Controllability (and Observability)

Lecture 15

Today: Look at meaning of "controllability" in more depth.

* Note, observability is a very analogous concept.

Review: From last time, recall that for a LINEAR dynamic system, there are two definitions we can (equivalently) use to test for controllability.

① If we can pick n gain values in K to set locations of n poles (in an n^{th}-order system) arbitrarily.
② If we can goto an arbitrarily new state vector (n states), given an appropriately chosen $u(t)$ (control action).

Let's look at each definition more closely.

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① Given a dynamic system described via state space equations,

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{(n\times 1)} ; \qquad \boxed{\underset{(n\times1)}{\dot{X}} = \underset{(n\times n)(n\times1)}{AX} + \underset{(n\times m)(m\times1)}{Bu}} \qquad \boxed{\underset{(k\times1)}{Y} = \underset{(k\times n)(n\times1)}{CX} + \underset{(k\times m)(m\times1)}{Du}}$$

$(n\times 1)$ states

n states
m inputs
K outputs

usually, $D=0$

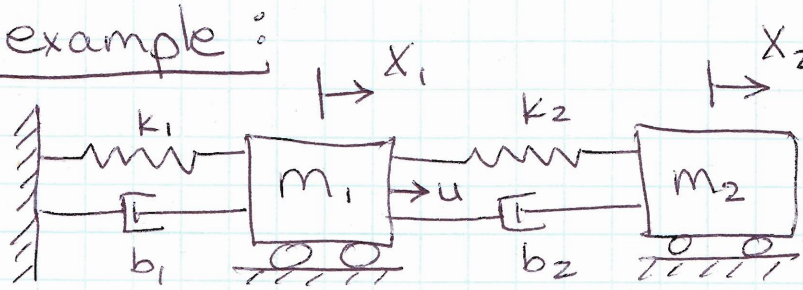
* Assume $m=1$ for now (single input).

a) Open-loop poles are at $\boxed{\text{eig}(A)}$.

b) From poles, we can create the characteristic eqn:

$$\boxed{s^n + a_1 s^{n-1} + \ldots + a_{n-1} = 0}$$

For example:



$$m_1\ddot{x}_1 + (b_1+b_2)\dot{x}_1 + (k_1+k_2)x = u + b_2\dot{x}_2 + k x_2$$
$$m_2\ddot{x}_2 + b_2\dot{x}_2 + k_2 x_2 = b_2\dot{x}_1 + k_2 x_1$$

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$; $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-(k_1+k_2)}{m_1}, & \frac{k_2}{m_1}, & \frac{-(b_1+b_2)}{m_1}, & \frac{k_2}{m_1} \\ \frac{k_2}{m_2}, & \frac{-k_2}{m_2}, & \frac{b_2}{m_2}, & \frac{-b_2}{m_2} \end{bmatrix}$

For $m_1 = 1$, $m_2 = 2$
$b_1 = 1$, $b_2 = 2$
$k_1 = 5$, $k_2 = 10$, $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -15 & 10 & -3 & 2 \\ 5 & -5 & 1 & -1 \end{bmatrix}$

and $B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$\boxed{s = \mathrm{eig}(A)} \longrightarrow$
$-1.9 + 3.9j$
$-1.9 - 3.9j$
$-.13 + 1.1j$
$-.13 - 1.1j$

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p=1;
for n=1:n
    p=conv(p,[1,-s(n)]);
end
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$\rightarrow \boxed{p=[1,4,21,10,25]}$

i.e., $\boxed{s^4 + 4s^3 + 21s^2 + 10s + 25 = 0}$ ← Characteristic Equation, given A matrix.

- Now, given open-loop poles, which determine a (unique) characteristic equation (if unique highest order in s has coef. 1),

...we can generally (unless n=1 state) define a new dynamic system w/ A such that eigenvalues are the same.

- For example, the CANONICAL form, "A_c", where top row of A_c is built directly from the char. poly.:

$$s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0$$

$$A_c = \begin{bmatrix} -a_1, & -a_2, & -a_3, & \dots, & -a_n \\ 1 & 0 & 0 & & 0 \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ & & & \ddots & \end{bmatrix}$$

$$B_c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Here, $X = \begin{bmatrix} \dddot{x} \\ \ddot{x} \\ \dot{x} \\ x \end{bmatrix}$ ← Not easy to interpret physically!

Controllability matrix

$$\mathbb{C} \equiv \underbrace{[B, AB, A^2B, \ldots, A^{n-1}B]}_{(n \times n)}$$

* If $\mathbb{C}$ is FULL RANK, then it is invertible, & then you can set pole arbitrarily by picking K.

Why?: Let $\boxed{T = \mathbb{C}\,\mathbb{C}_c^{-1}}$

Controllability matrix for A_c & B_c

$\boxed{\begin{aligned} A_c &= T^{-1}AT \\ B_c &= T^{-1}B \end{aligned}}$ $\boxed{\begin{aligned} A &= TA_cT^{-1} \\ B &= TB_c \end{aligned}}$

For the canonical system, control gains are:

$$K_c = [K_{1c}, K_{2c}, \ldots, K_{nc}]$$

Control law is:

$$u = -Kx$$

$$\therefore \dot{x} = A_cx + B_cu$$

$$= A_cx - B_cK_cx$$

$$\boxed{\dot{x} = (A_c - B_cK_c)x}$$

closed-loop poles are at $\boxed{s = \text{eig}(A_c - B_cK_c)}$

Since $B_c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $B_cK_c = \begin{bmatrix} K_1 & K_2 & K_3 & K_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$; $A_c - B_cK_c = \begin{bmatrix} -(a_1+K_1), & -(a_2+K_2), & (-a_3-K_3), & -(a_4+K_4) \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

our example

Say we <u>want</u> poles $s = \begin{cases} -1+j \\ -1-j \\ -10 \\ -10 \end{cases}$

Desired char. eq. is: $\boxed{s^4 + 22s^3 + 142s^2 + 240s + 200 = 0}$

So, since O.L. was : $s^4 + 4s^3 + 21s^2 + 10s + 25 = 0$

$a_1 \quad a_2 \quad a_3 \quad a_4$

We <u>need</u>:

$K_{c1} + a_1 = 22 \rightarrow \boxed{K_1 = 22 - 4 = 18}$

$K_{c2} + a_2 = 142 \rightarrow \boxed{K_2 = 142 - 21 = 121}$

$K_{c3} + a_3 = 240 \rightarrow \boxed{K_3 = 240 - 10 = 230}$

$K_{c4} + a_4 = 200 \rightarrow \boxed{K_4 = 200 - 25 = 175}$

Finally, to get from K_c (for the control canonical form) to K (for original A & B matrices):

$$A_c - B_c K_c$$

$$\downarrow$$

$$T(A_c)T^{-1} - T(B_c K_c)T^{-1}$$

$$A - B(K_c T^{-1})$$

$$A - BK$$

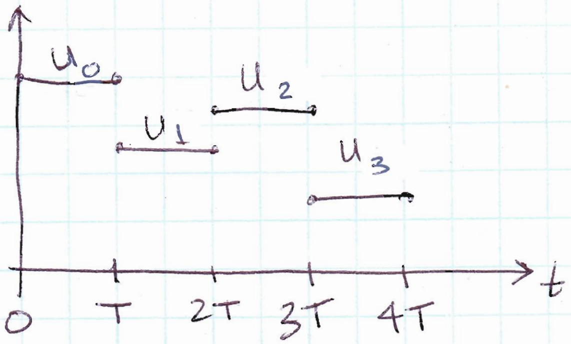
Each line will have <u>same</u> eigenvalues!

where $\boxed{K \equiv K_c T^{-1}}$

② Arbitrary states in finite time?

With 4 states, we expect to need 4 <u>tunable</u> <u>Variables</u> to set each state independently.

So, say we do the following: (Discrete-time control) "$\hat{DT}$"



Can we find values for each u_i to change all 4 states independently?

* <u>Note</u> - It's a lot easier to visualize this for the DT case, but same concept of "getting to a desired state" applies in CT (cont-time) case, too.

<u>DT</u>: $\boxed{X_{n+1} = A_d X_n + B_d u_n}$

So, for 4 time steps,

after u_0, $X_1 = AX_0 + Bu_0$

after u_1, $X_2 = AX_1 + Bu_1$

$= A(AX_0 + Bu_0) + Bu_1$

$= A^2X_0 + ABu_0 + Bu_1$

after u_2, $X_3 = A(A^2X_0 + ABu_0 + Bu_1) + Bu_2$

$= A^3X_0 + A^2Bu_0 + ABu_1 + Bu_2$

after u_3, $X_4 = AX_3 + Bu_3$

$X_4 = A^4X_0 + A^3Bu_0 + A^2Bu_1 + ABu_2 + Bu_3$

We want $X_4 = X_{des}$

$$X_{des} - A^4X_0 = [A^3B, A^2B, AB, B] \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$(x_{des} - A^4 x_0) = \underbrace{\begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}}_{(n\times n)} \begin{bmatrix} u_3 \\ u_2 \\ u_1 \\ u_0 \end{bmatrix}$$

This is $\mathbb{C}$! (Controllability)

To solve for vector u, use <u>matrix algebra</u>:

$$(x_{des} - A^4 x_0) = \mathbb{C}\, u$$

$$\boxed{\mathbb{C}^{-1}(x_{des} - A^4 x_0) = \mathbb{C}^{-1}\mathbb{C}\, u = u}$$

$\mathbb{C}$ must be INVERTIBLE for a solution to be guaranteed!

<u>Observability</u>: Given some initial vector of states, x_i, can we identify all values in x_i by observing the available outputs, y, over some finite time?

$$y = Cx + \cancel{Du}^{\,0}$$

$$\downarrow$$

$y_0 = Cx_0$ ← $x_0 = x_{init}$

$x_1 = Ax_0 + Bu_0$

$y_1 = C(Ax_0 + Bu_0)$

$x_2 = Ax_1 + Bu_1$

$\quad = A(Ax_0 + Bu_0) + Bu_1$

$y_2 = C[A(Ax_0 + Bu_0) + Bu_1]$

$x_3 = Ax_2 + Bu_2$

$\therefore y_3 = C\{A[A(Ax_0 + Bu_0) + Bu_1] + Bu_2\}$

Rearranging

x_0 is the unknown we want to observe!

$$
\begin{aligned}
y_0 &= \boxed{C x_0} \\
y_1 &= \boxed{CA x_0} + CBu_0 \\
y_2 &= \boxed{CA^2 x_0} + CABu_0 + CBu_1 \\
\underbrace{y_3}_{\uparrow} &= \boxed{CA^3 x_0} + \underbrace{CA^2Bu_0 + CABu_1 + CBu_2}
\end{aligned}
$$

All other terms ($y_0, y_1, y_2, y_3, C, A, B$) are measured or known at the start.

$$
\therefore \underbrace{\begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}}_{\mathcal{O}^T} x_0 = \underbrace{\begin{bmatrix} y_0 \\ y_1 - CBu_0 \\ y_2 - CABu_0 - CBu_1 \\ y_3 - CA^2Bu_0 - CABu_1 - CBu_2 \end{bmatrix}}_{\text{all known}} = \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix}
$$

$$
\therefore x_0 = \left(\mathcal{O}^T\right)^{-1} \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix}
$$

$\boxed{\text{Can be found if } \mathcal{O} \text{ is full rank, i.e., invertible.}}$

(eig(A-BK)) control poles

(eig(A-LC)) observer poles

Matlab to find K = place(A, B, pcon)

L = place(A', C', pobs)'

control gains

observer gain

note 3 transpose chars

eig(A-BK)

eig(A-LC)