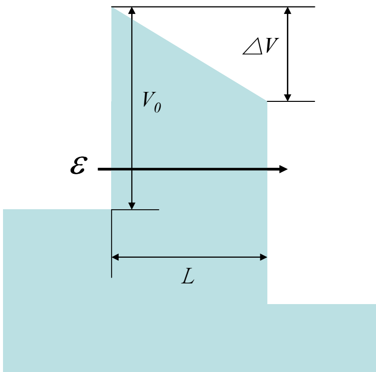
ECE/MAT 211A winter 07

HWK#6 solutions

#6•4-1: Tunneling through a Barrier with an Applied Voltage



The problem is best treated as a propagation matrix problem, similar to the treatment of tunneling through a square barrier on pp. 148-150, but with a WKB wavefunction inside the barrier of the form (6•1-14). We write the wave number k in the form

$$\kappa(x) = \kappa_0 \sqrt{1 - \frac{\Delta V}{V_0 - \varepsilon} \cdot \frac{x}{L}}\,, \tag{1a}$$

where $$\kappa_0 = \sqrt{\frac{2m_e(V_0 - \varepsilon)}{\hbar^2}}\,. \tag{1b}$$

In going through the barrier, the right-evanescent C-term in (6•1-14) is attenuated and left-evanescent D-term grows, giving us a propagation matrix for the space interval (x, L) similar to (5•3-21):

$$\hat{P}_\Delta = \begin{pmatrix} e^{+\delta} & 0 \\ 0 & e^{-\delta} \end{pmatrix}, \tag{2}$$

where

$$\delta = \int_0^L k(x)dx = \kappa_0 \int_0^L \sqrt{1 - \frac{\Delta V}{V_0 - \varepsilon} \cdot \frac{x}{L}} \cdot dx = \frac{2}{3}\kappa_0 L \cdot \frac{V_0 - \varepsilon}{\Delta V}\left[1 - \left(1 - \frac{\Delta V}{V_0 - \varepsilon}\right)^{\frac{3}{2}}\right] \tag{3}$$

Note that, although the problem statement refers to *ΔV* as a *voltage*, it is, strictly speaking, the energy equivalent of a voltage, measured in, say, eV rather than Volts.

In addition to $\hat{P}_\Delta$ we also need again propagation matrices for the steps at the beginning and the end of the barriers, which are of the form (5•3-20a,b), with appropriate(self-explanatory) values of K and κ:

$$\hat{P}_\uparrow = \frac{1}{2\sqrt{K_0 \kappa_0}} \begin{pmatrix} K_0 + i\kappa_0 & K_0 - i\kappa_0 \\ K_0 - i\kappa_0 & K_0 + i\kappa_0 \end{pmatrix} \tag{4a}$$

$$\hat{P}_\downarrow = \frac{1}{2i\sqrt{K_L \kappa_L}} \begin{pmatrix} i\kappa_L + K_L & i\kappa_L - K_L \\ i\kappa_L - K_L & i\kappa_L + K_L \end{pmatrix} \tag{4b}$$

From the product of the three propagation matrices, we obtain

$$P_{11} = \frac{1}{4i\sqrt{K_0\kappa_0 K_L\kappa_L}}\left[(K_0 + i\kappa_0)(i\kappa_L + K_L)e^{+\delta} + (K_0 + i\kappa_0)(i\kappa_L - K_L)e^{-\delta}\right] \tag{5}$$

In the limit that the total tunneling probability is small compared to unity, the $e^{-\delta}$ -term may be neglected next to the $e^{+\delta}$ -term.

With this, from (5):

$$|P_{11}|^2 = \frac{1}{16K_0\kappa_0 K_L\kappa_L}\left(K_0^2 + \kappa_0^2\right)\left(K_L^2 + \kappa_L^2\right)e^{+2\delta} = \frac{m_e^2 V_0^2}{4\hbar^4 K_0\kappa_0 K_L\kappa_L}e^{+2\delta}, \tag{6}$$

where, in the last equality, we have drawn on the fact that

$$\left(K_0^2 + \kappa_0^2\right) = \left(K_L^2 + \kappa_L^2\right) = \frac{2m_e V_0}{\hbar^2}. \tag{7}$$

From (1a,b) we know:

$$\kappa_0 = \sqrt{\frac{2m_e(V_0 - \varepsilon)}{\hbar^2}},\quad \kappa_L = \kappa_0\sqrt{1 - \frac{\Delta V}{V_0 - \varepsilon}} = \sqrt{\frac{2m_e(V_0 - \varepsilon)}{\hbar^2}}\sqrt{\frac{V_0 - \varepsilon - \Delta V}{V_0 - \varepsilon}} = \sqrt{\frac{2m_e(V_0 - \varepsilon - \Delta V)}{\hbar^2}},$$

$$K_0 = \sqrt{\frac{2m_e\varepsilon}{\hbar^2}},\quad K_L = \sqrt{\frac{2m_e(\varepsilon + \Delta V)}{\hbar^2}}$$

So $K_0\kappa_0 K_L\kappa_L = \dfrac{4m_e^2}{\hbar^4}\sqrt{\varepsilon(V_0 - \varepsilon)(\varepsilon + \Delta V)(V_0 - \varepsilon - \Delta V)}$ (8)

Replacing (8) into (6) yields the final expression for the transmission probability

$$T = \frac{1}{|P_{11}|^2} = \frac{16}{V_0^2}\sqrt{\varepsilon(V_0 - \varepsilon)(\varepsilon + \Delta V)(V_0 - \varepsilon - \Delta V)}\cdot e^{-2\delta}. \tag{9}$$

Inserting (1b) and (3) into (9) yields

$$\boxed{T = \frac{16}{V_0^2}\sqrt{\varepsilon(V_0 - \varepsilon)(\varepsilon + \Delta V)(V_0 - \varepsilon - \Delta V)}\cdot \exp\left[\frac{-4L\sqrt{2m_e}}{3\Delta V\cdot\hbar}\left((V_0 - \varepsilon)^{\frac{3}{2}} - (V_0 - \varepsilon - \Delta V)^{\frac{3}{2}}\right)\right]}. \tag{10}$$

If we define:

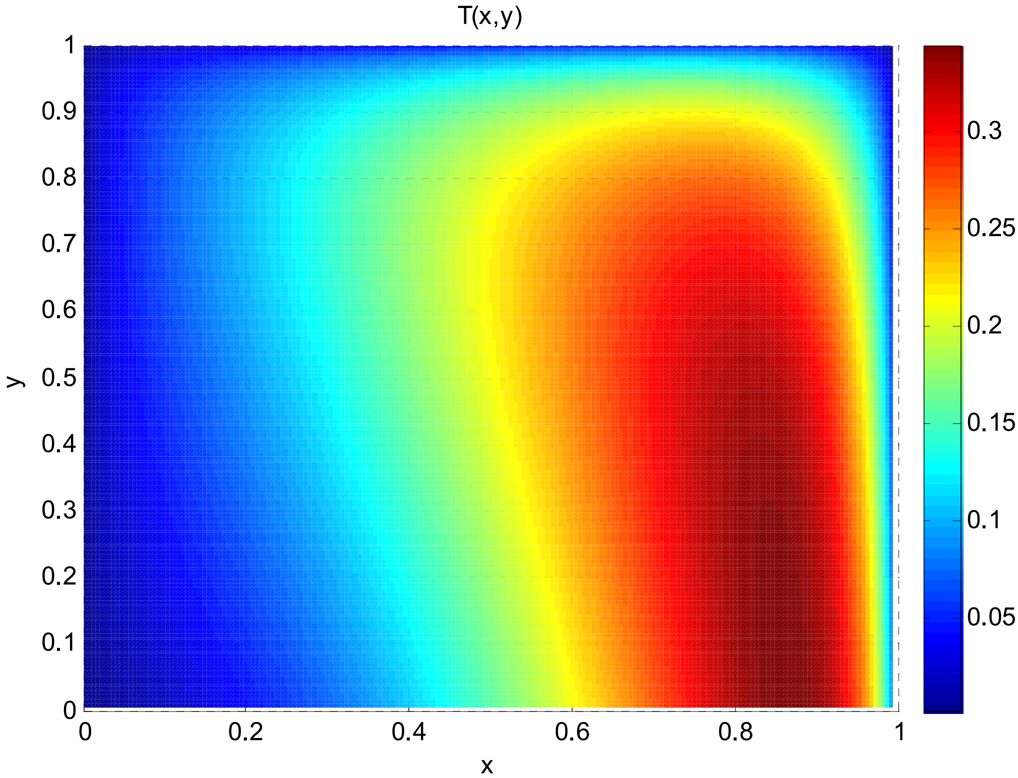
$$\begin{cases} \varepsilon = xV_0 \\ \Delta V = y(V_0 - \varepsilon) = (y - xy)V_0 \end{cases}$$

where x = 0~1 and y = 0~1,

we can re-write (10) as

$$T = 16\sqrt{x(1-x)(x+y-xy)(1-x-y+xy)} \cdot \exp\left[\frac{-4L\sqrt{2m_e V_0}}{3(y-xy)\hbar}\left((1-x)^{\frac{3}{2}} - (1-x-y+xy)^{\frac{3}{2}}\right)\right]$$

And then, we can try to plot T as a function of x and y:



As we can see, T has a maximum(0.3476) when x = 0.8643 and y = 0.

Assumptions are:

L = 1nm

V_0 = 0.2eV

Exercise on page 191

From (6•1-14) we know

$$(\Delta p)^2 = \left\langle p^2 \right\rangle - \left\langle p \right\rangle^2 .$$

And by definition,

$$\left\langle p^2 \right\rangle = \int \Psi^* (\hat{p})^2 \Psi dr^3 = \int \Psi^* (-i\hbar\nabla)^2 \Psi dr^3 = \int \Psi^* (-\hbar^2\nabla^2) \Psi dr^3 = 2M \int \Psi^* (\frac{-\hbar^2\nabla^2}{2M}) \Psi dr^3$$

$$= 2M \int \Psi^* \hat{\varepsilon}_{kin} \Psi dr^3 = 2M \left\langle \varepsilon_{kin} \right\rangle .$$

As a result,

$$\boxed{(\Delta p)^2 = 2M \left\langle \varepsilon_{kin} \right\rangle - \left\langle p \right\rangle^2} \qquad \text{QED}$$