

3

Mobile Robot Kinematics

3.1 Introduction

Kinematics is the most basic study of how mechanical systems behave. In mobile robotics, we need to understand the mechanical behavior of the robot both in order to design appropriate mobile robots for tasks and to understand how to create control software for an instance of mobile robot hardware.

Of course, mobile robots are not the first complex mechanical systems to require such analysis. Robot manipulators have been the subject of intensive study for more than thirty years. In some ways, manipulator robots are much more complex than early mobile robots: a standard welding robot may have five or more joints, whereas early mobile robots were simple differential-drive machines. In recent years, the robotics community has achieved a fairly complete understanding of the kinematics and even the dynamics (i.e., relating to force and mass) of robot manipulators [11, 32].

The mobile robotics community poses many of the same kinematic questions as the robot manipulator community. A manipulator robot's workspace is crucial because it defines the range of possible positions that can be achieved by its end effector relative to its fixture to the environment. A mobile robot's workspace is equally important because it defines the range of possible poses that the mobile robot can achieve in its environment. The robot arm's controllability defines the manner in which active engagement of motors can be used to move from pose to pose in the workspace. Similarly, a mobile robot's controllability defines possible paths and trajectories in its workspace. Robot dynamics places additional constraints on workspace and trajectory due to mass and force considerations. The mobile robot is also limited by dynamics; for instance, a high center of gravity limits the practical turning radius of a fast, car-like robot because of the danger of rolling.

But the chief difference between a mobile robot and a manipulator arm also introduces a significant challenge for *position estimation*. A manipulator has one end fixed to the environment. Measuring the position of an arm's end effector is simply a matter of understanding the kinematics of the robot and measuring the position of all intermediate joints. The manipulator's position is thus always computable by looking at current sensor data. But a

mobile robot is a self-contained automaton that can wholly move with respect to its environment. There is no direct way to measure a mobile robot's position instantaneously. Instead, one must integrate the motion of the robot over time. Add to this the inaccuracies of *motion estimation* due to slippage and it is clear that measuring a mobile robot's position precisely is an extremely challenging task.

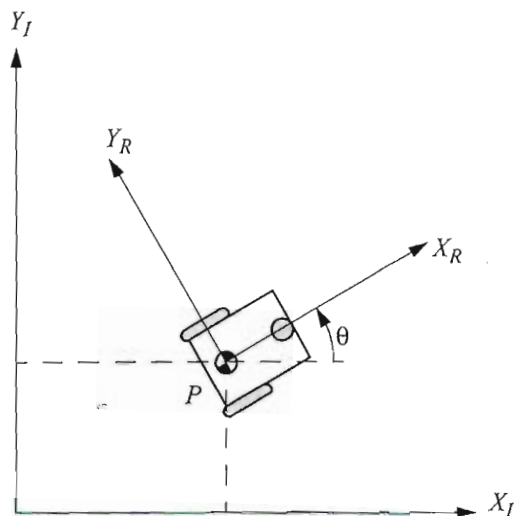
The process of understanding the motions of a robot begins with the process of describing the contribution each wheel provides for motion. Each wheel has a role in enabling the whole robot to move. By the same token, each wheel also imposes constraints on the robot's motion; for example, refusing to skid laterally. In the following section, we introduce notation that allows expression of robot motion in a global reference frame as well as the robot's local reference frame. Then, using this notation, we demonstrate the construction of simple forward kinematic models of motion, describing how the robot as a whole moves as a function of its geometry and individual wheel behavior. Next, we formally describe the kinematic constraints of individual wheels, and then combine these kinematic constraints to express the whole robot's kinematic constraints. With these tools, one can evaluate the paths and trajectories that define the robot's maneuverability.

3.2 Kinematic Models and Constraints

Deriving a model for the whole robot's motion is a bottom-up process. Each individual wheel contributes to the robot's motion and, at the same time, imposes constraints on robot motion. Wheels are tied together based on robot chassis geometry, and therefore their constraints combine to form constraints on the overall motion of the robot chassis. But the forces and constraints of each wheel must be expressed with respect to a clear and consistent reference frame. This is particularly important in mobile robotics because of its self-contained and mobile nature; a clear mapping between global and local frames of reference is required. We begin by defining these reference frames formally, then using the resulting formalism to annotate the kinematics of individual wheels and whole robots. Throughout this process we draw extensively on the notation and terminology presented in [52].

3.2.1 Representing robot position

Throughout this analysis we model the robot as a rigid body on wheels, operating on a horizontal plane. The total dimensionality of this robot chassis on the plane is three, two for position in the plane and one for orientation along the vertical axis, which is orthogonal to the plane. Of course, there are additional degrees of freedom and flexibility due to the wheel axles, wheel steering joints, and wheel castor joints. However by robot *chassis* we refer only to the rigid body of the robot, ignoring the joints and degrees of freedom internal to the robot and its wheels.

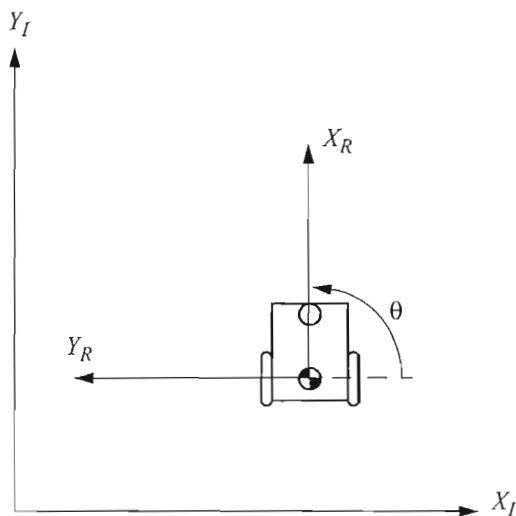
**Figure 3.1**

The global reference frame and the robot local reference frame.

In order to specify the position of the robot on the plane we establish a relationship between the global reference frame of the plane and the local reference frame of the robot, as in figure 3.1. The axes X_I and Y_I define an arbitrary inertial basis on the plane as the global reference frame from some origin O : $\{X_I, Y_I\}$. To specify the position of the robot, choose a point P on the robot chassis as its position reference point. The basis $\{X_R, Y_R\}$ defines two axes relative to P on the robot chassis and is thus the robot's local reference frame. The position of P in the global reference frame is specified by coordinates x and y , and the angular difference between the global and local reference frames is given by θ . We can describe the pose of the robot as a vector with these **three** elements. Note the use of the subscript I to clarify the basis of this pose as the global reference frame:

$$\xi_I = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} \quad (3.1)$$

To describe robot motion in terms of component motions, it will be necessary to map motion along the axes of the global reference frame to motion along the axes of the robot's local reference frame. Of course, the mapping is a function of the current pose of the robot. This mapping is accomplished using the *orthogonal rotation matrix*:

**Figure 3.2**

The mobile robot aligned with a global axis.

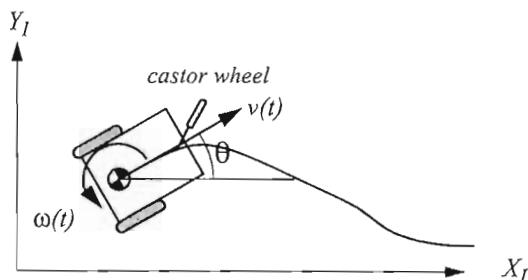
$$R(\theta) = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3.2)$$

This matrix can be used to map motion in the global reference frame $\{X_I, Y_I\}$ to motion in terms of the local reference frame $\{X_R, Y_R\}$. This operation is denoted by $R(\theta)\xi_I$ because the computation of this operation depends on the value of θ :

$$\xi_R = R\left(\frac{\pi}{2}\right)\xi_I \quad (3.3)$$

For example, consider the robot in figure 3.2. For this robot, because $\theta = \frac{\pi}{2}$ we can easily compute the instantaneous rotation matrix R :

$$R\left(\frac{\pi}{2}\right) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3.4)$$

**Figure 3.3**

A differential-drive robot in its global reference frame.

Given some velocity $(\dot{x}, \dot{y}, \dot{\theta})$ in the global reference frame we can compute the components of motion along this robot's local axes X_R and Y_R . In this case, due to the specific angle of the robot, motion along X_R is equal to $\dot{y}$ and motion along Y_R is $-\dot{x}$:

$$\dot{\xi}_R = R\left(\frac{\pi}{2}\right)\dot{\xi}_I = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{y} \\ -\dot{x} \\ \dot{\theta} \end{bmatrix} \quad (3.5)$$

3.2.2 Forward kinematic models

In the simplest cases, the mapping described by equation (3.3) is sufficient to generate a formula that captures the forward kinematics of the mobile robot: how does the robot move, given its geometry and the speeds of its wheels? More formally, consider the example shown in figure 3.3.

This differential drive robot has two wheels, each with diameter r . Given a point P centered between the two drive wheels, each wheel is a distance l from P . Given r , l , θ , and the spinning speed of each wheel, $\dot{\phi}_1$ and $\dot{\phi}_2$, a forward kinematic model would predict the robot's overall speed in the global reference frame:

$$\dot{\xi}_I = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = f(l, r, \theta, \dot{\phi}_1, \dot{\phi}_2) \quad (3.6)$$

From equation (3.3) we know that we can compute the robot's motion in the global reference frame from motion in its local reference frame: $\dot{\xi}_I = R(\theta)^{-1}\dot{\xi}_R$. Therefore, the strategy will be to first compute the contribution of each of the two wheels in the local reference

frame, $\dot{\xi}_R$. For this example of a differential-drive chassis, this problem is particularly straightforward.

Suppose that the robot's local reference frame is aligned such that the robot moves forward along $+X_R$, as shown in figure 3.1. First consider the contribution of each wheel's spinning speed to the translation speed at P in the direction of $+X_R$. If one wheel spins while the other wheel contributes nothing and is stationary, since P is halfway between the two wheels, it will move instantaneously with half the speed: $\dot{x}_{r1} = (1/2)r\dot{\phi}_1$ and $\dot{x}_{r2} = (1/2)r\dot{\phi}_2$. In a differential drive robot, these two contributions can simply be added to calculate the x_R component of $\dot{\xi}_R$. Consider, for example, a differential robot in which each wheel spins with equal speed but in opposite directions. The result is a stationary, spinning robot. As expected, $\dot{x}_R$ will be zero in this case. The value of $\dot{y}_R$ is even simpler to calculate. Neither wheel can contribute to sideways motion in the robot's reference frame, and so $\dot{y}_R$ is always zero. Finally, we must compute the rotational component $\dot{\theta}_R$ of $\dot{\xi}_R$. Once again, the contributions of each wheel can be computed independently and just added. Consider the right wheel (we will call this wheel 1). Forward spin of this wheel results in *counterclockwise* rotation at point P . Recall that if wheel 1 spins alone, the robot pivots around wheel 2. The rotation velocity ω_1 at P can be computed because the wheel is instantaneously moving along the arc of a circle of radius $2l$:

$$\omega_1 = \frac{r\dot{\phi}_1}{2l} \quad (3.7)$$

The same calculation applies to the left wheel, with the exception that forward spin results in *clockwise* rotation at point P :

$$\omega_2 = \frac{-r\dot{\phi}_2}{2l} \quad (3.8)$$

Combining these individual formulas yields a kinematic model for the differential-drive example robot:

$$\dot{\xi}_I = R(\theta)^{-1} \begin{bmatrix} \frac{r\dot{\phi}_1}{2} + \frac{r\dot{\phi}_2}{2} \\ 0 \\ \frac{r\dot{\phi}_1}{2l} + \frac{-r\dot{\phi}_2}{2l} \end{bmatrix} \quad (3.9)$$

We can now use this kinematic model in an example. However, we must first compute $R(\theta)^{-1}$. In general, calculating the inverse of a matrix may be challenging. In this case, however, it is easy because it is simply a transform from ξ_R to ξ_I rather than vice versa:

$$R(\theta)^{-1} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3.10)$$

Suppose that the robot is positioned such that $\theta = \pi/2$, $r = 1$, and $l = 1$. If the robot engages its wheels unevenly, with speeds $\dot{\phi}_1 = 4$ and $\dot{\phi}_2 = 2$, we can compute its velocity in the global reference frame:

$$\dot{\xi}_I = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \quad (3.11)$$

So this robot will move instantaneously along the y -axis of the global reference frame with speed 3 while rotating with speed 1. This approach to kinematic modeling can provide information about the motion of a robot given its component wheel speeds in straightforward cases. However, we wish to determine the space of possible motions for each robot chassis design. To do this, we must go further, describing formally the constraints on robot motion imposed by each wheel. Section 3.2.3 begins this process by describing constraints for various wheel types; the rest of this chapter provides tools for analyzing the characteristics and workspace of a robot given these constraints.

3.2.3 Wheel kinematic constraints

The first step to a kinematic model of the robot is to express constraints on the motions of individual wheels. Just as shown in section 3.2.2, the motions of individual wheels can later be combined to compute the motion of the robot as a whole. As discussed in chapter 2, there are four basic wheel types with widely varying kinematic properties. Therefore, we begin by presenting sets of constraints specific to each wheel type.

However, several important assumptions will simplify this presentation. We assume that the plane of the wheel always remains vertical and that there is in all cases one single point of contact between the wheel and the ground plane. Furthermore, we assume that there is no sliding at this single point of contact. That is, the wheel undergoes motion only under conditions of pure rolling and rotation about the vertical axis through the contact point. For a more thorough treatment of kinematics, including sliding contact, refer to [25].

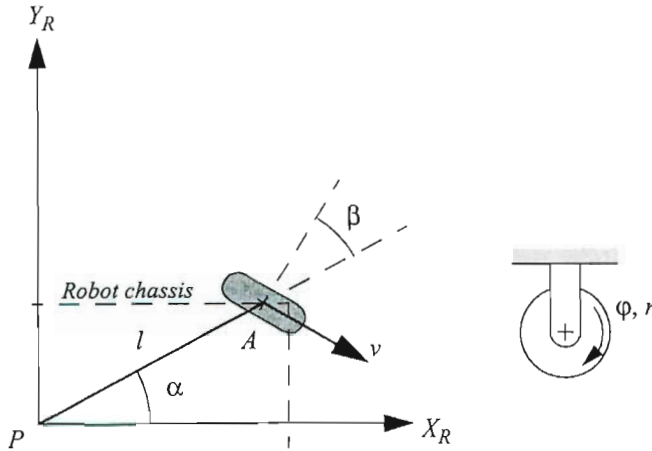


Figure 3.4
A fixed standard wheel and its parameters.

Under these assumptions, we present two constraints for every wheel type. The first constraint enforces the concept of rolling contact – that the wheel must roll when motion takes place in the appropriate direction. The second constraint enforces the concept of no lateral slippage – that the wheel must not slide orthogonal to the wheel plane.

3.2.3.1 Fixed standard wheel

The fixed standard wheel has no vertical axis of rotation for steering. Its angle to the chassis is thus fixed, and it is limited to motion back and forth along the wheel plane and rotation around its contact point with the ground plane. Figure 3.4 depicts a fixed standard wheel A and indicates its position pose relative to the robot's local reference frame $\{X_R, Y_R\}$. The position of A is expressed in polar coordinates by distance l and angle α . The angle of the wheel plane relative to the chassis is denoted by β , which is fixed since the fixed standard wheel is not steerable. The wheel, which has radius r , can spin over time, and so its rotational position around its horizontal axle is a function of time t : $\phi(t)$.

The rolling constraint for this wheel enforces that all motion along the direction of the wheel plane must be accompanied by the appropriate amount of wheel spin so that there is pure rolling at the contact point:

$$\begin{bmatrix} \sin(\alpha + \beta) & -\cos(\alpha + \beta) & (-l) \cos \beta \end{bmatrix} R(\theta) \dot{\xi}_I - r \dot{\phi} = 0 \quad (3.12)$$

The first term of the sum denotes the total motion along the wheel plane. The three elements of the vector on the left represent mappings from each of $\dot{x}, \dot{y}, \dot{\theta}$ to their contributions for motion along the wheel plane. Note that the $R(\theta)\xi_I$ term is used to transform the motion parameters ξ_I that are in the global reference frame $\{X_I, Y_I\}$ into motion parameters in the local reference frame $\{X_R, Y_R\}$ as shown in example equation (3.5). This is necessary because all other parameters in the equation, α, β, l , are in terms of the robot's local reference frame. This motion along the wheel plane must be equal, according to this constraint, to the motion accomplished by spinning the wheel, $r\dot{\phi}$.

The sliding constraint for this wheel enforces that the component of the wheel's motion orthogonal to the wheel plane must be zero:

$$\begin{bmatrix} \cos(\alpha + \beta) & \sin(\alpha + \beta) & l \sin \beta \end{bmatrix} R(\theta) \xi_I = 0 \quad (3.13)$$

For example, suppose that wheel A is in a position such that $\{(\alpha = 0), (\beta = 0)\}$. This would place the contact point of the wheel on X_I with the plane of the wheel oriented parallel to Y_I . If $\theta = 0$, then the sliding constraint [equation (3.13)] reduces to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = 0 \quad (3.14)$$

This constrains the component of motion along X_I to be zero and since X_I and X_R are parallel in this example, the wheel is constrained from sliding sideways, as expected.

3.2.3.2 Steered standard wheel

The steered standard wheel differs from the fixed standard wheel only in that there is an additional degree of freedom: the wheel may rotate around a vertical axis passing through the center of the wheel and the ground contact point. The equations of position for the steered standard wheel (figure 3.5) are identical to that of the fixed standard wheel shown in figure 3.4 with one exception. The orientation of the wheel to the robot chassis is no longer a single fixed value, β , but instead varies as a function of time: $\beta(t)$. The rolling and sliding constraints are

$$\begin{bmatrix} \sin(\alpha + \beta) & -\cos(\alpha + \beta) & (-l) \cos \beta \end{bmatrix} R(\theta) \xi_I - r\dot{\phi} = 0 \quad (3.15)$$

$$\begin{bmatrix} \cos(\alpha + \beta) & \sin(\alpha + \beta) & l \sin \beta \end{bmatrix} R(\theta) \xi_I = 0 \quad (3.16)$$

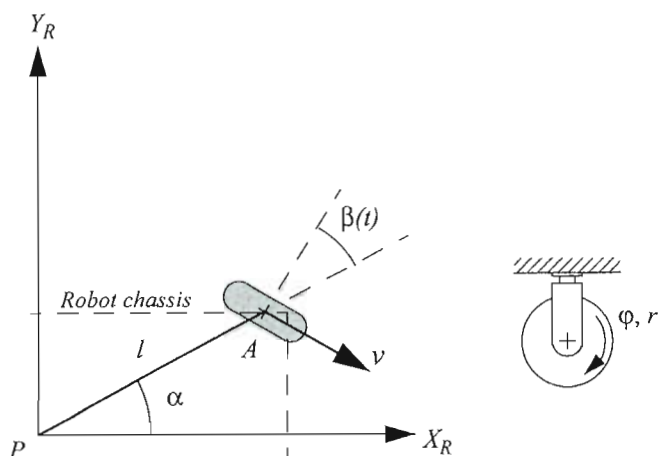


Figure 3.5

A steered standard wheel and its parameters.

These constraints are identical to those of the fixed standard wheel because, unlike $\dot{\phi}$, $\dot{\beta}$ does not have a direct impact on the instantaneous motion constraints of a robot. It is only by integrating over time that changes in steering angle can affect the mobility of a vehicle. This may seem subtle, but is a very important distinction between change in steering position, $\dot{\beta}$, and change in wheel spin, $\dot{\phi}$.

3.2.3.3 Castor wheel

Castor wheels are able to steer around a vertical axis. However, unlike the steered standard wheel, the vertical axis of rotation in a castor wheel does not pass through the ground contact point. Figure 3.6 depicts a castor wheel, demonstrating that formal specification of the castor wheel's position requires an additional parameter.

The wheel contact point is now at position B , which is connected by a rigid rod AB of fixed length d to point A , fixes the location of the vertical axis about which B steers, and this point A has a position specified in the robot's reference frame, as in figure 3.6. We assume that the plane of the wheel is aligned with AB at all times. Similar to the steered standard wheel, the castor wheel has two parameters that vary as a function of time. $\phi(t)$ represents the wheel spin over time as before. $\beta(t)$ denotes the steering angle and orientation of AB over time.

For the castor wheel, the rolling constraint is identical to equation (3.15) because the offset axis plays no role during motion that is aligned with the wheel plane:

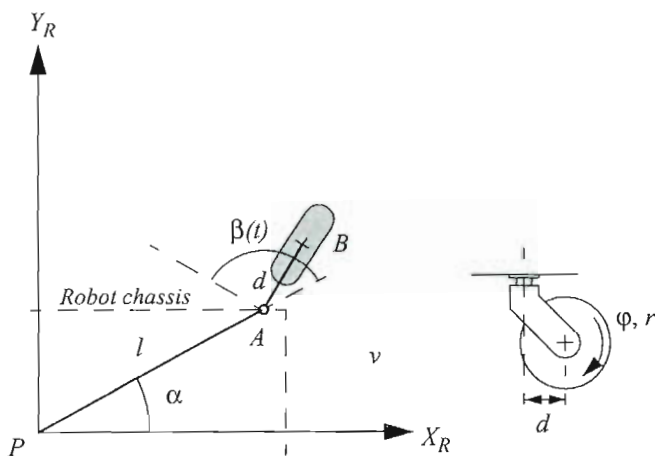


Figure 3.6
A castor wheel and its parameters.

$$[\sin(\alpha + \beta) - \cos(\alpha + \beta) (-l) \cos \beta] R(\theta) \dot{\xi}_l - r \dot{\phi} = 0 \quad (3.17)$$

The castor geometry does, however, have significant impact on the sliding constraint. The critical issue is that the lateral force on the wheel occurs at point A because this is the attachment point of the wheel to the chassis. Because of the offset ground contact point relative to A , the constraint that there be zero lateral movement would be wrong. Instead, the constraint is much like a rolling constraint, in that appropriate rotation of the vertical axis must take place:

$$[\cos(\alpha + \beta) \sin(\alpha + \beta) d + l \sin \beta] R(\theta) \dot{\xi}_l + d \dot{\beta} = 0 \quad (3.18)$$

In equation (3.18), any motion orthogonal to the wheel plane must be balanced by an equivalent and opposite amount of castor steering motion. This result is critical to the success of castor wheels because by setting the value of $\dot{\beta}$ any arbitrary lateral motion can be acceptable. In a steered standard wheel, the steering action does not by itself cause a movement of the robot chassis. But in a castor wheel the steering action itself moves the robot chassis because of the offset between the ground contact point and the vertical axis of rotation.



Figure 3.7
Office chair with five castor wheels.

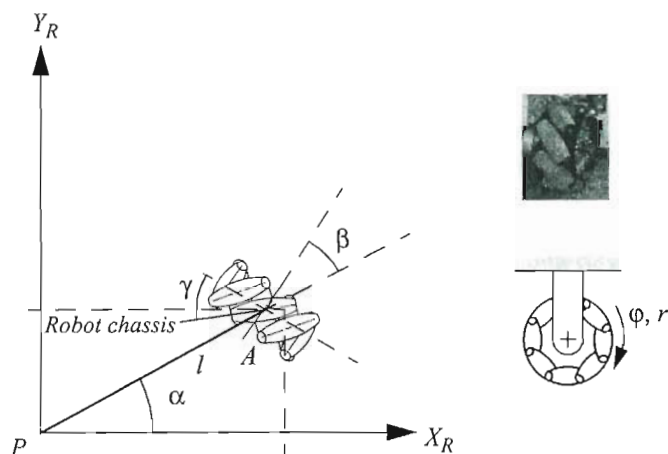
More concisely, it can be surmised from equations (3.17) and (3.18) that, given *any* robot chassis motion $\dot{\xi}_I$, there exists some value for spin speed $\dot{\phi}$ and steering speed $\dot{\beta}$ such that the constraints are met. Therefore, a robot with only castor wheels can move with any velocity in the space of possible robot motions. We term such systems *omnidirectional*.

A real-world example of such a system is the five-castor wheel office chair shown in figure 3.7. Assuming that all joints are able to move freely, you may select any motion vector on the plane for the chair and push it by hand. Its castor wheels will spin and steer as needed to achieve that motion without contact point sliding. By the same token, if each of the chair's castor wheels housed two motors, one for spinning and one for steering, then a control system would be able to move the chair along any trajectory in the plane. Thus, although the kinematics of castor wheels is somewhat complex, such wheels do not impose any real constraints on the kinematics of a robot chassis.

3.2.3.4 Swedish wheel

Swedish wheels have no vertical axis of rotation, yet are able to move *omnidirectionally* like the castor wheel. This is possible by adding a degree of freedom to the fixed standard wheel. Swedish wheels consist of a fixed standard wheel with rollers attached to the wheel perimeter with axes that are antiparallel to the main axis of the fixed wheel component. The exact angle γ between the roller axes and the wheel plane can vary, as shown in figure 3.8.

For example, given a Swedish 45-degree wheel, the motion vectors of the principal axis and the roller axes can be drawn as in figure 3.8. Since each axis can spin clockwise or counterclockwise, one can combine any vector along one axis with any vector along the other axis. These two axes are not necessarily independent (except in the case of the Swedish 90-degree wheel); however, it is visually clear that any desired direction of motion is achievable by choosing the appropriate two vectors.

**Figure 3.8**

A Swedish wheel and its parameters.

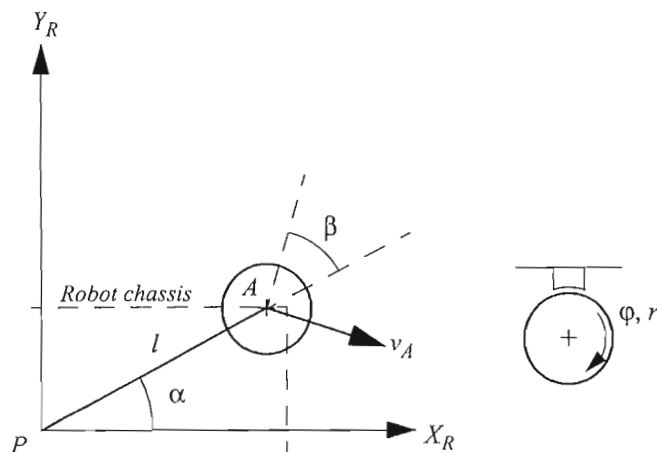
The pose of a Swedish wheel is expressed exactly as in a fixed standard wheel, with the addition of a term, γ , representing the angle between the main wheel plane and the axis of rotation of the small circumferential rollers. This is depicted in figure 3.8 within the robot's reference frame.

Formulating the constraint for a Swedish wheel requires some subtlety. The instantaneous constraint is due to the specific orientation of the small rollers. The axis around which these rollers spin is a zero component of velocity at the contact point. That is, moving in that direction without spinning the main axis is not possible without sliding. The motion constraint that is derived looks identical to the rolling constraint for the fixed standard wheel in equation (3.12) except that the formula is modified by adding γ such that the effective direction along which the rolling constraint holds is along this zero component plane rather than along the wheel plane:

$$[\sin(\alpha + \beta + \gamma) - \cos(\alpha + \beta + \gamma) (-l) \cos(\beta + \gamma)] R(\theta) \dot{\xi}_l - r \dot{\phi} \cos \gamma = 0 \quad (3.19)$$

Orthogonal to this direction the motion is not constrained because of the free rotation $\dot{\phi}_{sw}$ of the small rollers.

$$[\cos(\alpha + \beta + \gamma) \sin(\alpha + \beta + \gamma) l \sin(\beta + \gamma)] R(\theta) \dot{\xi}_l - r \dot{\phi} \sin \gamma - r_{sw} \dot{\phi}_{sw} = 0 \quad (3.20)$$

**Figure 3.9**

A spherical wheel and its parameters.

The behavior of this constraint and thereby the Swedish wheel changes dramatically as the value γ varies. Consider $\gamma = 0$. This represents the Swedish 90-degree wheel. In this case, the zero component of velocity is in line with the wheel plane and so equation (3.19) reduces exactly to equation (3.12), the fixed standard wheel rolling constraint. But because of the rollers, there is no sliding constraint orthogonal to the wheel plane [see equation (3.20)]. By varying the value of ϕ , any desired motion vector can be made to satisfy equation (3.19) and therefore the wheel is omnidirectional. In fact, this special case of the Swedish design results in fully decoupled motion, in that the rollers and the main wheel provide orthogonal directions of motion.

At the other extreme, consider $\gamma = \pi/2$. In this case, the rollers have axes of rotation that are parallel to the main wheel axis of rotation. Interestingly, if this value is substituted for γ in equation (3.19) the result is the fixed standard wheel sliding constraint, equation (3.13). In other words, the rollers provide no benefit in terms of lateral freedom of motion since they are simply aligned with the main wheel. However, in this case the main wheel never needs to spin and therefore the rolling constraint disappears. This is a degenerate form of the Swedish wheel and therefore we assume in the remainder of this chapter that $\gamma \neq \pi/2$.

3.2.3.5 Spherical wheel

The final wheel type, a ball or spherical wheel, places no direct constraints on motion (figure 3.9). Such a mechanism has no principal axis of rotation, and therefore no appropriate rolling or sliding constraints exist. As with castor wheels and Swedish wheels, the spherical