Lagrangian EOM – part 3

Lecture 14

- Absolute vs Relative DOFs: “Big Xi”, Ξ_i (non-conservative forces)
- State space (i.e., matrix) form for EOM
- State space control:
  - -Controllability
  - -Pole placement
  - -LQR: Linear Quadratic Regulator

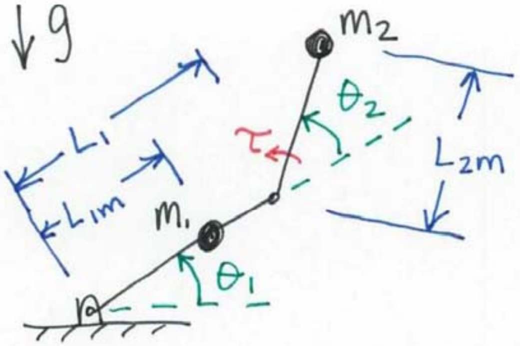
Three handouts (H.O.’s) :

1. Lecture 13/14 “supplement”. Gives general rules for determining Ξ_i (Xi). (8 pages)
2. Acrobot example. (2-page)
3. Segway example and introduction to state space. (10-page)

Acrobot Handout – Relative DOF

Acrobot, $\boxed{\text{relative}}$ "ξ_2" coordinate

Ⓡ



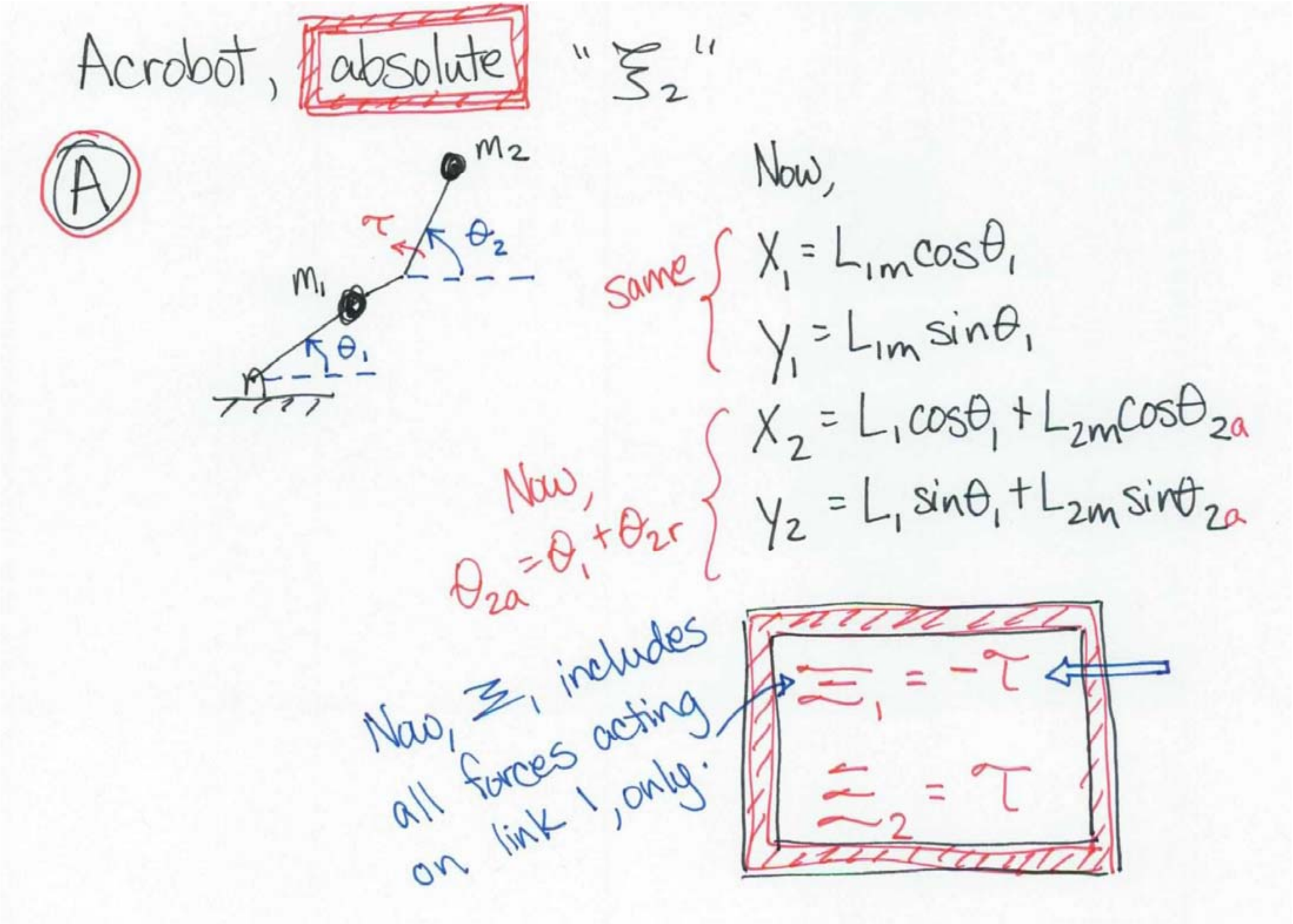
$$x_1 = L_{1m}\cos\theta_1$$
$$y_1 = L_{1m}\sin\theta_1$$
$$x_2 = L_1\cos\theta_1 + L_{2m}\cos(\theta_1+\theta_{2r})$$
$$y_2 = L_1\sin\theta_1 + L_{2m}\sin(\theta_1+\theta_{2r})$$

$$T^* = \frac{1}{2}m_1(\dot{x}_1^2+\dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2+\dot{y}_2^2)$$
$$V = m_1 g y_1 + m_2 g y_2$$
$$\mathcal{L} = T^* - V$$

$$\boxed{\begin{aligned}\tilde{\Xi}_1 &= 0\\ \tilde{\Xi}_2 &= \tau\end{aligned}}$$

Here, Ξ_1 only includes external torques on the *entire* system.

Acrobot Handout – Absolute DOF



Acrobot, $\boxed{\text{absolute}}$ "ξ_2"

Now,

same $\begin{cases} X_1 = L_{1m}\cos\theta_1 \\ Y_1 = L_{1m}\sin\theta_1 \end{cases}$

$$X_2 = L_1\cos\theta_1 + L_{2m}\cos\theta_{2a}$$

$$Y_2 = L_1\sin\theta_1 + L_{2m}\sin\theta_{2a}$$

Now, $\theta_{2a} = \theta_1 + \theta_{2r}$

Now, Ξ_1 includes all forces acting on link 1, only.

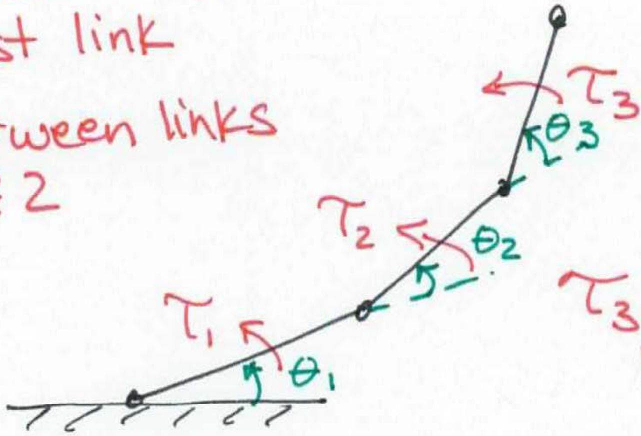
$$\boxed{\begin{aligned} \Xi_1 &= -\tau \\ \Xi_2 &= \tau \end{aligned}}$$

Acrobot Handout – Generalize Rel.

* Multi-link, relative Ξ's

τ_1 between base & first link

τ_2 between links 1 & 2

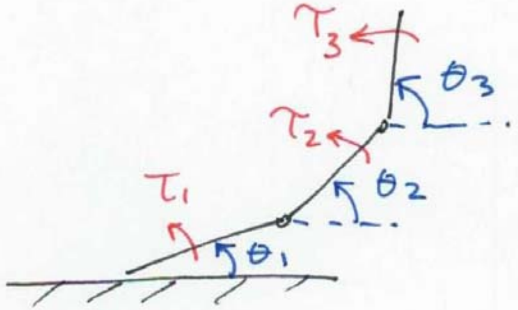


τ_3 between links 2 & 3

$$\boxed{\begin{aligned} \Xi_1 &= \tau_1 \\ \Xi_2 &= \tau_2 \\ \Xi_3 &= \tau_3 \end{aligned}}$$

Acrobot Handout – Generalize Abs.

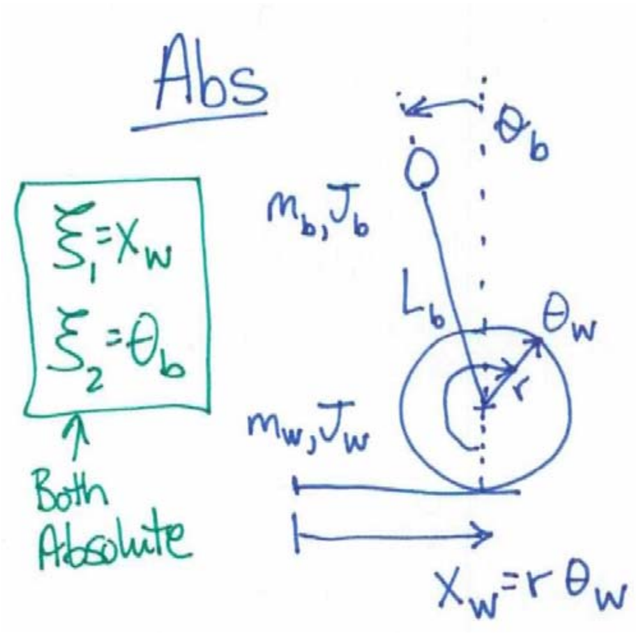
* Multi-link, **absolute** ξ's



$$\boxed{\begin{aligned} \tilde{\Xi}_1 &= \tau_1 - \tau_2 \\ \tilde{\Xi}_2 &= \tau_2 - \tau_3 \\ \tilde{\Xi}_3 &= \tau_3 \end{aligned}}$$

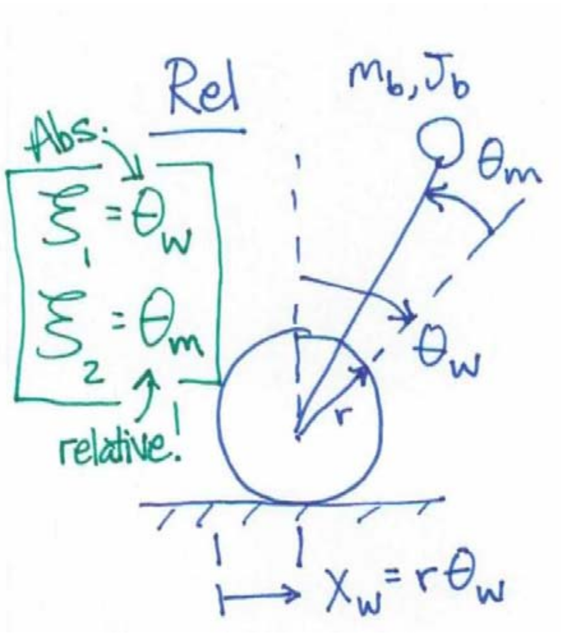
- Link one "sees" $+\tau_1$ & $-\tau_2$.
- Link two "sees" $+\tau_2$ (exerted against link one) & $-\tau_3$ (equal & opposite to $+\tau_3$ torque on link three)
- Link three "sees" only τ_3. No other torque is directly applied...

Recall: “Segway” style inverted pendulum



$$\boxed{\Xi_{1a} = -\tau / r}$$

$$\boxed{\Xi_{2a} = \tau}$$



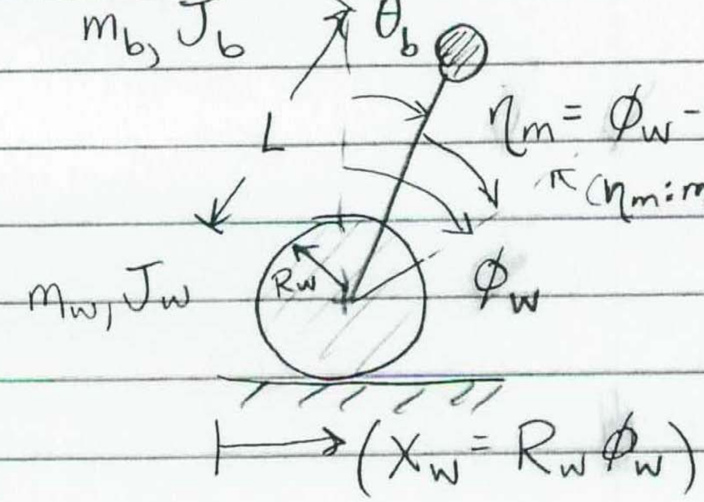
$$\boxed{\Xi_{1r} = 0}$$

$$\boxed{\Xi_{2r} = \tau}$$

Segway Handout

Segway example:

(neglects $n^2 J_m \ldots$)



$\eta_m = \phi_w - \theta_b$

(η_m: motor angle)

$\xi_1 = \phi_w =$ "ξ_{ϕ_w}"

$\xi_2 = \theta_b =$ "ξ_{θ_b}"

$+\tilde{\tau}_m \rightarrow$ tends to $\uparrow \theta_b$, $\downarrow \phi_w$

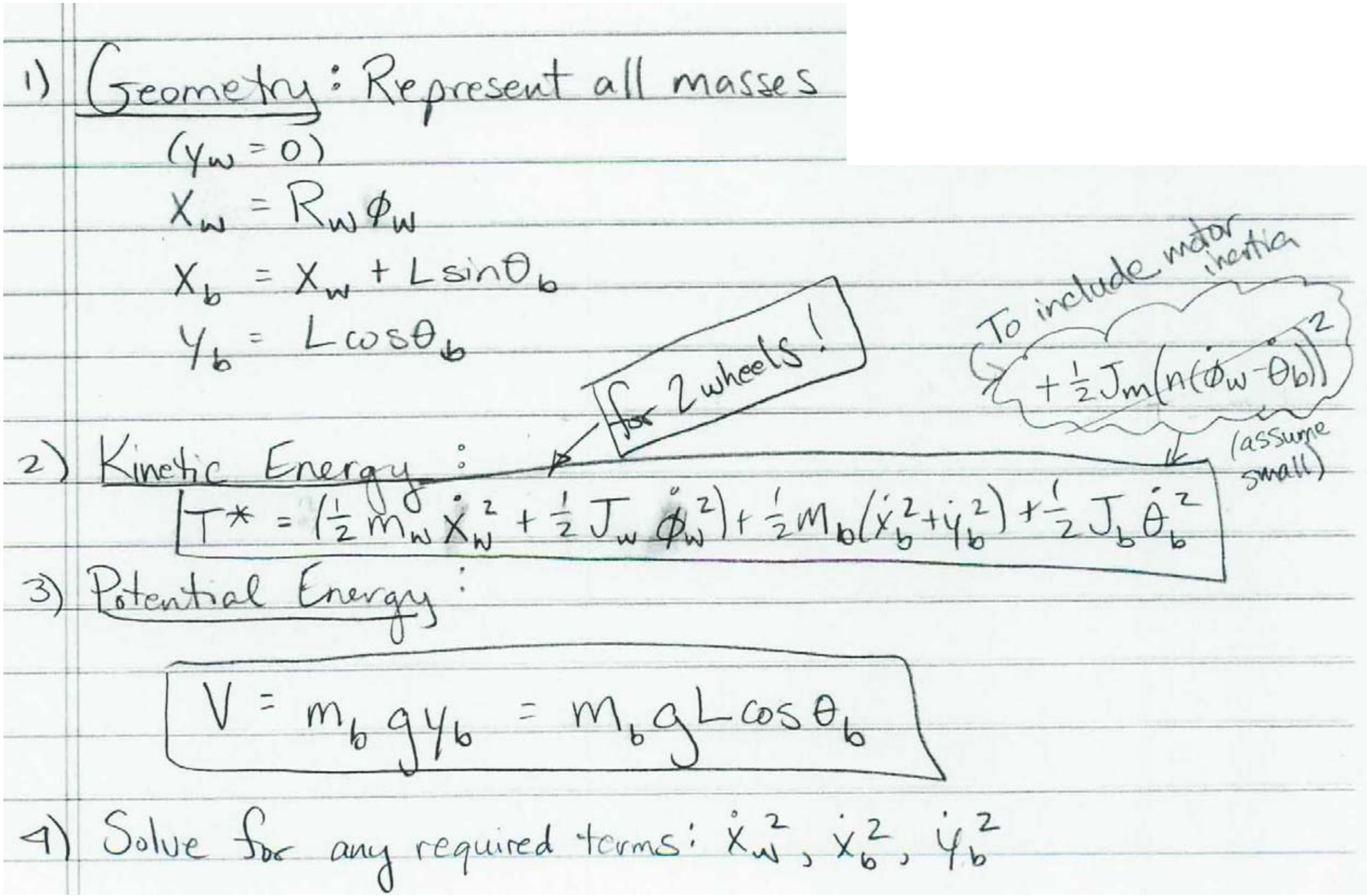
b due to back EMF, mech. damping

wheel: $\tilde{\Xi}_1 = \pm \tilde{\tau}_m - b(\dot{\phi}_w - \dot{\theta}_b)$

body: $\tilde{\Xi}_2 = \mp \tilde{\tau}_m + b(\dot{\phi}_w - \dot{\theta}_b)$

sign depends on (arbitrary) Lego wiring!

Segway Handout



1) Geometry: Represent all masses

$(y_w = 0)$

$x_w = R_w \phi_w$

$x_b = x_w + L\sin\theta_b$

$y_b = L\cos\theta_b$

2) Kinetic Energy:

$$\boxed{T^* = \left(\frac{1}{2} m_w \dot{x}_w^2 + \frac{1}{2} J_w \dot{\phi}_w^2\right) + \frac{1}{2} m_b\left(\dot{x}_b^2 + \dot{y}_b^2\right) + \frac{1}{2} J_b \dot{\theta}_b^2}$$

for 2 wheels!

To include motor inertia

$$+\frac{1}{2} J_m \left(n(\dot{\phi}_w - \dot{\theta}_b)\right)^2$$

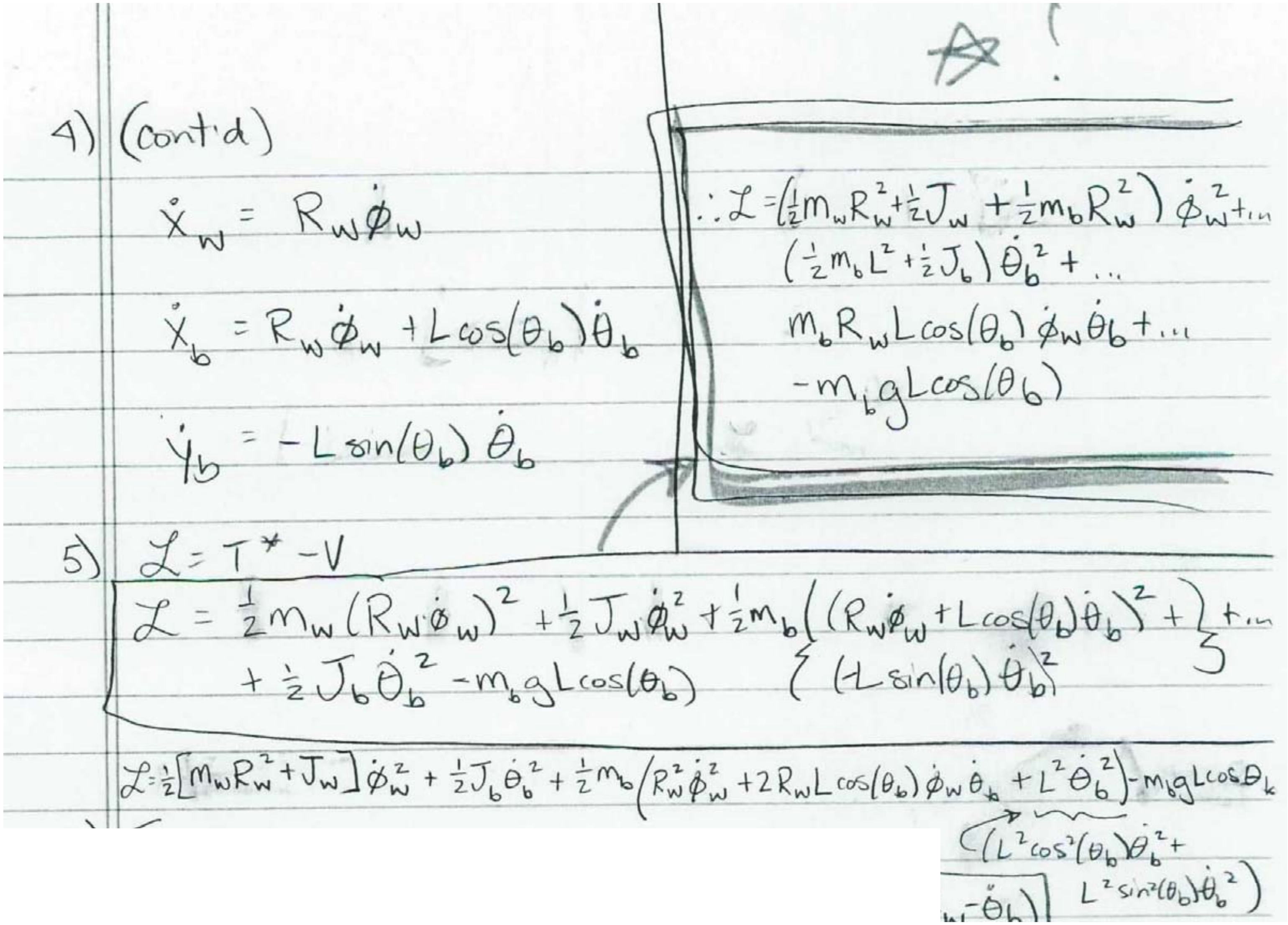
(assume small)

3) Potential Energy:

$$\boxed{V = m_b g y_b = m_b g L\cos\theta_b}$$

4) Solve for any required terms: $\dot{x}_w^2$, $\dot{x}_b^2$, $\dot{y}_b^2$

Segway Handout



4) (cont'd)

$$\dot{x}_w = R_w \dot{\phi}_w$$

$$\dot{x}_b = R_w \dot{\phi}_w + L\cos(\theta_b)\dot{\theta}_b$$

$$\dot{y}_b = -L\sin(\theta_b)\,\dot{\theta}_b$$

$$\boxed{\begin{aligned} \therefore \mathcal{L} = &\left(\tfrac{1}{2}m_w R_w^2 + \tfrac{1}{2}J_w + \tfrac{1}{2}m_b R_w^2\right)\dot{\phi}_w^2 + \ldots \\ &\left(\tfrac{1}{2}m_b L^2 + \tfrac{1}{2}J_b\right)\dot{\theta}_b^2 + \ldots \\ &m_b R_w L\cos(\theta_b)\,\dot{\phi}_w \dot{\theta}_b + \ldots \\ &-m_b g L\cos(\theta_b) \end{aligned}}$$

5) $\mathcal{L} = T^* - V$

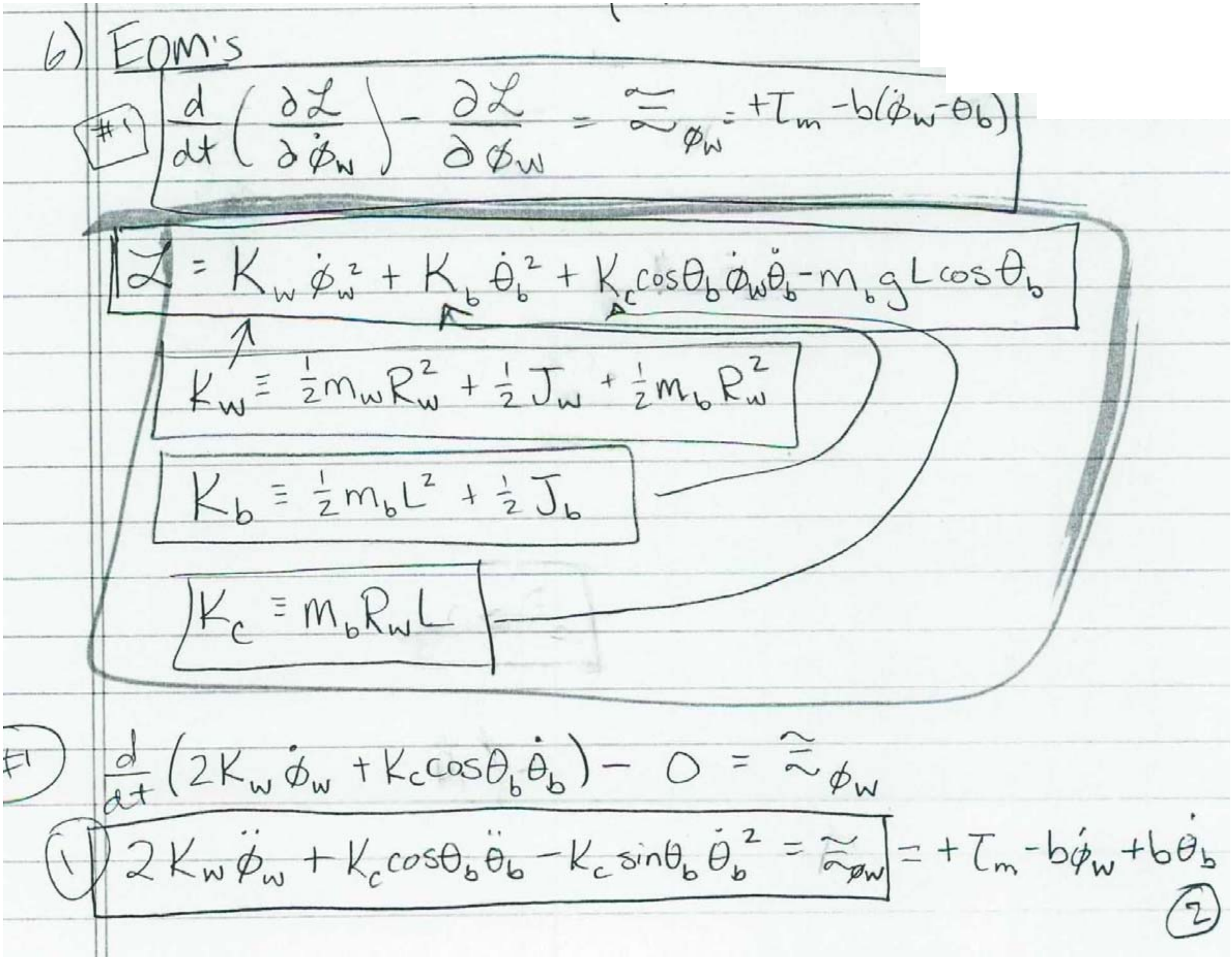
$$\boxed{\begin{aligned} \mathcal{L} = &\tfrac{1}{2}m_w (R_w \dot{\phi}_w)^2 + \tfrac{1}{2}J_w \dot{\phi}_w^2 + \tfrac{1}{2}m_b\Big((R_w\dot{\phi}_w + L\cos(\theta_b)\dot{\theta}_b)^2 + \Big\} + \ldots \\ &+ \tfrac{1}{2}J_b \dot{\theta}_b^2 - m_b g L\cos(\theta_b) \qquad \Big\{ (-L\sin(\theta_b)\dot{\theta}_b)^2 \end{aligned}}$$

$$\mathcal{L} = \tfrac{1}{2}\left[m_w R_w^2 + J_w\right]\dot{\phi}_w^2 + \tfrac{1}{2}J_b\dot{\theta}_b^2 + \tfrac{1}{2}m_b\left(R_w^2\dot{\phi}_w^2 + 2R_w L\cos(\theta_b)\,\dot{\phi}_w\dot{\theta}_b + \underbrace{L^2\dot{\theta}_b^2}\right) - m_b g L\cos\theta_b$$

$$\left(L^2\cos^2(\theta_b)\dot{\theta}_b^2 + L^2\sin^2(\theta_b)\dot{\theta}_b^2\right)$$

$$-\ddot{\theta}_b)$$

Segway Handout



6) EOM's

$$\#1 \quad \boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_w}\right) - \frac{\partial \mathcal{L}}{\partial \phi_w} = \tilde{\Xi}_{\phi_w} = +\tau_m - b(\dot{\phi}_w - \theta_b)}$$

$$\boxed{\mathcal{L} = K_w \dot{\phi}_w^2 + K_b \dot{\theta}_b^2 + K_c \cos\theta_b \dot{\phi}_w \ddot{\theta}_b - m_b g L \cos\theta_b}$$

$$\boxed{K_w \equiv \tfrac{1}{2} m_w R_w^2 + \tfrac{1}{2} J_w + \tfrac{1}{2} m_b R_w^2}$$

$$\boxed{K_b \equiv \tfrac{1}{2} m_b L^2 + \tfrac{1}{2} J_b}$$

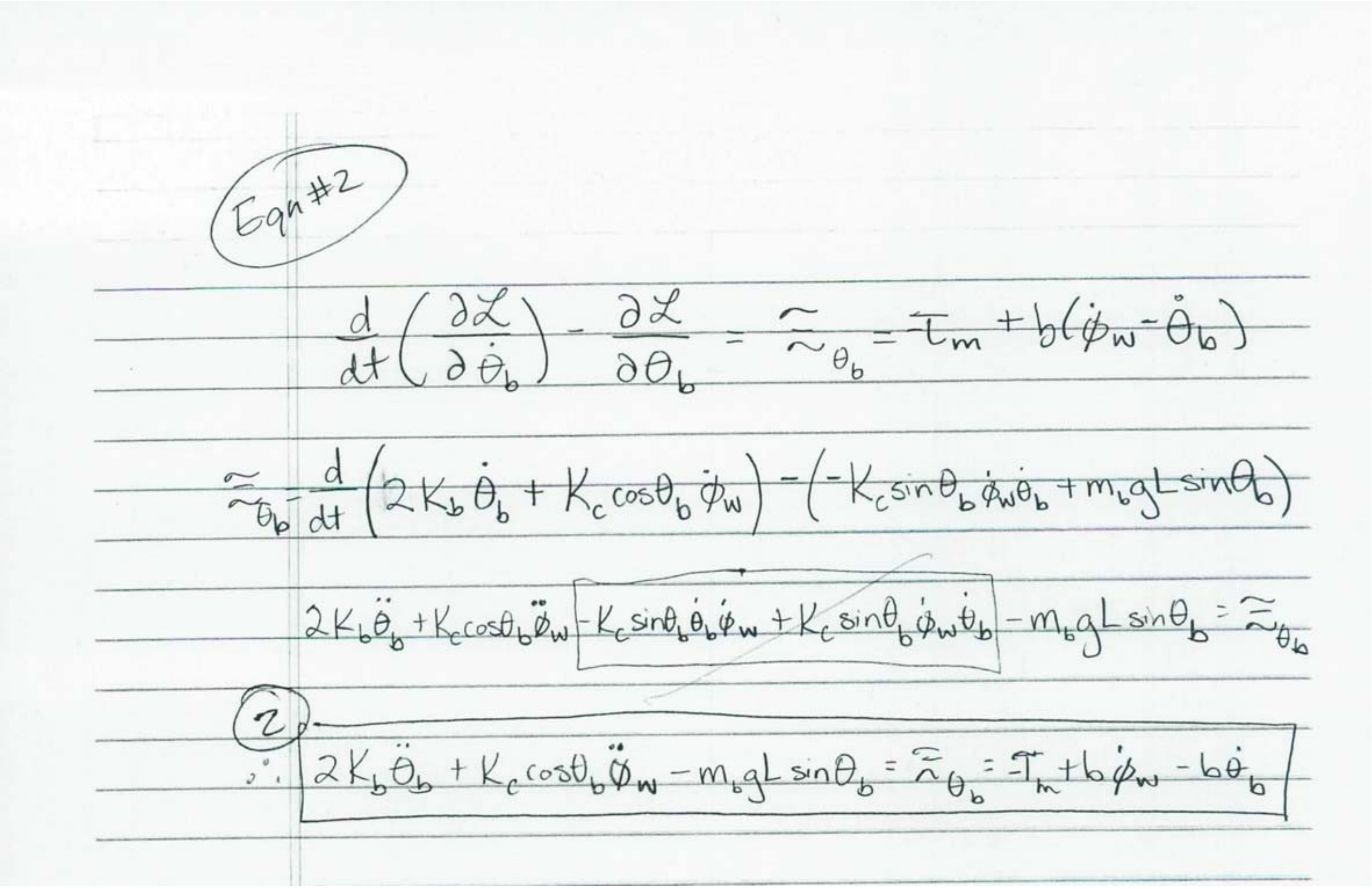
$$\boxed{K_c \equiv m_b R_w L}$$

$$\#1 \quad \frac{d}{dt}\left(2K_w \dot{\phi}_w + K_c \cos\theta_b \dot{\theta}_b\right) - 0 = \tilde{\Xi}_{\phi_w}$$

$$① \quad \boxed{2K_w \ddot{\phi}_w + K_c \cos\theta_b \ddot{\theta}_b - K_c \sin\theta_b \dot{\theta}_b^2 = \tilde{\Xi}_{\phi_w}} = +\tau_m - b\dot{\phi}_w + b\dot{\theta}_b$$

②

Segway Handout



Eqn #2

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_b}\right) - \frac{\partial \mathcal{L}}{\partial \theta_b} = \tilde{\tau}_{\theta_b} = -\tau_m + b(\dot{\phi}_w - \dot{\theta}_b)$$

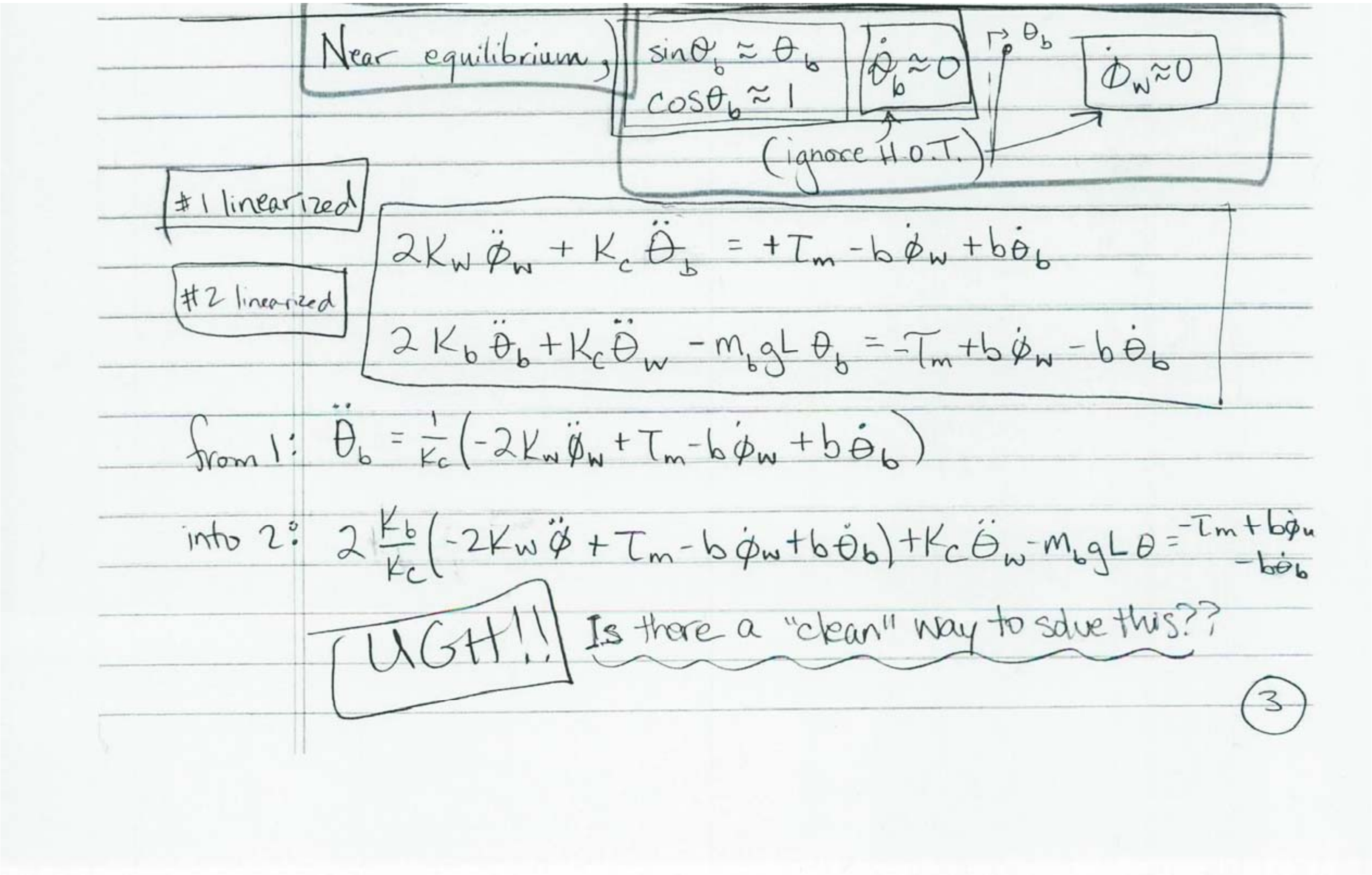
$$\tilde{\tau}_{\theta_b} = \frac{d}{dt}\left(2K_b\dot{\theta}_b + K_c\cos\theta_b\dot{\phi}_w\right) - \left(-K_c\sin\theta_b\dot{\phi}_w\dot{\theta}_b + m_b g L\sin\theta_b\right)$$

$$2K_b\ddot{\theta}_b + K_c\cos\theta_b\ddot{\phi}_w \boxed{\cancel{-K_c\sin\theta_b\dot{\theta}_b\dot{\phi}_w + K_c\sin\theta_b\dot{\phi}_w\dot{\theta}_b}} - m_b g L\sin\theta_b = \tilde{\tau}_{\theta_b}$$

②

$$\therefore \boxed{2K_b\ddot{\theta}_b + K_c\cos\theta_b\ddot{\phi}_w - m_b g L\sin\theta_b = \tilde{\tau}_{\theta_b} = -\tau_m + b\dot{\phi}_w - b\dot{\theta}_b}$$

Segway Handout



Near equilibrium, $\sin\theta_b \approx \theta_b$ $\cos\theta_b \approx 1$ $\dot{\theta}_b \approx 0$ $\dot{\phi}_w \approx 0$

θ_b

(ignore H.O.T.)

\#1 linearized

\#2 linearized

$$2K_w\ddot{\phi}_w + K_c\ddot{\theta}_b = +\tau_m - b\dot{\phi}_w + b\dot{\theta}_b$$

$$2K_b\ddot{\theta}_b + K_c\ddot{\theta}_w - m_b g L\,\theta_b = -\tau_m + b\dot{\phi}_w - b\dot{\theta}_b$$

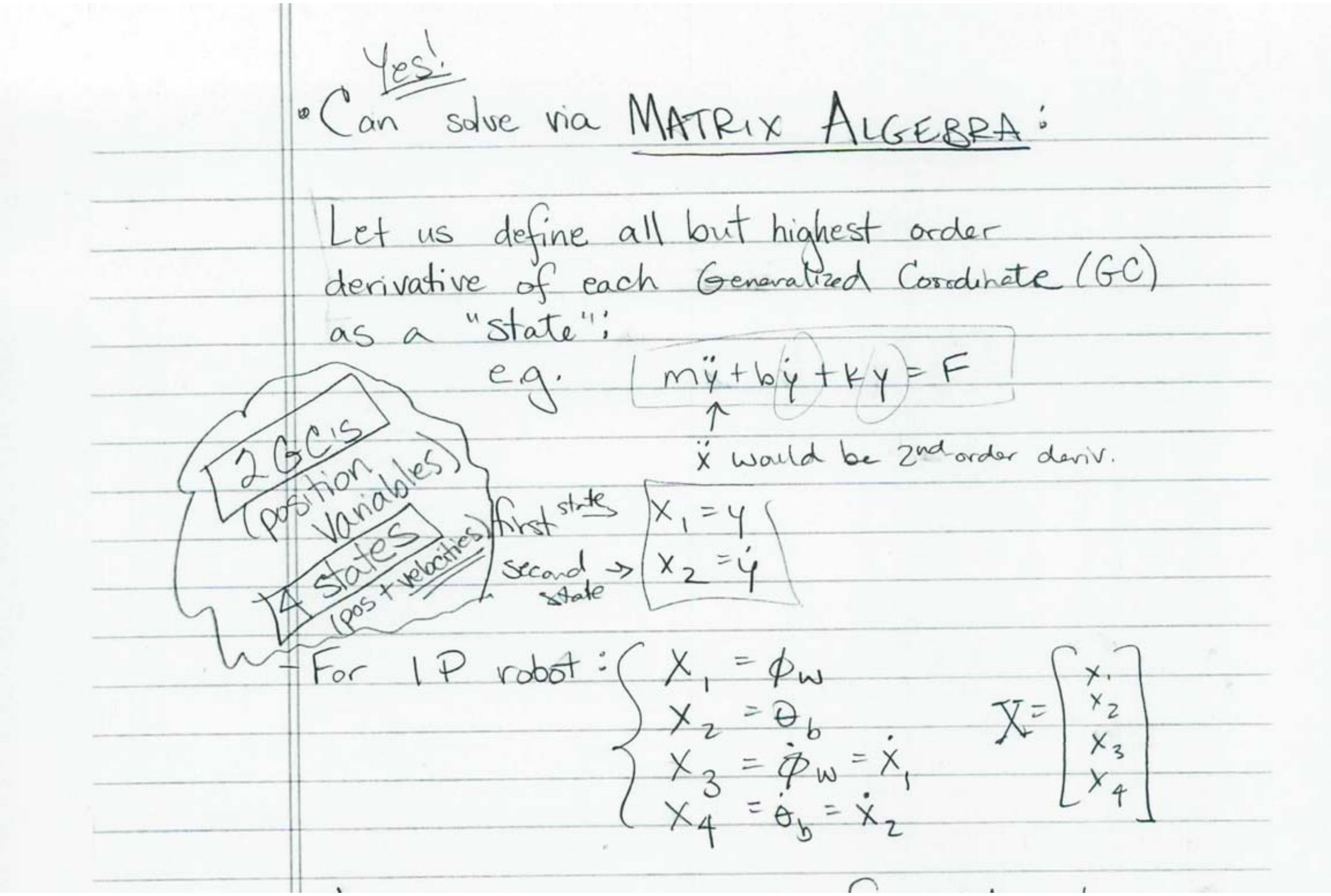
from 1: $\ddot{\theta}_b = \frac{1}{K_c}\left(-2K_w\ddot{\phi}_w + \tau_m - b\dot{\phi}_w + b\dot{\theta}_b\right)$

into 2: $2\frac{K_b}{K_c}\left(-2K_w\ddot{\phi} + \tau_m - b\dot{\phi}_w + b\dot{\theta}_b\right) + K_c\ddot{\theta}_w - m_b g L\theta = -\tau_m + b\dot{\phi}_w - b\dot{\theta}_b$

UGH!! Is there a "clean" way to solve this??

③

Segway Handout



Yes!

- Can solve via MATRIX ALGEBRA:

Let us define all but highest order derivative of each Generalized Coordinate (GC) as a "state";

e.g. $\boxed{m\ddot{y}+b\dot{y}+ky=F}$

$\uparrow$

$\ddot{x}$ would be 2nd order deriv.

2 GC's (position variables)

4 states (pos + velocities)

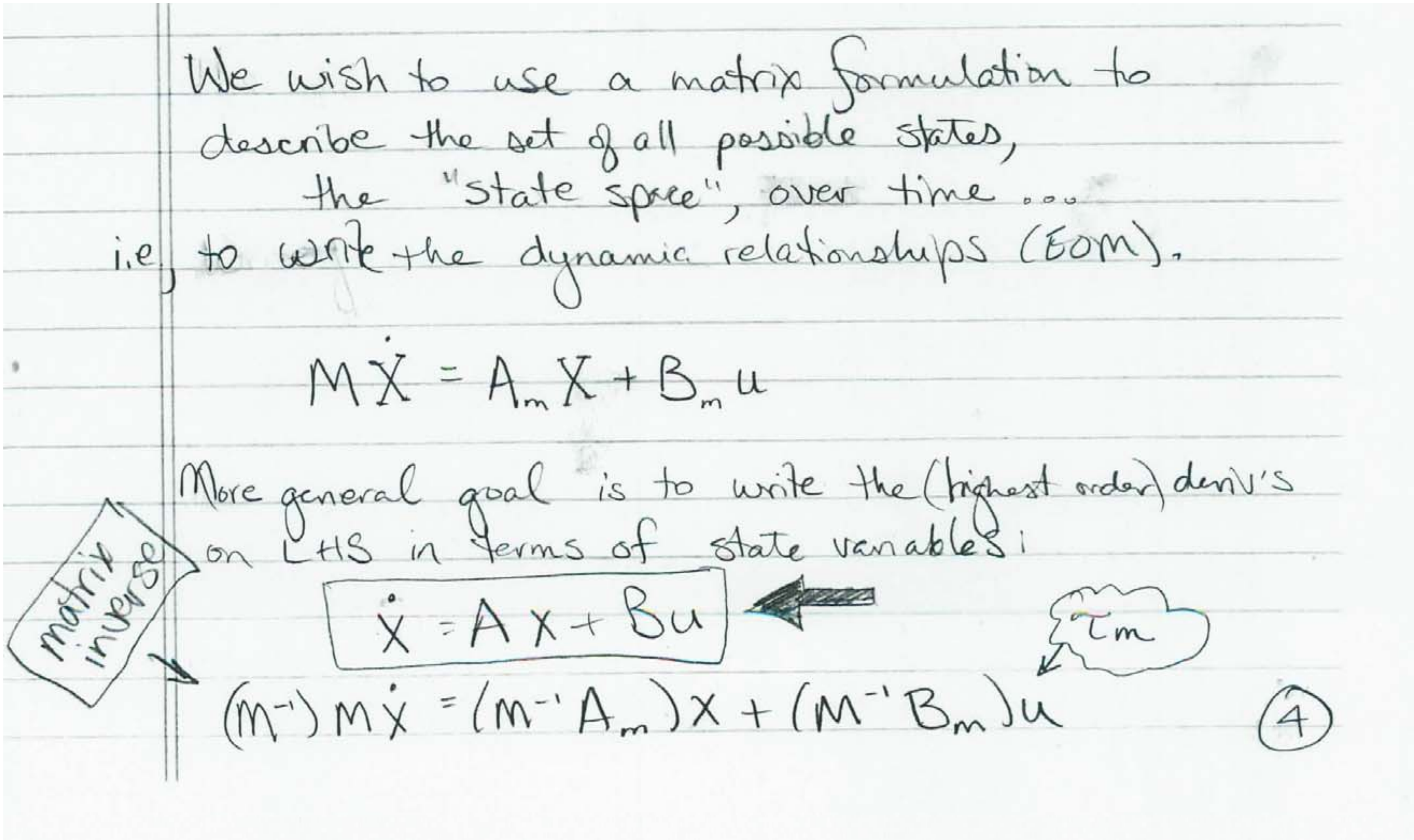
first state $x_1 = y$

second state $\rightarrow$ $x_2 = \dot{y}$

- For IP robot:

$$\begin{cases} x_1 = \phi_w \\ x_2 = \theta_b \\ x_3 = \dot{\phi}_w = \dot{x}_1 \\ x_4 = \dot{\theta}_b = \dot{x}_2 \end{cases} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Segway Handout



We wish to use a matrix formulation to describe the set of all possible states, the "state space", over time ...

i.e., to write the dynamic relationships (EOM).

$$M\dot{X} = A_m X + B_m u$$

More general goal is to write the (highest order) deriv's on LHS in terms of state variables:

$$\boxed{\dot{X} = A X + B u} \longleftarrow$$

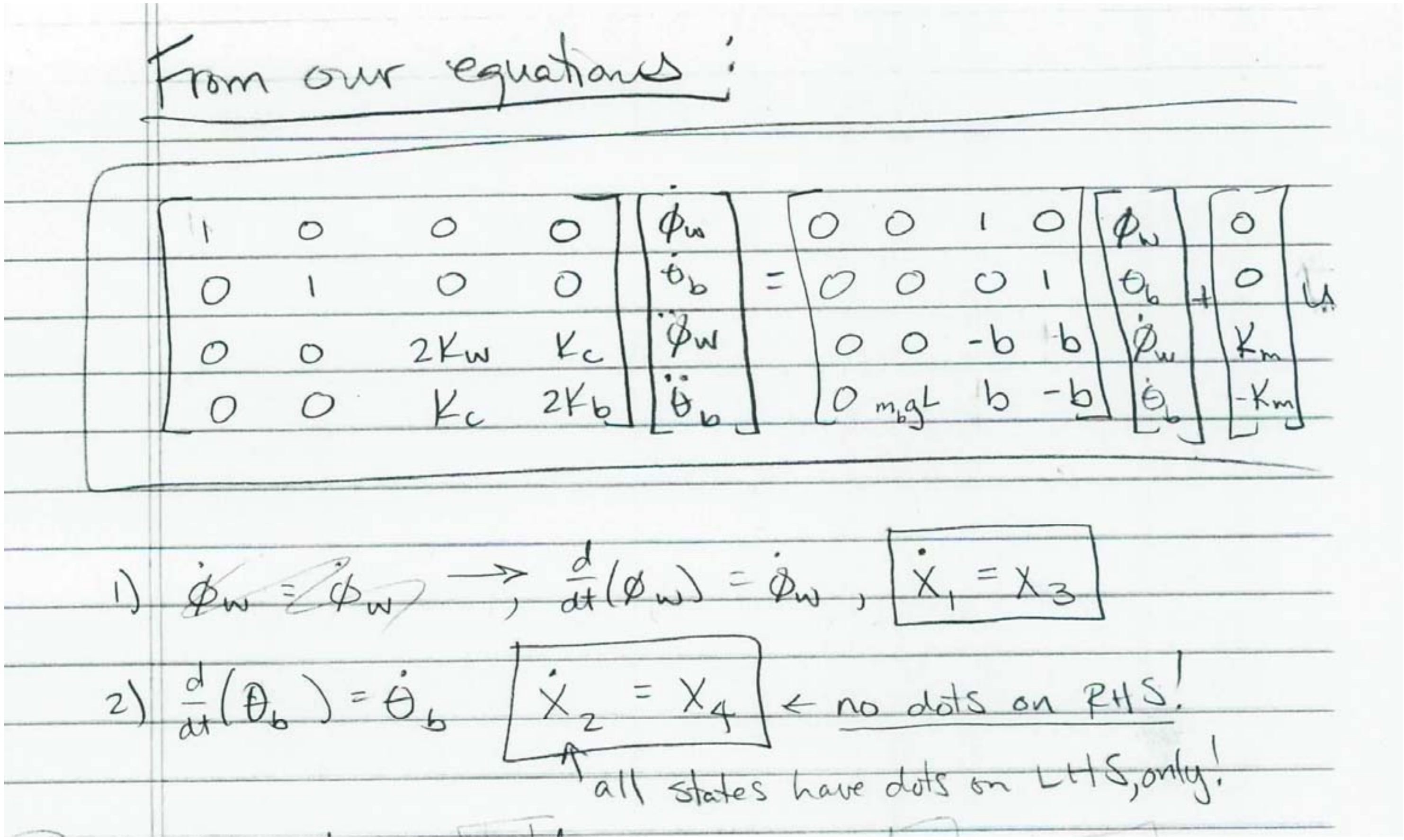
matrix inverse

τ_m

$$(M^{-1}) M\dot{x} = (M^{-1}A_m)x + (M^{-1}B_m)u$$

④

Segway Handout



From our equations:

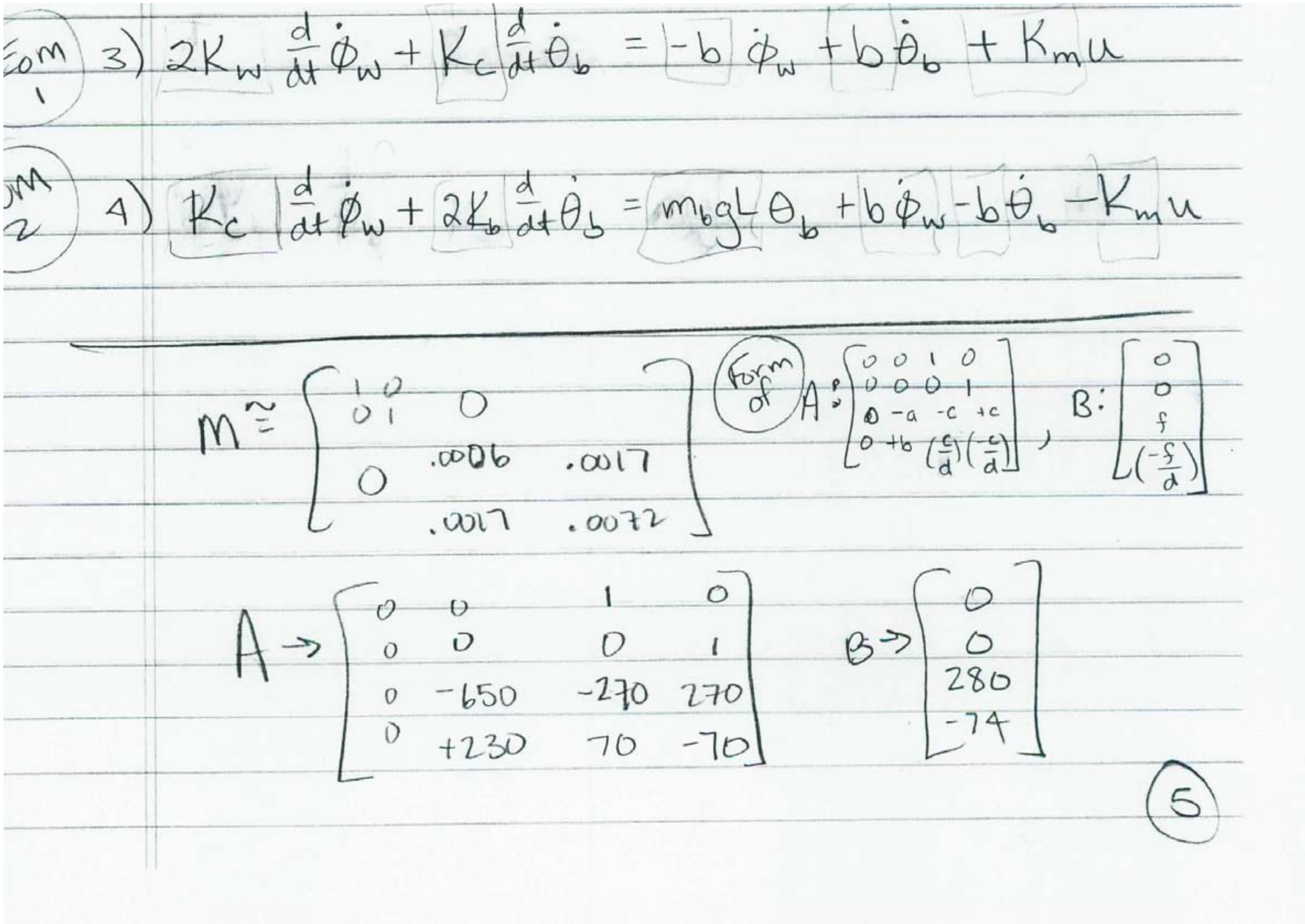
$$\boxed{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2K_w & K_c \\ 0 & 0 & K_c & 2K_b \end{bmatrix} \begin{bmatrix} \dot{\phi}_w \\ \dot{\theta}_b \\ \ddot{\phi}_w \\ \ddot{\theta}_b \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -b & b \\ 0 & m_b g L & b & -b \end{bmatrix} \begin{bmatrix} \phi_w \\ \theta_b \\ \dot{\phi}_w \\ \dot{\theta}_b \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ K_m \\ -K_m \end{bmatrix} u}$$

1) $\cancel{\dot{\phi}_w = \phi_w} \longrightarrow, \frac{d}{dt}(\phi_w) = \dot{\phi}_w$, $\boxed{\dot{x}_1 = x_3}$

2) $\frac{d}{dt}(\theta_b) = \dot{\theta}_b$ $\boxed{\dot{x}_2 = x_4}$ ← no dots on RHS!

↑ all states have dots on LHS, only!

Segway Handout



Eom 1 3) $2K_w \frac{d}{dt}\dot{\phi}_w + K_c \frac{d}{dt}\dot{\theta}_b = -b\,\dot{\phi}_w + b\,\dot{\theta}_b + K_m u$

M 2 4) $K_c \frac{d}{dt}\dot{\phi}_w + 2K_b \frac{d}{dt}\dot{\theta}_b = m_b g L\,\theta_b + b\,\dot{\phi}_w - b\,\dot{\theta}_b - K_m u$

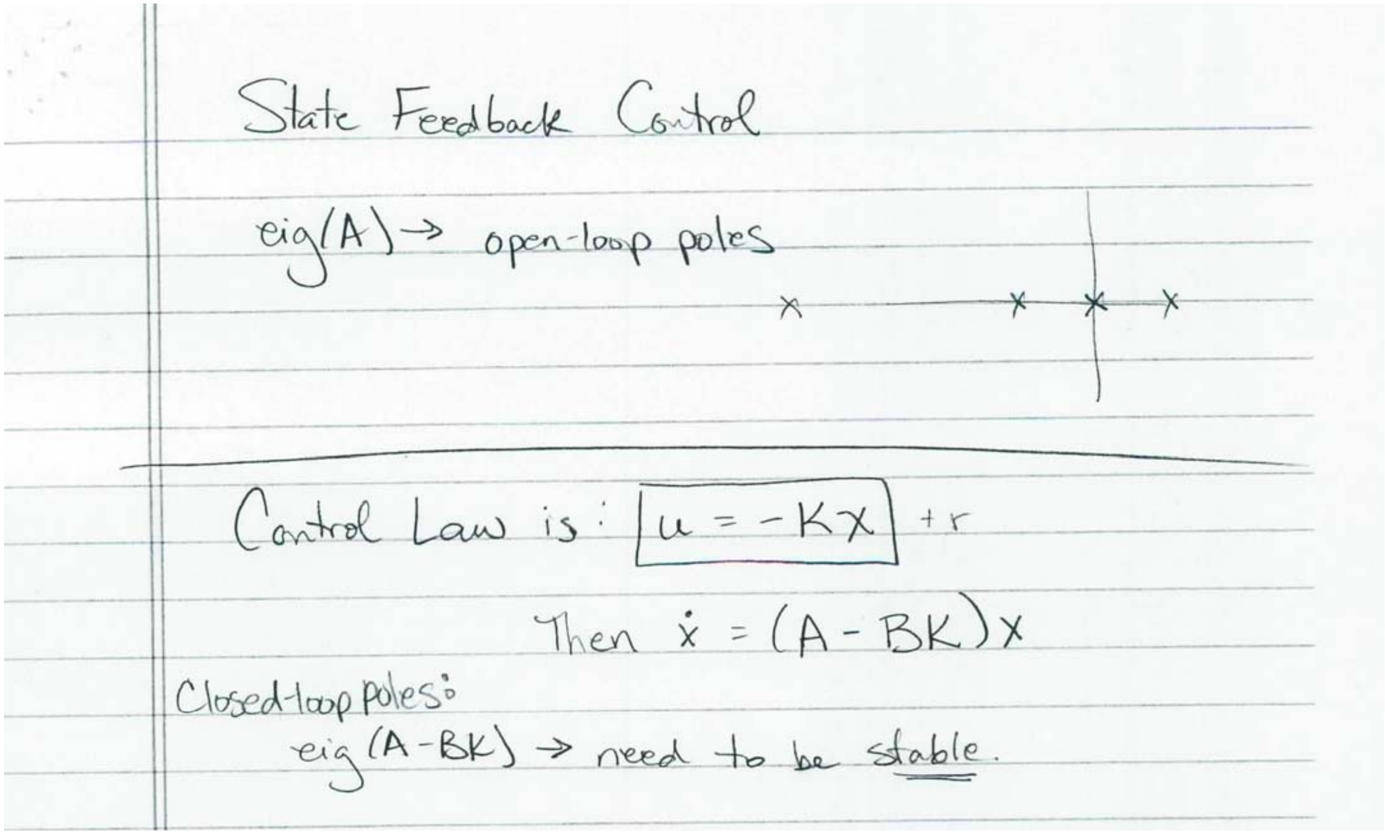
$$M \simeq \begin{bmatrix} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0 \\ 0 & \begin{matrix} .0006 & .0017 \\ .0017 & .0072 \end{matrix} \end{bmatrix}$$

Form of $A: \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -a & -c & +c \\ 0 & +b & \left(\frac{c}{d}\right) & \left(-\frac{c}{d}\right) \end{bmatrix}$, $B: \begin{bmatrix} 0 \\ 0 \\ f \\ \left(-\frac{f}{d}\right) \end{bmatrix}$

$$A \rightarrow \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -650 & -270 & 270 \\ 0 & +230 & 70 & -70 \end{bmatrix} \qquad B \rightarrow \begin{bmatrix} 0 \\ 0 \\ 280 \\ -74 \end{bmatrix}$$

⑤

Segway Handout



State Feedback Control

eig(A) → open-loop poles

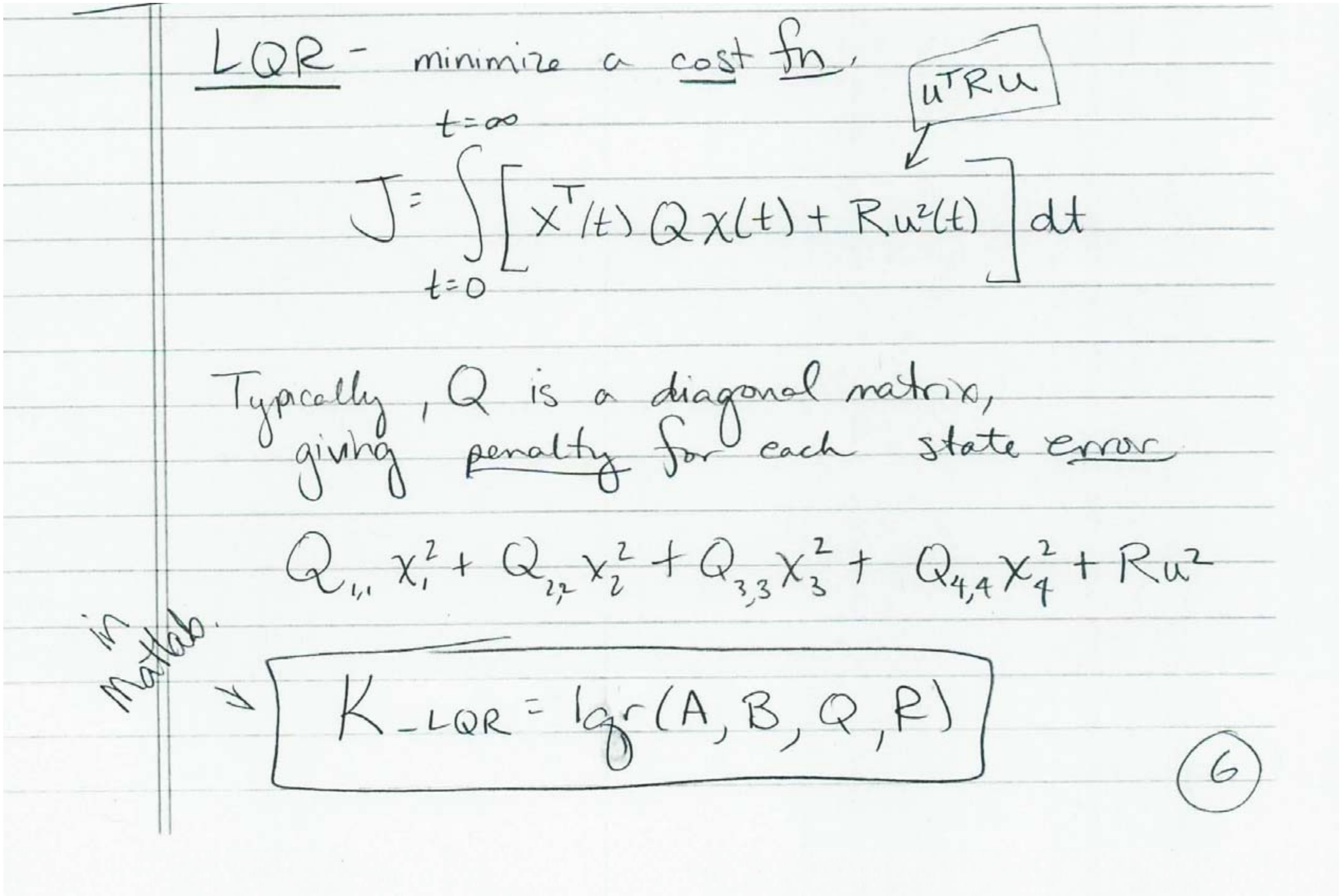
Control Law is: $\boxed{u = -Kx}$ +r

Then $\dot{x} = (A - BK)x$

Closed-loop poles:

eig(A - BK) → need to be $\underline{\underline{\text{stable}}}$.

Segway Handout



LQR - minimize a cost fn,

$u^T R u$

$$J = \int_{t=0}^{t=\infty} \left[x^T(t)\, Q\, x(t) + R u^2(t) \right] dt$$

Typically, Q is a diagonal matrix, giving penalty for each state error

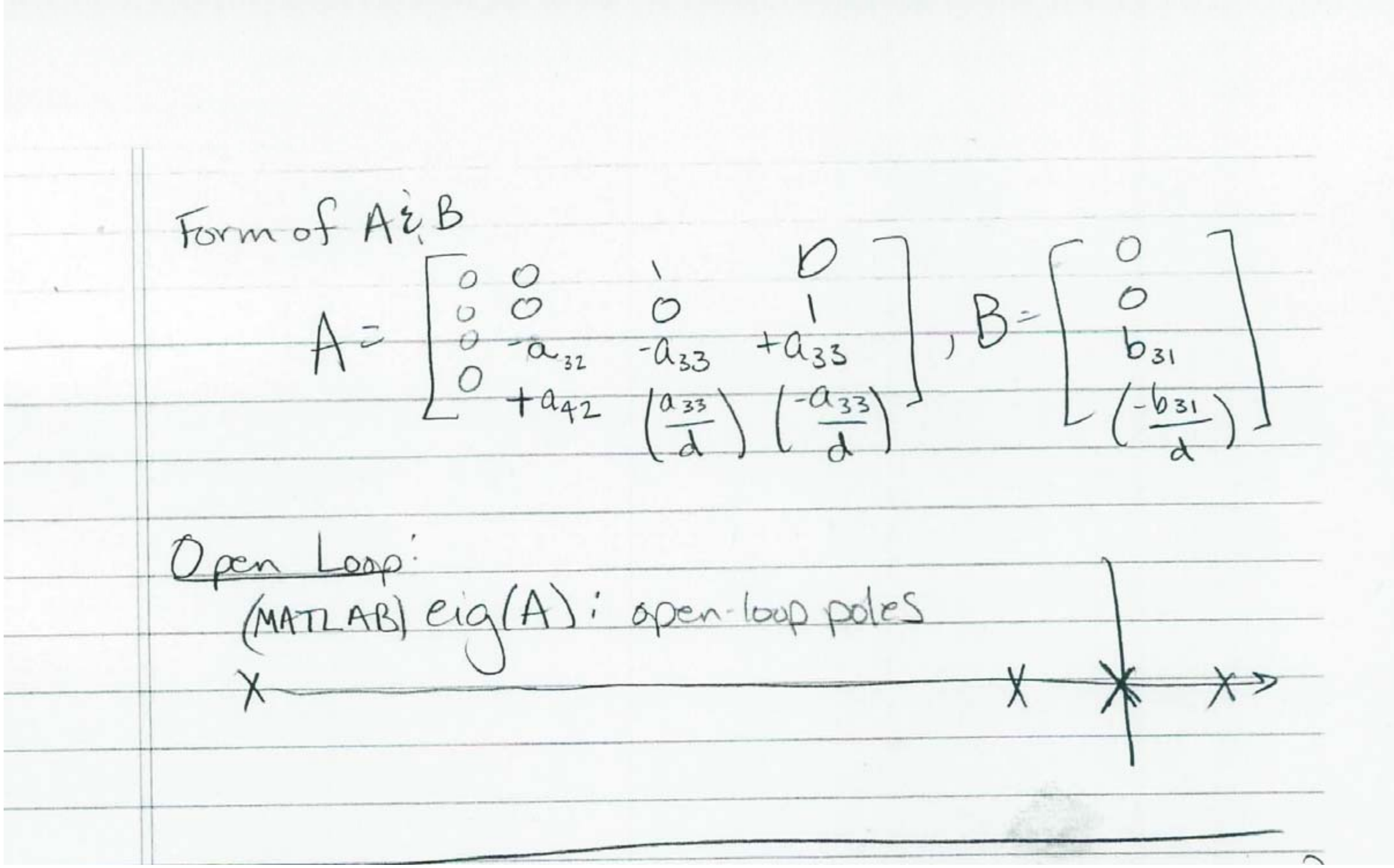
$$Q_{1,1} x_1^2 + Q_{2,2} x_2^2 + Q_{3,3} x_3^2 + Q_{4,4} x_4^2 + Ru^2$$

in matlab.

$$\boxed{K\_LQR = lqr(A, B, Q, R)}$$

⑥

Segway Handout



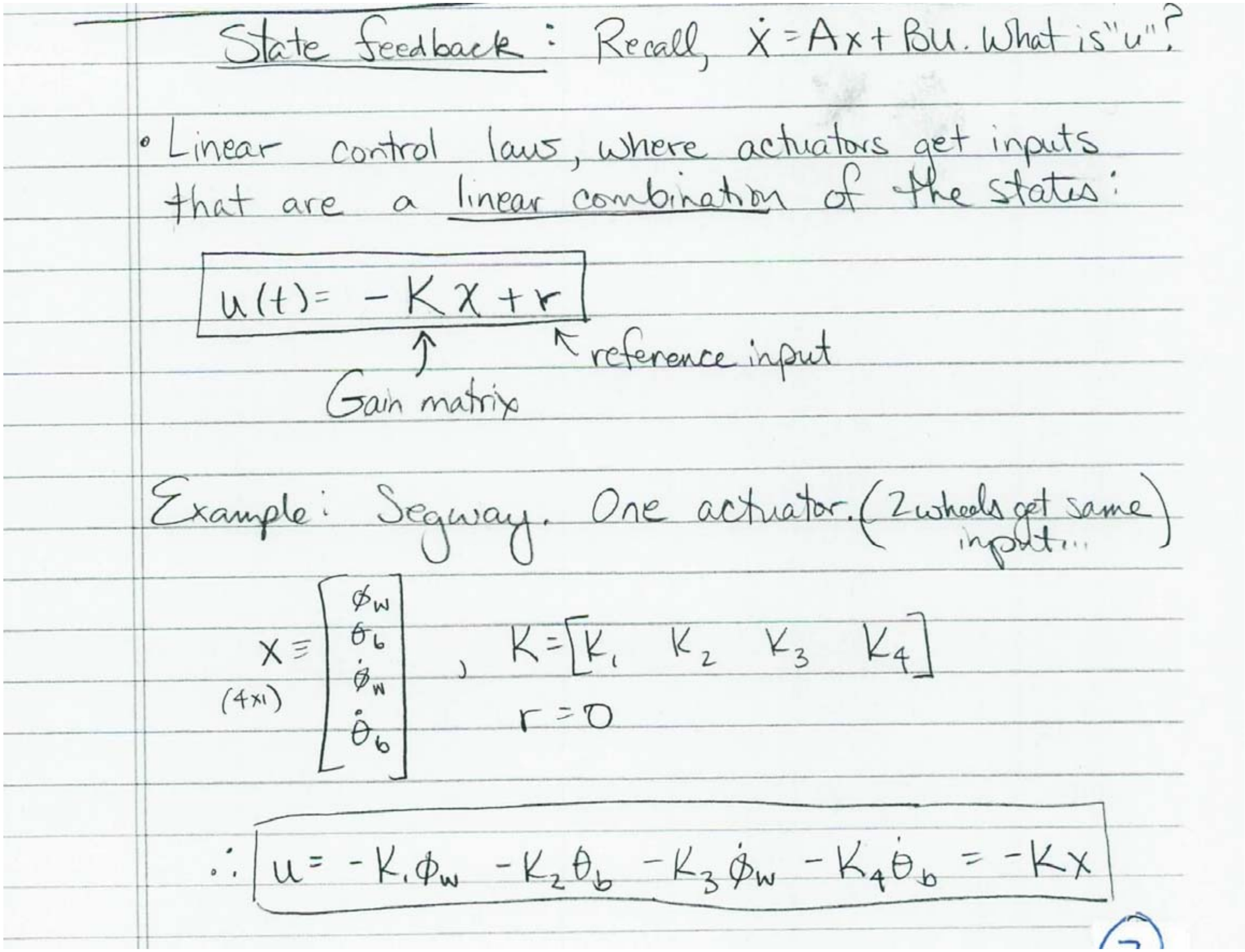
Form of A & B

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -a_{32} & -a_{33} & +a_{33} \\ 0 & +a_{42} & \left(\frac{a_{33}}{d}\right) & \left(\frac{-a_{33}}{d}\right) \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ b_{31} \\ \left(\frac{-b_{31}}{d}\right) \end{bmatrix}$$

Open Loop:

(MATLAB) eig(A): open-loop poles

Segway Handout



State feedback: Recall, $\dot{x} = Ax + Bu$. What is "u"?

- Linear control laws, where actuators get inputs that are a linear combination of the states:

$$\boxed{u(t) = -\underset{\substack{\uparrow\\ \text{Gain matrix}}}{K}x + \underset{\substack{\nwarrow\\ \text{reference input}}}{r}}$$

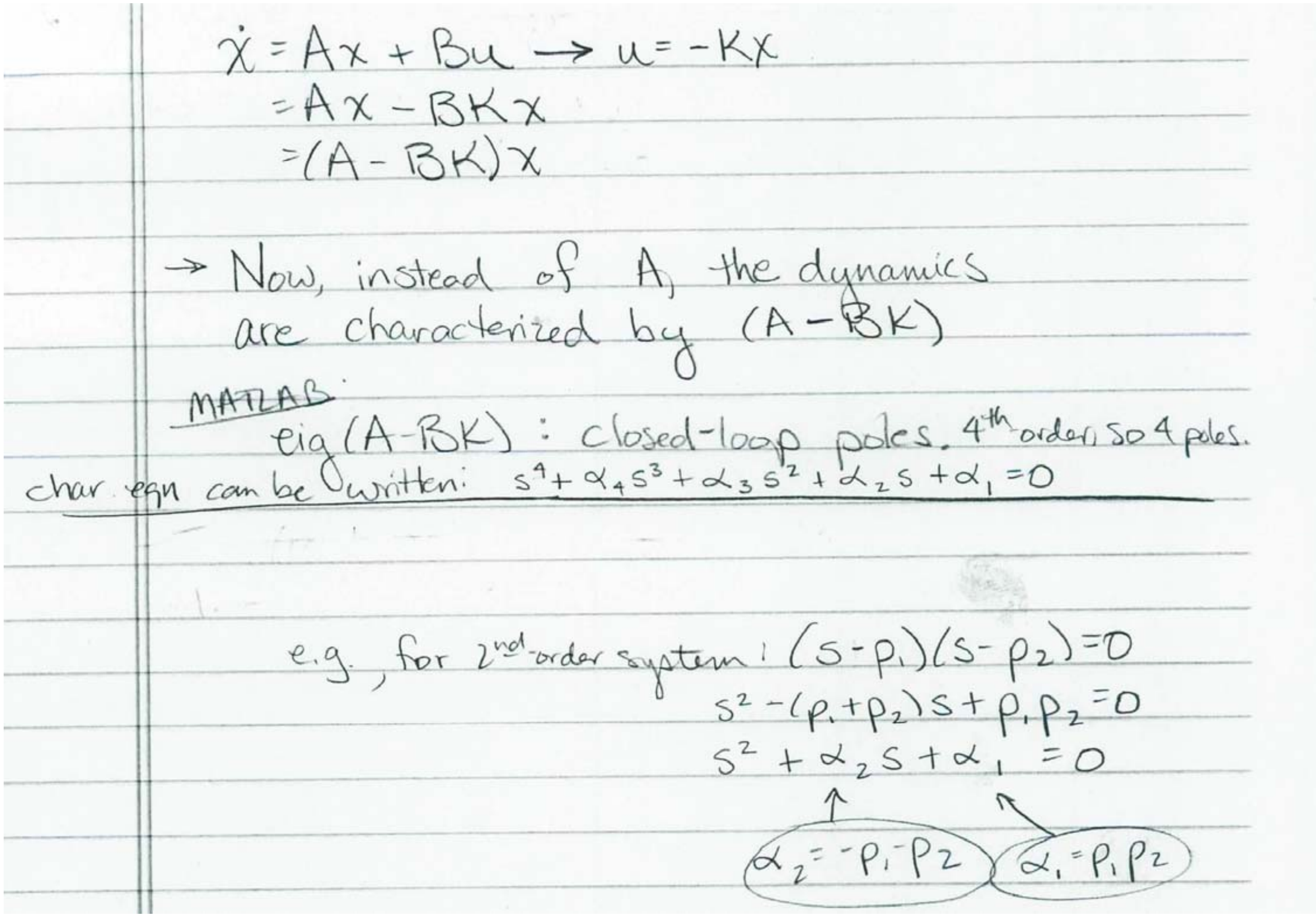
Example: Segway. One actuator. (2 wheels get same input...)

$$\underset{(4\times1)}{x} \equiv \begin{bmatrix}\phi_w\\ \theta_b\\ \dot{\phi}_w\\ \dot{\theta}_b\end{bmatrix}, \quad K = [K_1 \quad K_2 \quad K_3 \quad K_4]$$

$$r = 0$$

$$\therefore \boxed{u = -K_1\phi_w - K_2\theta_b - K_3\dot{\phi}_w - K_4\dot{\theta}_b = -Kx}$$

Segway Handout



$$\dot{x} = Ax + Bu \rightarrow u = -Kx$$
$$= Ax - BKx$$
$$= (A - BK)x$$

→ Now, instead of A, the dynamics are characterized by $(A - BK)$

MATLAB:

eig(A-BK) : closed-loop poles. 4th order, so 4 poles.

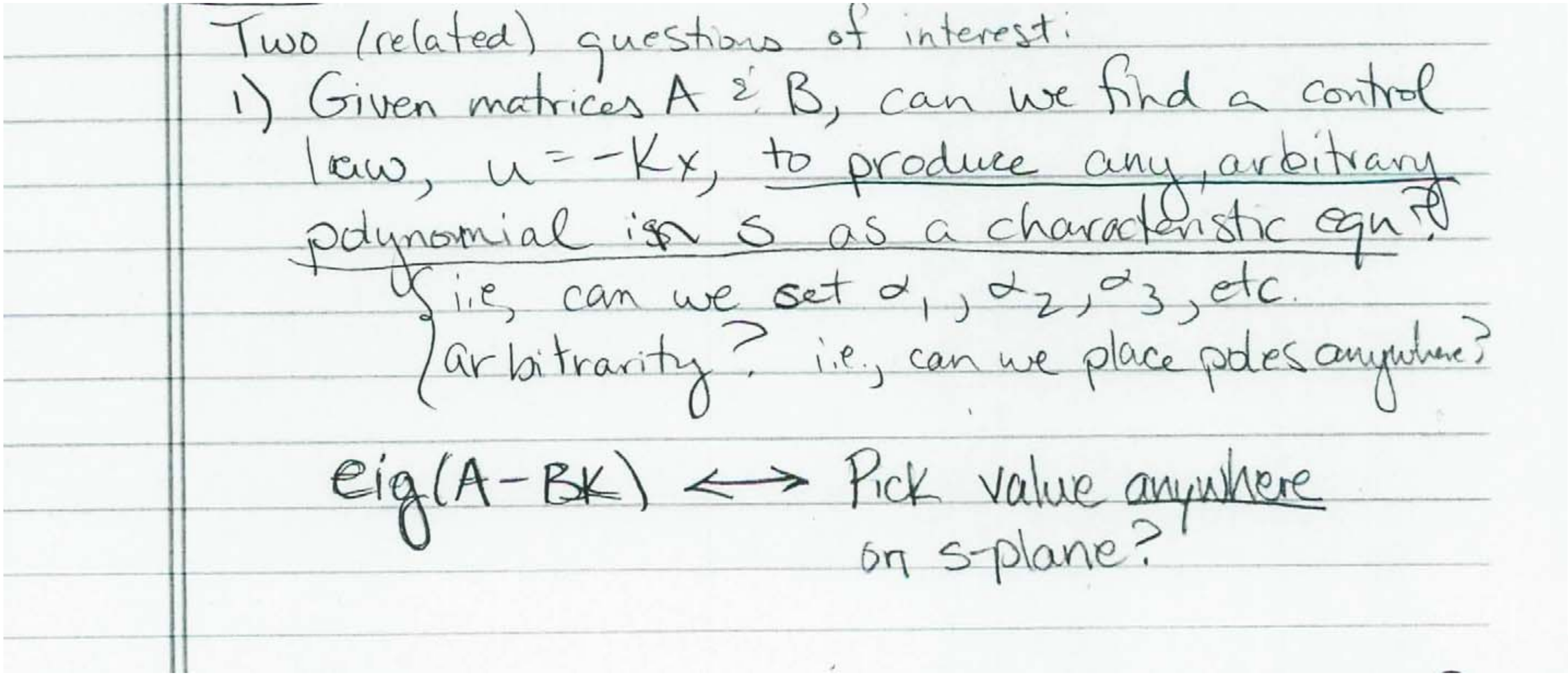
char. eqn can be written: $s^4 + \alpha_4 s^3 + \alpha_3 s^2 + \alpha_2 s + \alpha_1 = 0$

e.g., for 2nd-order system: $(s - p_1)(s - p_2) = 0$

$$s^2 - (p_1 + p_2)s + p_1 p_2 = 0$$
$$s^2 + \alpha_2 s + \alpha_1 = 0$$

$\boxed{\alpha_2 = -p_1 - p_2}$ $\boxed{\alpha_1 = p_1 p_2}$

Segway Handout



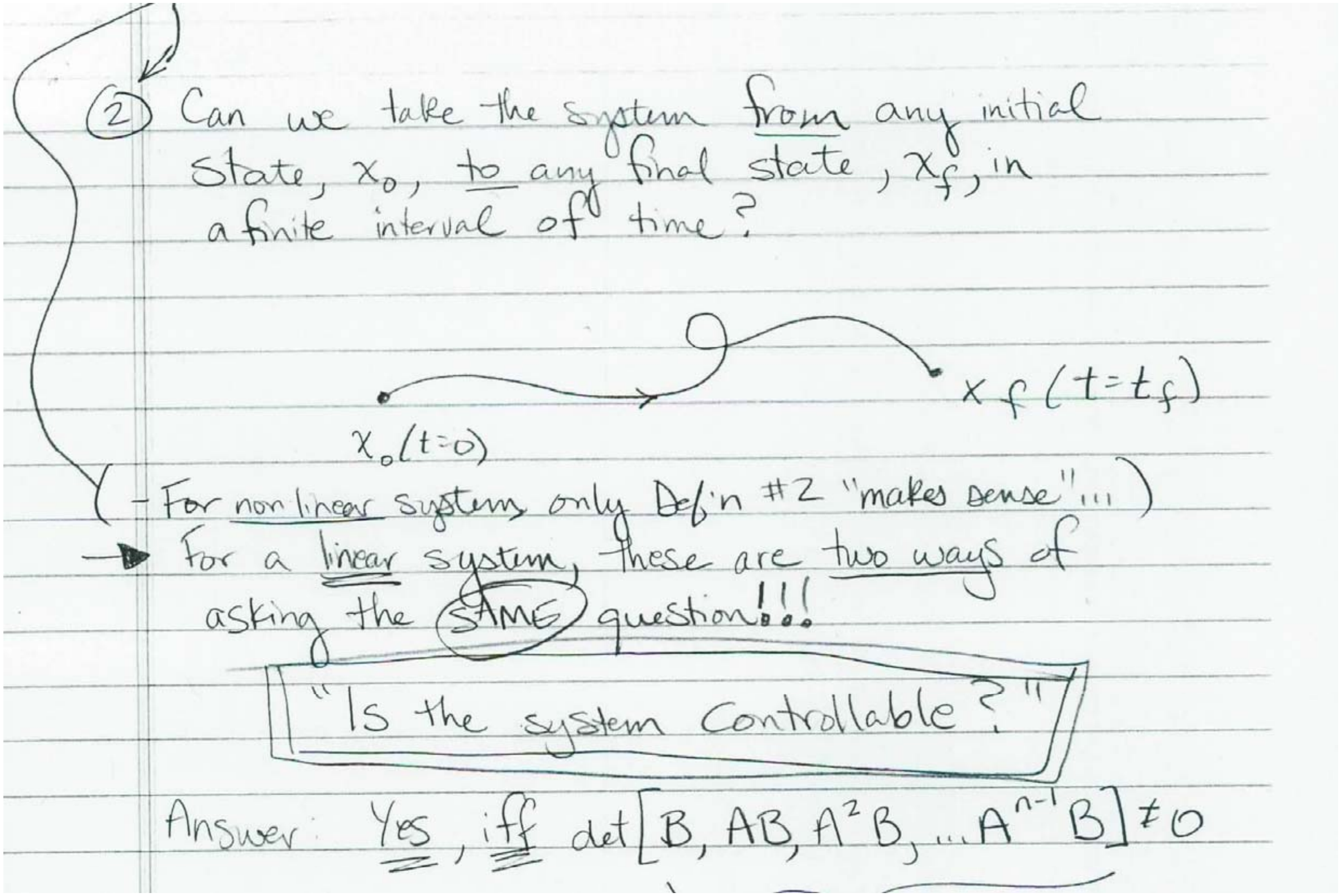
Two (related) questions of interest:

1) Given matrices A & B, can we find a control law, $u = -Kx$, to produce any, arbitrary polynomial in s as a characteristic eqn?

{ i.e., can we set α_1, α_2, α_3, etc. arbitrarity? i.e., can we place poles anywhere?

eig(A−BK) ⟷ Pick value anywhere on s-plane?

Segway Handout



② Can we take the system from any initial state, x_0, to any final state, x_f, in a finite interval of time?

$x_o(t=0)$

$x_f(t=t_f)$

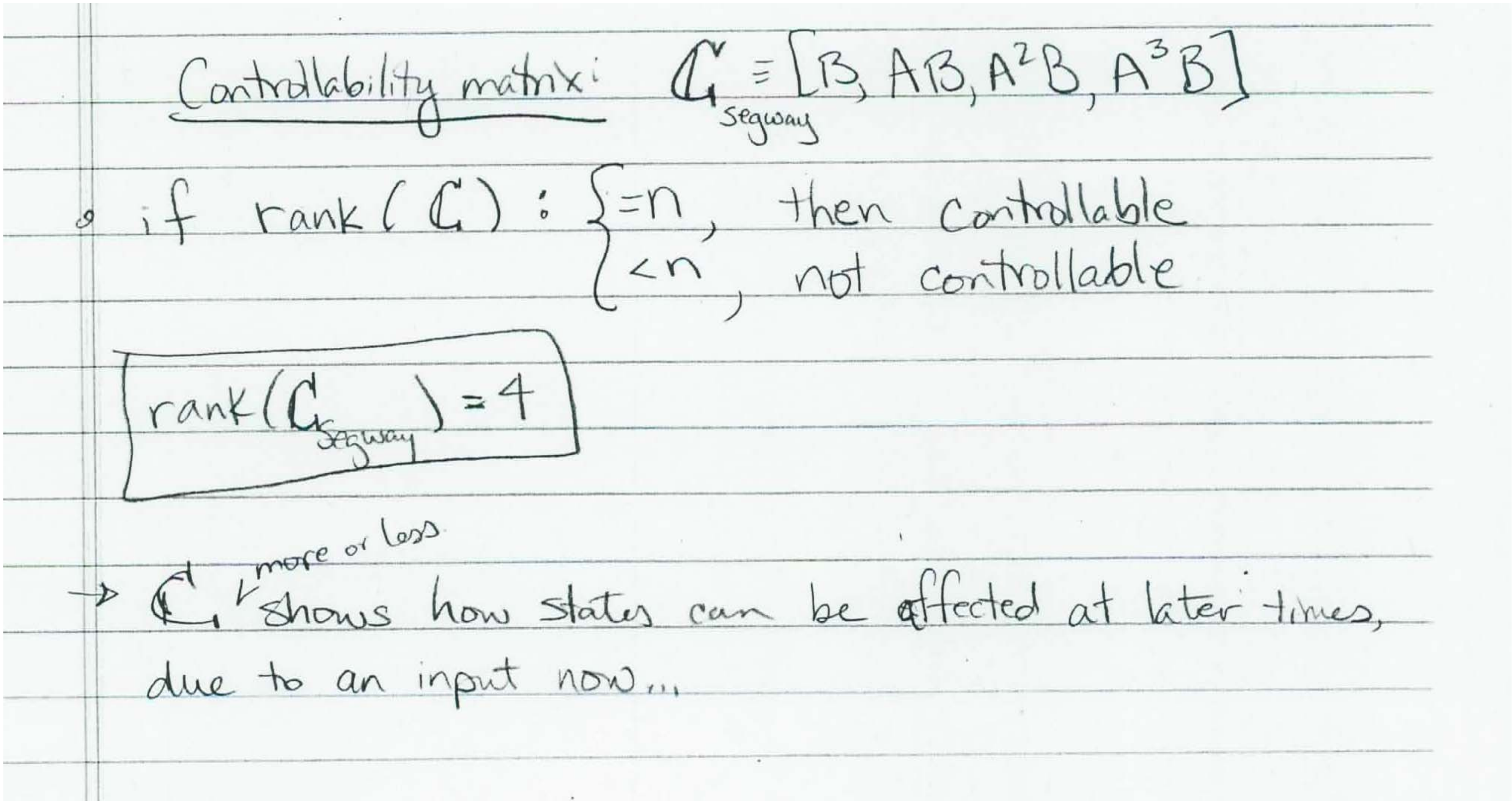
(- For non linear systems only Def'n #2 "makes sense"...)

→ For a linear system, these are two ways of asking the SAME question!!!

"Is the system Controllable?"

Answer: Yes, iff $\det[B, AB, A^2B, \ldots A^{n-1}B] \neq 0$

Segway Handout



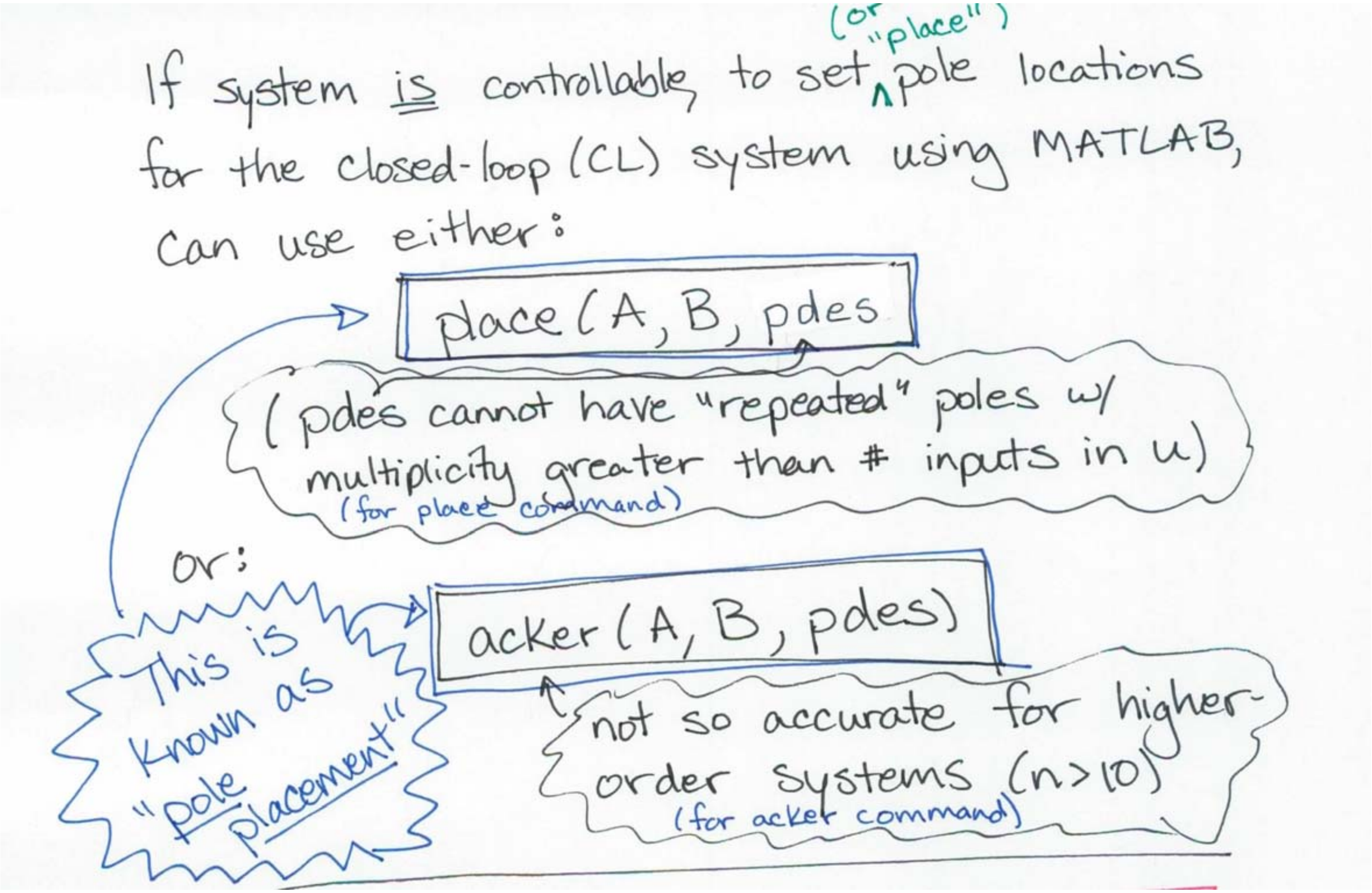
Controllability matrix: $\mathcal{C}_{\text{Segway}} \equiv [B, AB, A^2B, A^3B]$

- if $\text{rank}(\mathcal{C}): \begin{cases} = n, & \text{then controllable} \\ < n, & \text{not controllable} \end{cases}$

$$\boxed{\text{rank}(\mathcal{C}_{\text{Segway}}) = 4}$$

→ $\mathcal{C}$, more or less shows how states can be affected at later times, due to an input now...

Segway Handout



If system is controllable to set (or "place") pole locations for the closed-loop (CL) system using MATLAB, can use either:

place(A, B, pdes

(pdes cannot have "repeated" poles w/ multiplicity greater than # inputs in u)

(for place command)

or:

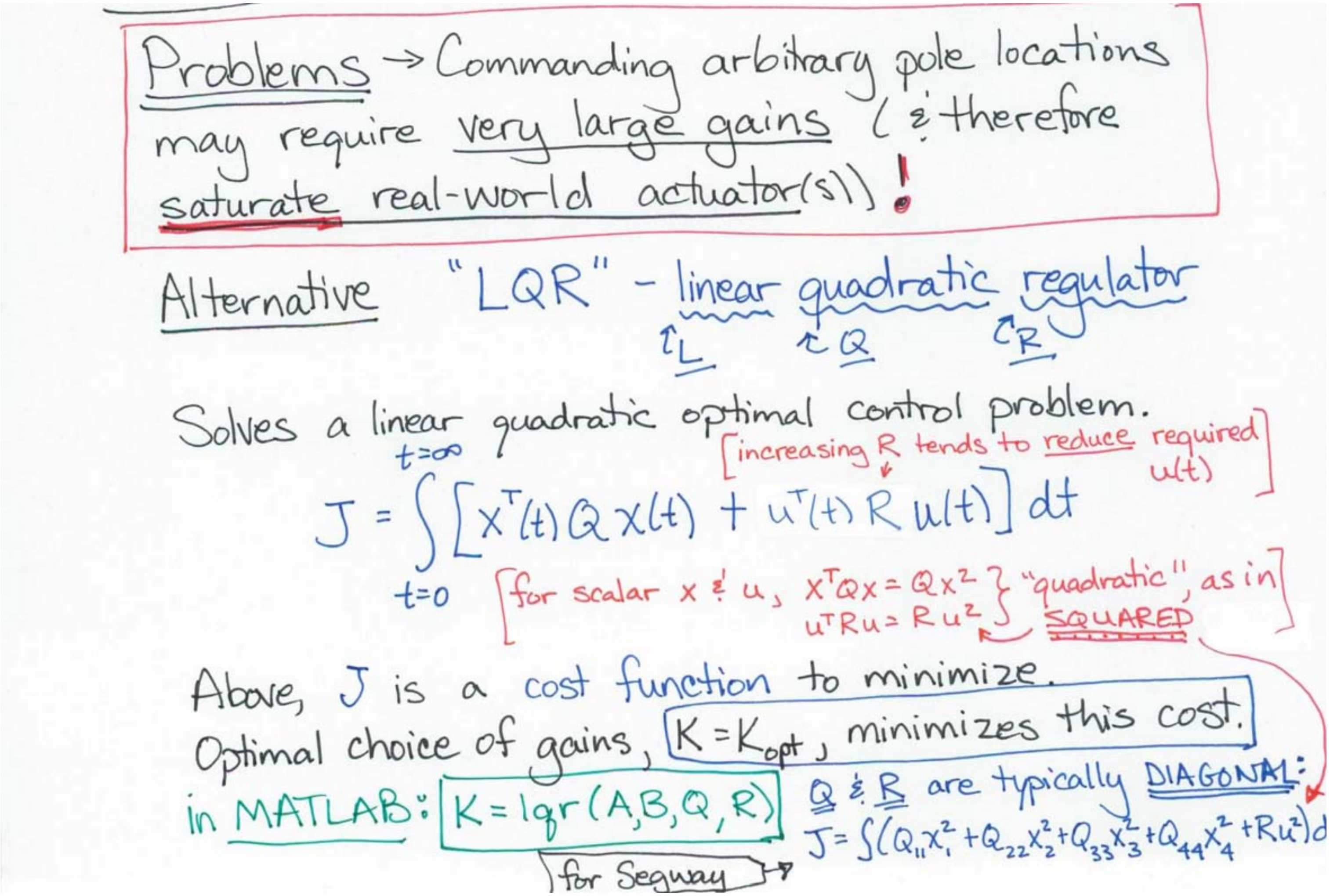
acker(A, B, pdes)

not so accurate for higher-order systems (n>10)

(for acker command)

This is known as "pole placement"

Segway Handout



Problems → Commanding arbitrary pole locations may require very large gains (& therefore saturate real-world actuator(s))!

Alternative "LQR" - linear quadratic regulator

↑L ↑Q ↑R

Solves a linear quadratic optimal control problem.

[increasing R tends to reduce required u(t)]

$$J = \int_{t=0}^{t=\infty} \left[x^T(t)\, Q\, x(t) + u^T(t)\, R\, u(t) \right] dt$$

[for scalar x & u, $x^TQx = Qx^2$, $u^TRu = Ru^2$ } "quadratic", as in SQUARED]

Above, J is a cost function to minimize.

Optimal choice of gains, $\boxed{K = K_{opt}}$, minimizes this cost.

in MATLAB: $\boxed{K = lqr(A,B,Q,R)}$

for Segway →

Q & R are typically DIAGONAL:

$$J = \int (Q_{11}x_1^2 + Q_{22}x_2^2 + Q_{33}x_3^2 + Q_{44}x_4^2 + Ru^2)d$$