} = \Xi_{2r}} = m_2\ddot{x}_2 + k(x_{2a} - x_1)$$
$$= m_2(\ddot{x}_1 + \ddot{x}_{2r}) + kx_{2r}$$

So, Ξ_{1r} = Sum of all forces on the entire system (m_1 AND m_2)

Ξ_{2r} = Forces exerted on m_2

non-conservative forces

Generalize



Ⓐ

$$\boxed{\begin{aligned} \Xi_{1a} &= F_1 = m_1\ddot{x}_1 + K_a(x_1 - x_{2a}) \\ \Xi_{2a} &= F_2 = m_2\ddot{x}_2 + K_a(x_{2a} - x_1) + K_b(x_{2a} - x_3) \\ \Xi_{3a} &= F_3 = m_3\ddot{x}_3 + K_b(x_3 - x_{2a}) \end{aligned}}$$

Ⓡ for Relative x_{2r} & x_{3r},

$$x_{1r} = x_{1a} = x_1$$
$$x_{2r} = x_{2a} - x_1$$
$$x_{3r} = x_{3a} - x_{2a}$$

$$T^* = \tfrac{1}{2}m_1\dot{x}_1^2 + \tfrac{1}{2}m_2(\dot{x}_1 + \dot{x}_{2r})^2 + \tfrac{1}{2}m_3(\dot{x}_1 + \dot{x}_{2r} + \dot{x}_{3r})^2$$

$$V = \tfrac{1}{2}K_a x_{2r}^2 + \tfrac{1}{2}K_b x_{3r}^2$$

$$\mathcal{L} = T^* - V$$

$$\Xi_{1r} = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}_1}\right) - \frac{\partial\mathcal{L}}{\partial x_1} = \frac{d}{dt}\Big(m_1\dot{x}_1 + m_2(\dot{x}_1 + \dot{x}_{2r}) + m_3(\dot{x}_1 + \dot{x}_{2r} + \dot{x}_{3r})\Big)$$

$$\boxed{\Xi_{1r} = m_1\ddot{x}_1 + m_2(\ddot{x}_1 + \ddot{x}_{2r}) + m_3(\ddot{x}_1 + \ddot{x}_2 + \ddot{x}_3)} = \Xi_{1a} + \Xi_{2a} + \Xi_{3a} = F_1 + F_2 + F_3$$

Ξ_{1r}

$$\Xi_{2r} = \frac{d}{dt}\Big(m_2(\dot{x}_1 + \dot{x}_{2r}) + m_3(\dot{x}_1 + \dot{x}_2 + \dot{x}_3)\Big) + K_a x_{2r}$$

$$\boxed{\Xi_{2r} = m_2(\ddot{x}_1 + \ddot{x}_{2r}) + m_3(\ddot{x}_1 + \ddot{x}_2 + \ddot{x}_3) + K_a x_{2r}} = \Xi_{2a} = F_2 + F_3$$

Ξ_{2r}

$$\Xi_{3r} = \frac{d}{dt}\Big(m_3(\dot{x}_1 + \dot{x}_2 + \dot{x}_3)\Big) + K_b x_{3r}$$

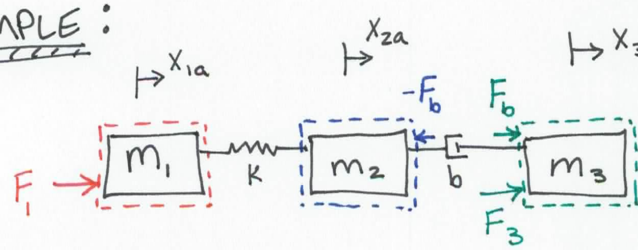
$$\boxed{\Xi_{3r} = m_3(\ddot{x}_1 + \ddot{x}_2 + \ddot{x}_3) + K_b x_{3r}} = \Xi_{3a} = F_3$$

Ξ_{3r}

Conclusions

1. When using <u>only</u> **ABSOLUTE** coordinates, each Ξ_i is simply the sum of all non-conservative forces (or torques) acting on the mass (or inertia) associated w/ the i^{th} **Generalized Coordinate**.

EXAMPLE:



$$\boxed{\Xi_{1a} = F_1}$$ ← Do <u>not</u> included <u>conservative</u> forces, like the <u>spring</u>

$$\boxed{\Xi_{2a} = -F_b}$$ ← Where $\boxed{F_b = -b(\dot{x}_{3a} - \dot{x}_{2a})}$

$$\boxed{\Xi_{3a} = F_b + F_3}$$ ← F_3 is some "external force" applied to m_3

2. When using 1 Absolute & N-1 Relative coordinates $(x_{nr} = x_{na} - x_{(n-1)a})$, $(x_{1r} = x_{1a})$,

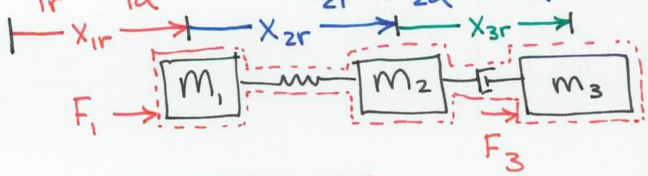
Coordinate n is RELATIVE to coordinate n-1 here!

$$\boxed{\Xi_{nr} = \sum_{i=n}^{N} \Xi_{ia}},$$ e.g. $\begin{cases} \Xi_{3r} = \Xi_{3a} \\ \Xi_{2r} = \Xi_{2a} + \Xi_{3a} \\ \Xi_{1r} = \Xi_{1a} + \Xi_{2a} + \Xi_{3a} \end{cases}$

for example above.

SAME EXAMPLE, Relative Coords:

$X_{1r} = X_{1a}$ $X_{2r} = X_{2a} - X_{1a}$ $X_{3r} = X_{3a} - X_{2a}$



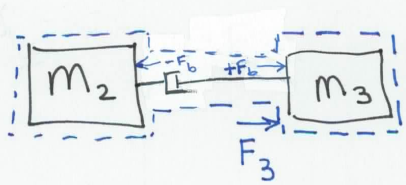
$$\boxed{\tilde{\Xi}_{1r} = F_1 + F_3}$$

↑ $\tilde{\Xi}_{1a} + \tilde{\Xi}_{2a} + \tilde{\Xi}_{3a}$

includes ALL external forces, acting on any of the masses.
Note: Internal forces (here "F_b" & "$-F_b$") will always cancel out (by definition)

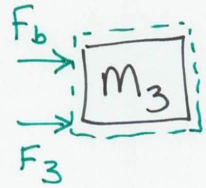
$$\boxed{\tilde{\Xi}_{2r} = F_3}$$

↑ $\tilde{\Xi}_{2a} + \tilde{\Xi}_{3a}$



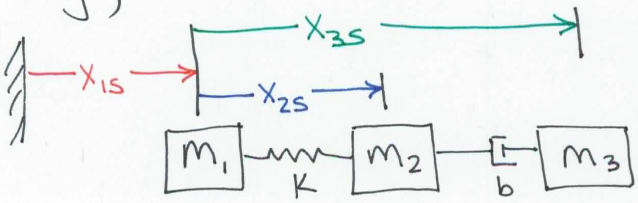
$$\boxed{\tilde{\Xi}_{3r} = F_b + F_3}$$

↑ $\tilde{\Xi}_{3a}$



Alternate G.C. Definitions are Possible...

e.g.,



$x_{1s} = x_{1a}$

$x_{2s} = x_{2a} - x_{1a}$

$x_{3s} = x_{3a} - x_{1a}$

Here,

$$T^* = \tfrac{1}{2} m_1 \dot{x}_{1s}^2 + \tfrac{1}{2} m_2 (\dot{x}_{1s} + \dot{x}_{2s})^2 + \tfrac{1}{2} m_3 (\dot{x}_{1s} + \dot{x}_{3s})^2$$

$$V = \tfrac{1}{2} K x_{2s}^2$$

$$\mathcal{L} = T^* - V$$

$$\Xi_{1s} = m_1 \ddot{x}_{1s} + m_2(\ddot{x}_{1s} + \ddot{x}_{2s}) + m_3(\ddot{x}_{1s} + \ddot{x}_{3s})$$

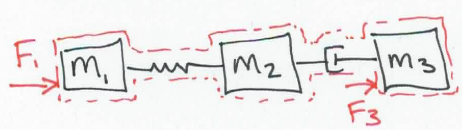
$$\boxed{\Xi_{1s} = \Xi_{1a} + \Xi_{2a} + \Xi_{3a}} \leftarrow = F_1 + F_3$$

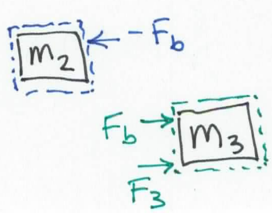
$$\Xi_{2s} = m_2(\ddot{x}_{1s} + \ddot{x}_{2s}) + K x_{2s}$$

$$\boxed{\Xi_{2s} = \Xi_{2a}} \leftarrow = -F_b$$

$$\Xi_{3s} = m_3(\ddot{x}_{1s} + \ddot{x}_{3s})$$

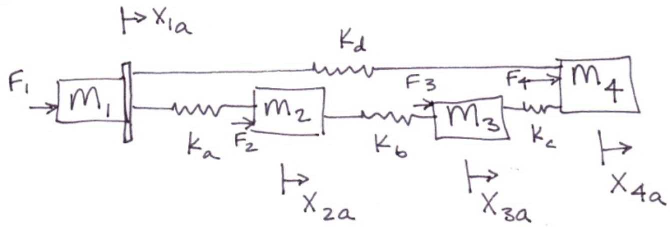
$$\boxed{\Xi_{3s} = \Xi_{3a}} \leftarrow = F_b + F_3$$





} ← <u>Both</u> x_{2s} & x_{3s} are RELATIVE to x_{1s} here...

Formal Technique to Find Ξ_i



Let's (arbitrarily) define our Generalized Coordinates as:

$$\begin{aligned} x_{1b} &= x_{1a} \\ x_{2b} &= x_{2a} - x_{1a} \\ x_{3b} &= x_{3a} - x_{2a} \\ x_{4b} &= x_{4a} - x_{1a} \end{aligned} \iff \boxed{\begin{aligned} x_{1a} &= x_{1b} \\ x_{2a} &= x_{1b} + x_{2b} \\ x_{3a} &= x_{1b} + x_{2b} + x_{3b} \\ x_{4a} &= x_{1b} + x_{4b} \end{aligned}}$$

write in matrix form

Any terms in T^* or V involving x_{3a} (and its derivatives) must be represented using "$x_{1b} + x_{2b} + x_{3b}$" (").

|| So, these terms show up in the ξ_1, ξ_2 & ξ_3 EOMs. Each EOM in this "b" GC definition is the <u>sum</u> of one or more EOM's in the absolute "a" GC system, related just as mentioned above... e.g. Ξ_{3a} must appear as an element of the expressions for Ξ_{1b}, Ξ_{2b}, & Ξ_{3b}

$$\begin{bmatrix} x_{1a} \\ x_{2a} \\ x_{3a} \\ x_{4a} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1b} \\ x_{2b} \\ x_{3b} \\ x_{4b} \end{bmatrix}$$

and so...

$$\begin{bmatrix} \Xi_{1b} \\ \Xi_{2b} \\ \Xi_{3b} \\ \Xi_{4b} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Xi_{1a} \\ \Xi_{2a} \\ \Xi_{3a} \\ \Xi_{4a} \end{bmatrix}$$

$\boxed{X_a = M\ X_b}$ or, $\boxed{\xi_a = M \xi_b}$ $\qquad$ $\boxed{\Xi_b = M^T\ \Xi_b}$