

Controllability (and Observability)

Lecture 14
ECE/ME 179D

Today: Look at meaning of "controllability" in more depth.

* Note, observability is a very analogous concept.

Review: From last time, recall that for a LINEAR dynamic system, there are two definitions we can (equivalently) use to test for controllability.

① If we can pick n gain values in K to set locations of n poles (in an n^{th} -order system) arbitrarily.

② If we can go to an arbitrarily new state vector (n states), given an appropriately chosen $u(t)$ (control action).

Let's look at each definition more closely.

① Given a dynamic system described via state space equations,

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} ; \quad \begin{matrix} (n \times 1) \\ \text{states} \end{matrix}$$

$$\dot{X} = AX + Bu$$

$$\begin{matrix} (n \times 1) & (n \times n) & (n \times 1) & (n \times m) & (m \times 1) \end{matrix}$$

$$Y = CX + Du$$

$$\begin{matrix} (k \times 1) & (k \times n) & (n \times 1) & (k \times m) & (m \times 1) \end{matrix}$$

usually, $D=0$

n states
 m inputs
 k outputs

* Assume $m=1$ for now (single input).

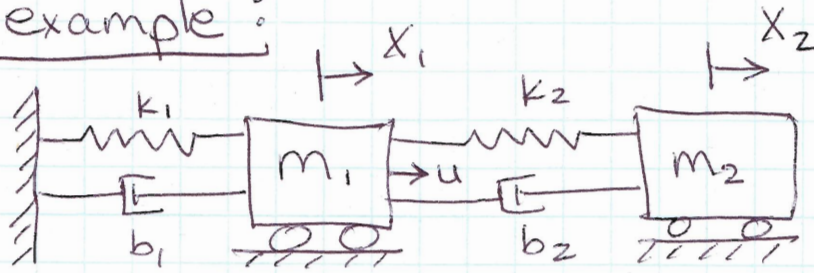
a) Open-loop poles are at $\text{eig}(A)$.

b) From poles, we can create the characteristic eqn:

$$s^n + a_1 s^{n-1} + \dots + a_{n-1} = 0$$

①

For example :



$$m_1 \ddot{x}_1 + (b_1 + b_2) \dot{x}_1 + (k_1 + k_2) x_1 = u + b_2 \dot{x}_2 + k_2 x_2$$

$$m_2 \ddot{x}_2 + b_2 \dot{x}_2 + k_2 x_2 = b_2 \dot{x}_1 + k_2 x_1$$

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$; $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-(k_1+k_2)}{m_1} & \frac{k_2}{m_1} & \frac{-(b_1+b_2)}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & \frac{-k_2}{m_2} & \frac{b_2}{m_2} & \frac{-b_2}{m_2} \end{bmatrix}$

For $m_1 = 1$, $m_2 = 2$
 $b_1 = 1$, $b_2 = 2$
 $k_1 = 5$, $k_2 = 10$, $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -15 & 10 & -3 & 2 \\ 5 & -5 & 1 & -1 \end{bmatrix}$

and $B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$S = \text{eig}(A) \rightarrow \begin{matrix} -1.9 + 3.9j \\ -1.9 - 3.9j \\ -0.13 + 1.1j \\ -0.13 - 1.1j \end{matrix}$

$p = 1$;
for $n = 1:n$
 $p = \text{conv}(p, [1, -s(n)])$;
end

$p = [1, 4, 21, 10, 25]$

i.e., $s^4 + 4s^3 + 21s^2 + 10s + 25 = 0$

Characteristic Equation given A matrix. (2)

- Now, given open-loop poles, which determine a (unique) characteristic equation ^(if highest order in s has coef. 1) _(unless $n=1$ state)

... we can generally define a new dynamic system w/ A such that eigenvalues are the same.

- For example, the CANONICAL form ^{" A_c "} where top row of A_c is built directly from the char. poly.:

$$s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0$$

$$A_c = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \dots & -a_n \\ 1 & 0 & 0 & & 0 \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

$$B_c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Here, $X = \begin{bmatrix} \ddots \\ \ddot{x} \\ \dot{x} \\ x \\ x \end{bmatrix}$

← Not easy to interpret physically!

Controllability matrix

$$\mathbf{C} \equiv \underbrace{[B, AB, A^2B, \dots, A^{n-1}B]}_{(n \times n)}$$

* If $\mathbf{C}$ is FULL RANK, then it is invertible, $\therefore$
then you can set pole arbitrarily by picking K_c .

Why?: Let $\boxed{T = C C_c^{-1}}$ $\begin{cases} \text{Controllability} \\ \text{matrix for} \\ A_c \text{ \& } B_c \end{cases}$

$A_c = T^{-1}AT$	$A = TA_cT^{-1}$
$B_c = T^{-1}B$	$B = TB_c$

For the canonical system, control gains are:

$$K_c = [K_{1c}, K_{2c}, \dots, K_{nc}]$$

Control law is:

$$u = -Kx$$

$$\begin{aligned} \therefore \dot{X} &= A_c X + B_c u \\ &= A_c X - B_c K_c X \end{aligned}$$

$$\boxed{\dot{X} = (A_c - B_c K_c) X}$$

closed-loop poles
are at

$$\boxed{s = \text{eig}(A_c - B_c K_c)}$$

Since $B_c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $B_c K_c = \begin{bmatrix} K_1 & K_2 & K_3 & K_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$; $A_c - B_c K_c = \begin{bmatrix} -(a_1 + K_1) & -(a_2 + K_2) & \downarrow & -(a_4 + K_4) \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

our
example

Say we want poles $s = \begin{cases} -1+j \\ -1-j \\ -10 \\ -10 \end{cases}$

Desired char. eq. is: $s^4 + 22s^3 + 142s^2 + 240s + 200 = 0$

So, since O.L. was: $s^4 + \underset{(a_1)}{4}s^3 + \underset{(a_2)}{21}s^2 + \underset{(a_3)}{10}s + \underset{(a_4)}{25} = 0$

We need:

$$K_1 + a_1 = 22 \rightarrow K_1 = 22 - 4 = 18$$

$$K_2 + a_2 = 142 \rightarrow K_2 = 142 - 21 = 121$$

$$K_3 + a_3 = 240 \rightarrow K_3 = 240 - 10 = 230$$

$$K_4 + a_4 = 200 \rightarrow K_4 = 200 - 25 = 175$$

Finally, to get from K_c (for the control canonical form) to K (for original A & B matrices):

$$A_c - B_c K_c$$

$$\downarrow$$

$$T(A_c)T^{-1} - T(B_c K_c)T^{-1}$$

$$A - B(K_c T^{-1})$$

$$A - B K$$

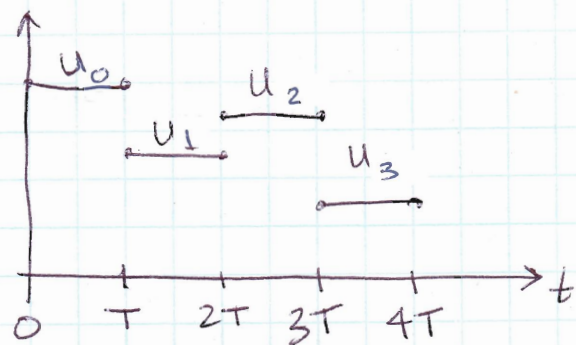
where $K \equiv K_c T^{-1}$

Each line will have same eigenvalues!

② Arbitrary states in finite time?

With 4 states, we expect to need 4 tunable variables to set each state independently.

So, say we do the following: (Discrete-time control) ^{"DT"}



Can we find values for each u_i to change all 4 states independently?

* Note - It's a lot easier to visualize this for the DT case, but same concept of "getting to a desired state" applies in CT (cont-time) case, too.

DT:
$$X_{n+1} = A_d X_n + B_d u_n$$

So, for 4 time steps,

after u_0 , $X_1 = A X_0 + B u_0$

after u_1 ,
$$\begin{aligned} X_2 &= A X_1 + B u_1 \\ &= A (A X_0 + B u_0) + B u_1 \\ &= A^2 X_0 + A B u_0 + B u_1 \end{aligned}$$

after u_2 ,
$$\begin{aligned} X_3 &= A (A^2 X_0 + A B u_0 + B u_1) + B u_2 \\ &= A^3 X_0 + A^2 B u_0 + A B u_1 + B u_2 \end{aligned}$$

after u_3 ,
$$\begin{aligned} X_4 &= A X_3 + B u_3 \\ X_4 &= A^4 X_0 + A^3 B u_0 + A^2 B u_1 + A B u_2 + B u_3 \end{aligned}$$

We want $X_4 = X_{des}$

$$X_{des} - A^4 X_0 = [A^3 B, A^2 B, A B, B] \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$(X_{des} - A^4 X_0) = \underbrace{\begin{bmatrix} B & AB & A^2 B & A^3 B \end{bmatrix}}_{(n \times n)} \begin{bmatrix} u_3 \\ u_2 \\ u_1 \\ u_0 \end{bmatrix}$$

This is $\mathcal{C}$!
(Controllability)

To solve for vector u , use matrix algebra:

$$(X_{des} - A^4 X_0) = \mathcal{C} u$$

$$\mathcal{C}^{-1} (X_{des} - A^4 X_0) = \mathcal{C}^{-1} \mathcal{C} u = u$$

$\mathcal{C}$ must be INVERTIBLE for a solution to be guaranteed!

Observability: Given some initial vector of states, x_i ,

$$y = Cx + Du$$

Can we identify all values in x_i by observing the available outputs, y , over some finite time?

$$y_0 = Cx_0 \leftarrow x_0 = x_{init}$$

$$x_1 = Ax_0 + Bu_0$$

$$y_1 = C(Ax_0 + Bu_0)$$

$$x_2 = Ax_1 + Bu_1$$

$$= A(Ax_0 + Bu_0) + Bu_1$$

$$y_2 = C[A(Ax_0 + Bu_0) + Bu_1]$$

$$x_3 = Ax_2 + Bu_2$$

$$\therefore y_3 = C\{A[A(Ax_0 + Bu_0) + Bu_1] + Bu_2\}$$

Rearranging

x_0 is the unknown we want to observe!

$$y_0 = Cx_0$$

$$y_1 = CAX_0 + CBu_0$$

$$y_2 = CA^2X_0 + CABu_0 + CBu_1$$

$$y_3 = CA^3X_0 + CA^2Bu_0 + CABu_1 + CBu_2$$

All other terms ($y_0, y_1, y_2, y_3, C, A, B$) are measured or known at the start.

$$\therefore \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} x_0 = \begin{bmatrix} y_0 \\ y_1 - CBu_0 \\ y_2 - CABu_0 - CBu_1 \\ y_3 - CA^2Bu_0 - CABu_1 - CBu_2 \end{bmatrix} = \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

all known

$$\mathcal{O}^T$$

$$\therefore x_0 = (\mathcal{O}^T)^{-1} \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

Can be found if $\mathcal{O}$ is full rank, i.e., invertible.

$\text{eig}(A-BK)$
 $\text{eig}(A-LC)$

Matlab

to find $K = \text{place}(A, B, p_{\text{con}})$

$L = \text{place}(A', C', p_{\text{obs}})$

control gain

observer gain

$(\text{eig}(A-BK))$
control poles

$(\text{eig}(A-LC))$
observer poles

note 3 transpose chars