ECE/MAT 211A winter 07

HWK#3 solutions

#4•4-1: Gaussian Wave Packet under the Influence of a Uniform Force, in the Position Representation

From (4•4-25)

$$\Phi(p') = \Phi_0 \exp\left[-\left(\frac{p'}{2\Delta p}\right)^2 + \frac{i(p'^3 - p^3)}{6\hbar MF}\right] \tag{1}$$

$$p' = p - Ft$$
$$p'^3 = p^3 - 3p^2 Ft + 3p(Ft)^2 - (Ft)^3$$

From (4•1-4), (4•1-8) and (4•1-8),

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} A(k,t)e^{ikx}dk\ , \quad \Phi(p,t) = \frac{1}{\sqrt{\hbar}}A(k,t)\ , \quad dp = \hbar dk\ ,$$

we can get

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\sqrt{\hbar}\Phi(p,t)e^{i\frac{p}{\hbar}x}\frac{dp}{\hbar} = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty}\Phi(p',t)e^{i\frac{p'+Ft}{\hbar}x}dp' \tag{2}$$

$$\Psi(x,t) = \frac{\Phi_0}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty}\exp\left[-\left(\frac{p'}{2\Delta p}\right)^2 - \frac{i(p'^3 - p^3)}{6\hbar MF} + \frac{i(p'+Ft)x}{\hbar}\right]dp' \tag{3}$$

$$p'^3 - p^3 = p'^3 - (p'+Ft)^3 = -3p'^2 Ft - 3p'(Ft)^2 - (Ft)^3$$

$$\Psi(x,t) = \frac{\Phi_0}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty}\exp\left[-\left(\frac{p'}{2\Delta p}\right)^2 + \frac{itp'^2}{2\hbar M} + \frac{ip'}{\hbar}\left(x - \frac{3(Ft)^2}{6MF}\right) + \frac{iFtx}{\hbar} - \frac{i(Ft)^3}{6\hbar MF}\right]dp' \tag{4}$$

Let $t_0 = \dfrac{2\hbar M}{(2\Delta p)^2}$, $x' = x - \dfrac{Ft^2}{2M}$, $\dfrac{F}{M} = a$,

$$\Psi(x,t) = \frac{\Phi_0}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty}\exp\left[\left(\frac{ip'}{2\Delta p}\sqrt{1+\frac{it}{t_0}} + \frac{\Delta p x'}{\hbar\sqrt{1+\frac{it}{t_0}}}\right)^2 - \frac{(\Delta p)^2 x'^2}{\hbar^2\left(1+\frac{it}{t_0}\right)} + \frac{iFt}{\hbar}\left(x - \frac{Ft^2}{6M}\right)\right]dp'$$

$$\Psi(x,t)=\frac{\Phi_0}{\sqrt{2\pi\hbar}}\exp\left[\frac{-(\Delta p)^2(x-\frac{at^2}{2})^2}{\hbar^2\left(1+\frac{it}{t_0}\right)}\right]\exp\left[\frac{iFt}{\hbar}\left(x-\frac{at^2}{6}\right)\right]\int_{-\infty}^{+\infty}\exp\left(\frac{ip'}{2\Delta p}\sqrt{1+\frac{it}{t_0}}+\frac{\Delta px'}{\hbar\sqrt{1+\frac{it}{t_0}}}\right)^2 dp' \qquad (5)$$

Let $\frac{p'}{2\Delta p}\sqrt{1+\frac{it}{t_0}}+\frac{\Delta px'}{i\hbar\sqrt{1+\frac{it}{t_0}}}=y$, $\qquad \frac{dp'}{2\Delta p}\sqrt{1+\frac{it}{t_0}}=dy$ (6a,b)

Substituting (6a,b) into (5) yields

$$\Psi(x,t)=\frac{\Phi_0\sqrt{2}\Delta p}{\sqrt{\pi\hbar}\sqrt{1+\frac{it}{t_0}}}\exp\left(\frac{-(\Delta p)^2(x-\frac{at^2}{2})^2}{\hbar^2\left(1+\frac{it}{t_0}\right)}\right)\exp\left(\frac{iFt}{\hbar}\left(x-\frac{at^2}{6}\right)\right)\int_{-\infty}^{+\infty}e^{-y^2}\,dy \qquad (7)$$

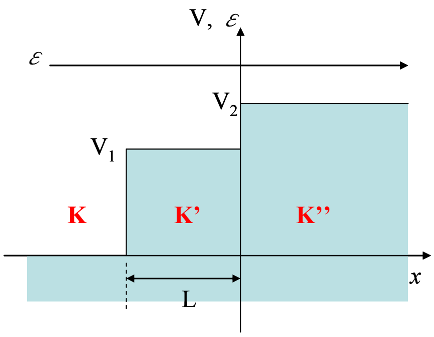
Replacing $\frac{(\Delta p)^2}{\hbar^2}=(\Delta k)^2=\frac{1}{(2\Delta x)^2}=\frac{1}{(2\Delta x')^2}\left(1+(\frac{t}{t_0})^2\right)$ in (7) yields

$$\boxed{\Psi(x,t)=\frac{c_0}{\sqrt{1+\frac{it}{t_0}}}\exp\left(-\left(\frac{x-\frac{at^2}{2}}{2\Delta x'}\right)^2\left(1-\frac{it}{t_0}\right)\right)\exp\left(\frac{iFt}{\hbar}\left(x-\frac{at^2}{6}\right)\right)} \qquad (8)$$

When F =0 hence a = 0, (8) becomes

$$\boxed{\Psi(x,t)=\frac{c_0}{\sqrt{1+\frac{it}{t_0}}}\exp\left(-\left(\frac{x}{2\Delta x'}\right)^2\left(1-\frac{it}{t_0}\right)\right)}$$, which is the same as (4•2-19)

#5·3-1: Zero-Reflection Conditions at a Double-Step Barrier



The condition of zero reflection is

$$P_{21} = 0 \tag{1}$$

We obtain the propagation matrix for our problem by replacing the downward-step propagation matrix in (5•3-4) by another upward-step matrix in which we have replaced K by K", the wave number in the region x>0. This yields the overall propagation matrix:

$$\hat{P} = \frac{1}{4K'\sqrt{K''K}} \begin{pmatrix} K+K' & K-K' \\ K-K' & K+K' \end{pmatrix} \begin{pmatrix} e^{-iK'L} & 0 \\ 0 & e^{+iK'L} \end{pmatrix} \begin{pmatrix} K'+K'' & K'-K'' \\ K'-K'' & K'+K'' \end{pmatrix} \tag{2}$$

From (1) and (2), we know

$$(K-K')(K'+K'')e^{-iK'L} + (K+K')(K'-K'')e^{iK'L} = 0\,. \tag{3}$$

This has both a real and an imaginary part. The real part is

$$\left[(K-K')(K'+K'') + (K+K')(K'-K'')\right]\cdot \cos K'L = 2K'\cdot(K-K'')\cdot \cos K'L = 0\,. \tag{4}$$

Because all three K-s are non-zero, and $K'' \neq K$, this condition requires

$$\boxed{\cos K'L = 0}, \text{ or equivalently, } \boxed{K'L = (2n+1)\frac{\pi}{2}} \tag{5a,b}$$

That is, the intermediate section must be an *odd* number of the quarter-wavelengths long, a result familiar from anti-reflection coatings in optics. The condition (5b) may be interpreted as a condition for the destructive interference between the two partial waves reflected at the two steps, just like the T = 1 condition (5•3-14a) for square barrier – with one important difference: The condition (5•3-14a) implies an *even* number of quarter-wavelengths for zero reflection. The difference between the two cases is a consequence of the difference in phase shift between the reflection at an upward and a downward step, as discussed on p. 147 of the text.

The imaginary part of (3) is

$$\left[(K-K')(K'+K'') - (K+K')(K'-K'')\right]\cdot \sin K'L = 2(KK''-K'^2)\cdot \sin K'L = \pm 2(KK''-K'^2) = 0 \tag{6}$$

where the last equality follows from (5a,b). Hence we obtain the second condition

$$\boxed{K'^2 = KK''}, \text{ or equivalently, } \boxed{\varepsilon - V_1 = \sqrt{\varepsilon\cdot(\varepsilon - V_2)}}, \tag{7}$$

the quantum-mechanical analog of the classical-optics requirement that the refractive index of the anti-reflection layer must be the geometric mean of the refractive indices of the two media.

#5·3-2: Pair of δ-Functions

For compactness we introduce the abbreviations

$$\alpha = KL = \sqrt{2M\varepsilon} \cdot \frac{L}{\hbar}, \quad \beta = \frac{\kappa_0}{K} = \frac{MS}{\hbar\sqrt{2M\varepsilon}} \quad . \tag{1a,b}$$

From (5•3-29a,b) and the conditions (5•2-17a,b), the propagation matrix of the δ-function barrier then has the form

$$\hat{P}_\delta = \begin{pmatrix} 1+i\beta & i\beta \\ -i\beta' & 1-i\beta' \end{pmatrix}, \tag{2}$$

and the overall propagation matrix for the δ-function pair is obtained via

$$\hat{P} = \begin{pmatrix} 1+i\beta & i\beta \\ -i\beta' & 1-i\beta' \end{pmatrix} \begin{pmatrix} e^{-i\alpha} & 0 \\ 0 & e^{+i\alpha} \end{pmatrix} \begin{pmatrix} 1+i\beta & i\beta \\ -i\beta' & 1-i\beta' \end{pmatrix}. \tag{3}$$

Hence the only overall matrix element of interest, P_{11}, would be

$$P_{11} = (1+i\beta)^2 e^{-i\alpha} + \beta^2 e^{+i\alpha} = [\cos\alpha + 2\beta\sin\alpha] + i[2\beta\cos\alpha - (1-2\beta^2)\sin\alpha] \tag{4}$$

$$|P_{11}|^2 = (1+4\beta^2)\cos^2\alpha + 8\beta^3 \sin\alpha\cos\alpha + (1+4\beta^4)\sin^2\alpha = 1 + 4\beta^2[\cos\alpha + \beta\sin\alpha]^2 . \tag{5}$$

Inserting α and β from (1a,b) into the transmission probability $T = \frac{1}{|P_{11}|^2}$ yields the latter as a function of K or ε, whichever is desired.

The plot below shows T for the values S = 1×10^{-7}eV•cm, L = 2×10^{-8}cm, as a function of , from 0 to 40eV, extended beyond the problem specification in order to go beyond the second peak.