

Lagrangian Approach: Recall

Little "xi" is the set of DOF. Since we assume system is holonomic, there are the same number of GCs and DOFs.

ξ_j ($j = 1, 2, 3, \dots$)

Big "Xi" represents all non-conservative forces, for each DOF. This includes both the ACTUATION and LOSSES (due to damping, most typically).

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\xi}_i} \right) - \frac{\partial \mathcal{L}}{\partial \xi_i} = \Xi_i$$

Fancy "L" is the "Lagrangian"

$$\mathcal{L} = T^* - V$$

V is potential energy

T* is kinetic CO-ENERGY

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Lagrangian Approach: Procedure

3. Define potential energy, V.

Potential energy can be stored in a spring:

$$V_{\text{spring}} = \frac{1}{2} k (x - x_o)^2$$

...and also includes any "mgh" of particles in a gravitational field:

$$V_{\text{gravity}} = mgh$$

Note: "h" is relative to any arbitrary (but fixed) inertial reference frame.

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Lagrangian Approach: Procedure

0. Assume system is **holonomic**:

$$\#GC = \xi_j = \delta \xi_k = \#DOF \longleftrightarrow \text{Holonomic}$$

1. Pick **generalized coordinates** (DOFs) that are:

- **Complete**
- **Independent**

ξ_j

Little xi: GCs (gen. coord's)

- Possible GC set(s) **might not be unique**!
- **Any** independent and complete set is OK.
- Either **absolute** or **relative** coords is OK.

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Lagrangian Approach: Procedure

4. Lagrangian is **kinetic minus potential** energy:

$$\mathcal{L} = T^* - V$$

5. For each GC (DOF), solve for an equation of motion:

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\xi}_i} \right) - \frac{\partial \mathcal{L}}{\partial \xi_i} = \Xi_i$$

Note: A system with **N** generalized coordinates (GCs) will have **N** equations of motion.

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Lagrangian Approach: Procedure

2. Define kinetic (co-)energy, T. (Technically, T*)

Kinetic energy includes the sum of "one-half mass time velocity-squared" for every particle:

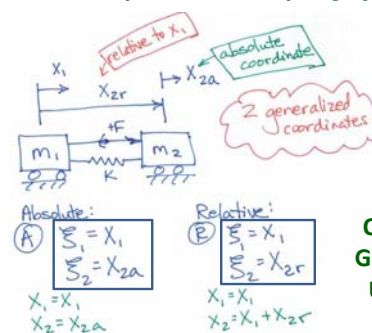
$$T = \sum_i \frac{1}{2} m_i v_i^2$$

For a **rigid rotating body**, we can use moment of inertia, J:

$$T = \frac{1}{2} J \dot{\theta}^2$$

(Note: We are **ignoring** electrical kinetic or potential energy here, for now and focusing on **MECHANICAL energy**.)

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Example: Two-cart spring system

Choice of GCs is **NOT** UNIQUE

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Absolute Coord's

$$T^* = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_{2a}^2$$

$$V = \frac{1}{2} k (x_1 - x_{2a})^2$$

$$\mathcal{L} = T^* - V$$

$$= \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_{2a}^2 + \dots$$

$$- \frac{1}{2} k x_1^2 - \frac{1}{2} k x_{2a}^2 + k x_1 x_{2a}$$

$$\tilde{F}_{1a} = -F$$

$$\tilde{F}_{2a} = +F$$

Note the difference

$$\textcircled{1} -F = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1} \right) - \frac{\partial \mathcal{L}}{\partial x_1}$$

$$-F = \frac{d}{dt} (m_1 \dot{x}_1) - (-kx_1 + kx_{2a})$$

$$\boxed{1a} \quad -F = m_1 \ddot{x}_1 + kx_1 - kx_{2a}$$

$$\textcircled{2} F = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_{2a}} \right) - \frac{\partial \mathcal{L}}{\partial x_{2a}}$$

$$F = \frac{d}{dt} (m_2 \dot{x}_{2a}) - (-kx_{2a} + kx_1)$$

$$\boxed{2a} \quad F = m_2 \ddot{x}_{2a} + kx_{2a} - kx_1$$

Relative Coord's

$$T^* = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 (\dot{x}_1 + \dot{x}_2)^2$$

$$V = \frac{1}{2} k x_{2r}^2$$

$$\mathcal{L} = T^* - V$$

$$= \frac{1}{2} m_1 \dot{x}_1^2 + \dots$$

$$+ \frac{1}{2} m_2 (\dot{x}_1^2 + 2\dot{x}_1 \dot{x}_{2r} + \dot{x}_{2r}^2) + \dots$$

$$- \frac{1}{2} k x_{2r}^2$$

$$\tilde{F}_{1r} = 0$$

$$\tilde{F}_{2r} = +F$$

$$\textcircled{1} 0 = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1} \right) - \frac{\partial \mathcal{L}}{\partial x_1}$$

$$0 = \frac{d}{dt} (m_1 \dot{x}_1 + m_2 \dot{x}_1 + m_2 \dot{x}_{2r})$$

$$\boxed{1r} \quad 0 = (m_1 + m_2) \ddot{x}_1 + m_2 \ddot{x}_{2r}$$

$$\textcircled{2} F = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_{2r}} \right) - \frac{\partial \mathcal{L}}{\partial x_{2r}}$$

$$F = \frac{d}{dt} (m_2 \dot{x}_1 + m_2 \dot{x}_{2r}) + \dots$$

$$- (-kx_{2r})$$

$$\boxed{2r} \quad F = m_2 \ddot{x}_1 + m_2 \ddot{x}_{2r} + kx_{2r}$$

Let's compare!!!

• Write $\boxed{1a}$ in terms of x_{1r} & x_{2r} :

$$-F = m_1 \ddot{x}_1 + kx_1 - k(x_1 + x_{2r})$$

$$\therefore -F = m_1 \ddot{x}_1 - kx_{2r} \quad \boxed{B-1}$$

• Write $\boxed{2a}$ in terms of x_{1r} & x_{2r} :

$$F = m_2 \ddot{x}_1 + m_2 \ddot{x}_{2r} + (kx_1 + kx_{2r}) - kx_1$$

$$\therefore F = m_2 \ddot{x}_1 + m_2 \ddot{x}_{2r} + kx_{2r} \quad \boxed{B-2}$$

• Equation $\boxed{B-2}$ matches

$$\boxed{2r}.$$

→ Now, add $\boxed{B-1}$ and $\boxed{B-2}$

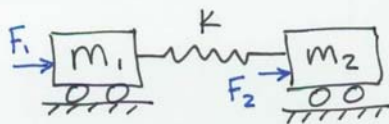
$$0 = (m_1 + m_2) \ddot{x}_1 + m_2 \ddot{x}_{2r}$$

Matches $\boxed{1r}$!

$\boxed{1a}$ & $\boxed{2a}$ can be written as $\boxed{1r}$ & $\boxed{2r}$!

Another Perspective:

- ① Absolute coordinates require Ξ_i ; includes forces (or torques) wrt the inertial reference frame.



$$\begin{aligned}\Xi_{1a} &= F_1 = m_1 \ddot{x}_1 + k(x_1 - x_{2a}) \\ \Xi_{2a} &= F_2 = m_2 \ddot{x}_2 + k(x_{2a} - x_1)\end{aligned}$$

Same derivation as before, but
 $F_1 \leftrightarrow -F$
 $F_2 \leftrightarrow +F$
 $\boxed{1a} \leftrightarrow \boxed{2a}$

- ② To match when using x_{2r} (relative coord) requires:

$$\begin{aligned}\Xi_{1r} &= (m_1 + m_2) \ddot{x}_1 + m_2 \ddot{x}_{2r} \\ \Xi_{2r} &= m_2 (\ddot{x}_1 + \ddot{x}_{2r}) + k x_{2r}\end{aligned}$$

See eqns $\boxed{1r}$ & $\boxed{2r}$

$$\begin{aligned}\therefore \Xi_{1a} + \Xi_{2a} &= \Xi_{1r} = (m_1 \ddot{x}_1 + k(x_1 - x_{2a})) + \dots \\ &\quad (m_2 \ddot{x}_2 + k(x_{2a} - x_1)) \\ &= (m_1 + m_2) \ddot{x}_1 + m_2 \ddot{x}_{2r}\end{aligned}$$

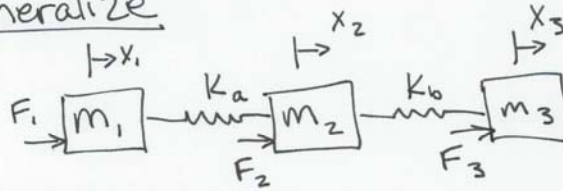
And:

$$\begin{aligned}\Xi_{2a} &= \Xi_{2r} = m_2 \ddot{x}_2 + k(x_{2a} - x_1) \\ &= m_2 (\ddot{x}_1 + \ddot{x}_{2r}) + k x_{2r}\end{aligned}$$

So, $\Xi_{1r} =$ Sum of all forces on the entire system (m_1 AND m_2)

$\Xi_{2r} =$ Forces exerted on m_2

non-conservative forces

Generalize

Ⓐ

$$\Xi_{1a} = F_1 = m_1 \ddot{x}_1 + K_a (x_1 - x_{2a})$$

$$\Xi_{2a} = F_2 = m_2 \ddot{x}_2 + K_a (x_{2a} - x_1) + K_b (x_{2a} - x_3)$$

$$\Xi_{3a} = F_3 = m_3 \ddot{x}_3 + K_b (x_3 - x_{2a})$$

Ⓑ for Relative x_{2r} & x_{3r} ,

$$x_{1r} = x_{1a} = x_1$$

$$x_{2r} = x_{2a} - x_1$$

$$x_{3r} = x_{3a} - x_{2a}$$

$$T^* = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 (\dot{x}_1 + \dot{x}_{2r})^2 + \frac{1}{2} m_3 (\dot{x}_1 + \dot{x}_{2r} + \dot{x}_{3r})^2$$

$$V = \frac{1}{2} K_a x_{2r}^2 + \frac{1}{2} K_b x_{3r}^2$$

$$\mathcal{L} = T^* - V$$

$$\Xi_{1r} = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1} \right) - \frac{\partial \mathcal{L}}{\partial x_1} = \frac{d}{dt} (m_1 \dot{x}_1 + m_2 (\dot{x}_1 + \dot{x}_{2r}) + m_3 (\dot{x}_1 + \dot{x}_{2r} + \dot{x}_{3r}))$$

$$\Xi_{1r} = m_1 \ddot{x}_1 + m_2 (\ddot{x}_1 + \ddot{x}_{2r}) + m_3 (\ddot{x}_1 + \ddot{x}_2 + \ddot{x}_3) = \Xi_{1a} + \Xi_{2a} + \Xi_{3a} = F_1 + F_2 + F_3$$

$$\Xi_{2r} = \frac{d}{dt} (m_2 (\dot{x}_1 + \dot{x}_{2r}) + m_3 (\dot{x}_1 + \dot{x}_2 + \dot{x}_3)) + K_a x_{2r}$$

$$\Xi_{2r} = m_2 (\ddot{x}_1 + \ddot{x}_{2r}) + m_3 (\ddot{x}_1 + \ddot{x}_2 + \ddot{x}_3) + K_a x_{2r} = \Xi_{2a} (F_2 + F_3)$$

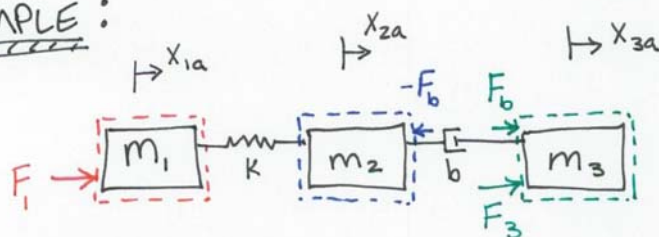
$$\Xi_{3r} = \frac{d}{dt} (m_3 (\dot{x}_1 + \dot{x}_2 + \dot{x}_3)) + K_b x_{3r}$$

$$\Xi_{3r} = m_3 (\ddot{x}_1 + \ddot{x}_2 + \ddot{x}_3) + K_b x_{3r} = \Xi_{3a} = F_3$$

Conclusions

- When using only ABSOLUTE coordinates, each $\tilde{\Xi}_i$ is simply the sum of all non-conservative forces (or torques) acting on the mass (or inertia) associated w/ the i^{th} Generalized Coordinate.

EXAMPLE:



$$\tilde{\Xi}_{1a} = F_1 \quad \leftarrow \text{Do not include conservative forces, like the spring}$$

$$\tilde{\Xi}_{2a} = -F_b \quad \leftarrow \text{Where } F_b = -b(\dot{x}_{3a} - \dot{x}_{2a})$$

$$\tilde{\Xi}_{3a} = F_b + F_3 \quad \leftarrow F_3 \text{ is some "external force" applied to } m_3$$

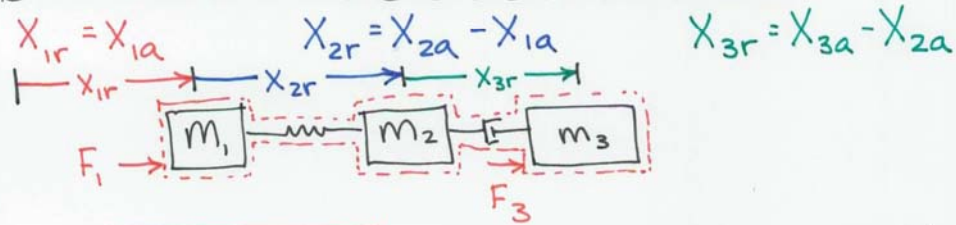
- When using 1 Absolute & N-1 Relative coordinates ($x_{nr} = x_{na} - x_{(n-1)a}$), ($x_{1r} = x_{1a}$),

Coordinate n is RELATIVE to coordinate n-1 here!

$$\tilde{\Xi}_{nr} = \sum_{i=n}^N \tilde{\Xi}_{ia}$$

$$\begin{cases} \tilde{\Xi}_{3r} = \tilde{\Xi}_{3a} \\ \tilde{\Xi}_{2r} = \tilde{\Xi}_{2a} + \tilde{\Xi}_{3a} \\ \tilde{\Xi}_{1r} = \tilde{\Xi}_{1a} + \tilde{\Xi}_{2a} + \tilde{\Xi}_{3a} \end{cases} \quad \text{e.g., for example above.}$$

SAME EXAMPLE, Relative Coords:



$$\tilde{\Xi}_{1r} = F_1 + F_3$$

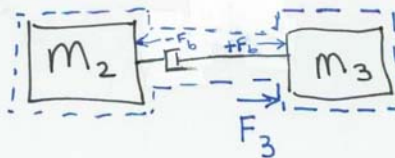
$$\uparrow \tilde{\Xi}_{1a} + \tilde{\Xi}_{2a} + \tilde{\Xi}_{3a}$$

includes ALL external forces, acting on any of the masses.

Note: Internal forces (here " F_b " & " $-F_b$ ") will always cancel out (by definition)

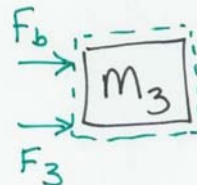
$$\tilde{\Xi}_{2r} = F_3$$

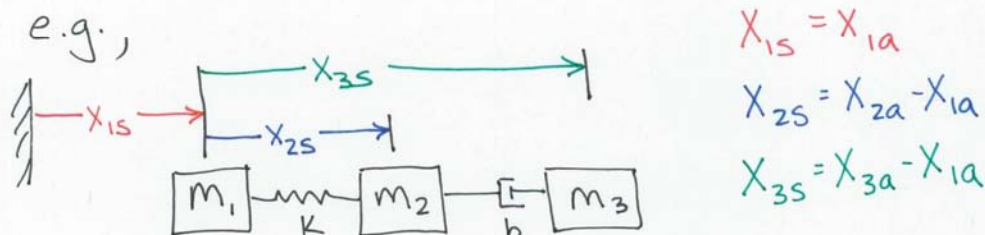
$$\uparrow \tilde{\Xi}_{2a} + \tilde{\Xi}_{3a}$$



$$\tilde{\Xi}_{3r} = F_b + F_3$$

$$\uparrow \tilde{\Xi}_{3a}$$



Alternate G.C. Definitions are Possible...

Here,

$$T^* = \frac{1}{2} m_1 \dot{x}_{1s}^2 + \frac{1}{2} m_2 (\dot{x}_{1s} + \dot{x}_{2s})^2 + \frac{1}{2} m_3 (\dot{x}_{1s} + \dot{x}_{3s})^2$$

$$V = \frac{1}{2} K x_{2s}^2$$

$$\mathcal{L} = T^* - V$$

$$\tilde{\tau}_{1s} = m_1 \ddot{x}_{1s} + m_2 (\ddot{x}_{1s} + \ddot{x}_{2s}) + m_3 (\ddot{x}_{1s} + \ddot{x}_{3s})$$

$$\tilde{\tau}_{1s} = \tilde{\tau}_{1a} + \tilde{\tau}_{2a} + \tilde{\tau}_{3a} \leftarrow = F_1 + F_3$$

$$\tilde{\tau}_{2s} = m_2 (\ddot{x}_{1s} + \ddot{x}_{2s}) + K x_{2s}$$

$$\tilde{\tau}_{2s} = \tilde{\tau}_{2a} \leftarrow = -F_b$$

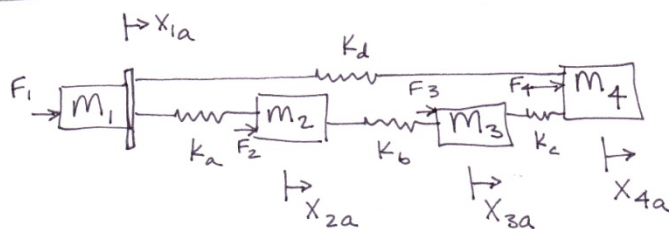
$$\tilde{\tau}_{3s} = m_3 (\ddot{x}_{1s} + \ddot{x}_{3s})$$

$$\tilde{\tau}_{3s} = \tilde{\tau}_{3a} \leftarrow = F_b + F_3$$



Both x_{2s} & x_{3s}
are RELATIVE to
 x_{1s} here...

Formal Technique to Find $\tilde{z}_i$



Let's (arbitrarily) define our Generalized Coordinates as:

$$X_{1b} = X_{1a}$$

$$X_{2b} = X_{2a} - X_{1a}$$

$$X_{3b} = X_{3a} - X_{2a}$$

$$X_{4b} = X_{4a} - X_{1a}$$



Write
in matrix
form

$$X_{1a} = X_{1b}$$

$$X_{2a} = X_{1b} + X_{2b}$$

$$X_{3a} = X_{1b} + X_{2b} + X_{3b}$$

$$X_{4a} = X_{1b} + X_{4b}$$

Any terms in T^* or V involving X_{3a} (and its derivatives) must be represented using " $X_{1b} + X_{2b} + X_{3b}$ " (" ").

|| So, these terms show up in the ξ_1, ξ_2, ξ_3 EOMs.

Each EOM in this "b" GC definition is the sum of one or more EOMs in the absolute "a" GC system, related just as mentioned above... e.g. $\tilde{z}_{3a}$ must appear as an element of the expressions for $\tilde{z}_{1b}, \tilde{z}_{2b}, \tilde{z}_{3b}$.

$$\begin{bmatrix} X_{1a} \\ X_{2a} \\ X_{3a} \\ X_{4a} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_{1b} \\ X_{2b} \\ X_{3b} \\ X_{4b} \end{bmatrix} \quad \text{and so...} \quad \begin{bmatrix} \tilde{z}_{1b} \\ \tilde{z}_{2b} \\ \tilde{z}_{3b} \\ \tilde{z}_{4b} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{z}_{1a} \\ \tilde{z}_{2a} \\ \tilde{z}_{3a} \\ \tilde{z}_{4a} \end{bmatrix}$$

$\boxed{X_a = M X_b}$ or $\boxed{\xi_a = M \xi_b}$
 $\boxed{\tilde{z}_b = M^T \tilde{z}_a}$