ECE/MAT 211A winter 07

HWK#7 solutions

#7•4-1: The Radial Component of $\hat{p} = -i\hbar\nabla$ in Spherical Polar Coordinates

(a)

Written as an integral in spherical polar coordinates, the hermiticity condition (7•4-3) has the form

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \cdot d\theta \int_0^{\infty} \Phi^* \hat{A}\Psi \cdot r^2 dr = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \cdot d\theta \int_0^{\infty} (\hat{A}\Phi)^* \Psi \cdot r^2 dr \tag{1}$$

Insert now

$$\hat{A} = -i\hbar \frac{\partial}{\partial r} \tag{2}$$

on the left-hand side, and perform an integration by parts of the radial integral (the other two integrals are irrelevant here):

$$\int_0^{\infty} \Phi^* \left[-i\hbar \frac{\partial}{\partial r} \Psi \right] \cdot r^2 dr = -i\hbar \Phi^* \Psi r^2 \Big|_0^{\infty} + \int_0^{\infty} \left[-i\hbar \frac{\partial}{\partial r} (\Phi r^2) \right]^* \Psi \cdot dr \tag{3}$$

The first term on the right-hand side vanishes, but the remainder is evidently not at all the same as inserting (2) on the right-hand side of (1):

$$\int_0^{\infty} \left[-i\hbar \frac{\partial}{\partial r} \Phi \right]^* \Psi \cdot r^2 dr \tag{4}$$

Hence, the operator defined in (2) is not Hermitian. QED

(b)

If we insert the alternate form (7•4-22), we obtain

$$\int_0^{\infty} \Phi^* \left[-i\hbar \frac{1}{r} \frac{\partial}{\partial r} r\Psi \right] \cdot r^2 dr = -i\hbar \Phi^* \Psi r^2 \Big|_0^{\infty} + \int_0^{\infty} \left[-i\hbar \frac{1}{r} \frac{\partial}{\partial r} (r\Phi) \right]^* \Psi \cdot r^2 dr \tag{5}$$

which evidently *does* satisfy (1). QED

To give a more *physical* justification for the choice (7•4-22) as the operator for the radial component of the momentum, consider a spherically symmetric non-stationary outward-expanding "exploding" state Ψ(r, t). Associated with it is a probability current density with the radial component

$$j_r(r) = -\frac{i\hbar}{2M}\left[\Psi^*\frac{\partial\Psi}{\partial r} - \Psi\frac{\partial\Psi^*}{\partial r}\right] \tag{6}$$

This may also be written as

$$Mj_r(r) = -\frac{i\hbar}{2}\left[\Psi^*\frac{1}{r}\frac{\partial(r\Psi)}{\partial r} - \Psi\frac{1}{r}\frac{\partial(r\Psi^*)}{\partial r}\right] = \frac{1}{2}\left[\Psi^*\hat{p}_r\Psi - \Psi\hat{p}_r\Psi^*\right], \tag{7}$$

where Mj_r has the physical meaning of the radial component of a momentum density. The integral of this over all space,

$$I_r = \int_0^\infty Mj_r(r)\cdot 4\pi r^2 dr \tag{8}$$

has the physical meaning of the total *radial impulse* as it might be absorbed by a spherical shell at some faraway distance R,

$$I_r = 4\pi R^2\int_0^\infty P(t)\cdot dt\,, \tag{9}$$

where P is the force per unit area, that is, the pressure on the shell. Inserting (7) and drawing on the already demonstrated Hermiticity of $\hat{p}_r$, yields

$$I_r = \frac{1}{2}\int_0^\infty\left[\psi^*\hat{p}_r\psi - \psi\hat{p}_r\psi^*\right]\cdot 4\pi r^2 dr = \int_0^\infty \psi^*\hat{p}_r\psi\cdot 4\pi r^2 dr = \langle p_r\rangle, \tag{10}$$

as it should be.

We finally note that

$$\left[\hat{r},\hat{p}_r\right]\psi = -i\hbar\left[\frac{\partial(r\Psi)}{\partial r} - \frac{1}{r}\frac{\partial(r^2\psi)}{\partial r}\right] = -i\hbar\Psi\,, \tag{11}$$

implying the commutation relation

$$\left[\hat{r},\hat{p}_r\right] = -i\hbar \tag{12}$$

the same as for the Cartesian position-momentum paris.

#7•5-1: Angular Momentum Operators in Spherical Polar Coordinates

From (7•5-3):

$$\frac{\partial x}{\partial \theta} = +r \cdot \cos\theta \cos\phi , \qquad \frac{\partial x}{\partial \phi} = -r \cdot \sin\theta \sin\phi \tag{1a,b}$$

$$\frac{\partial y}{\partial \theta} = +r \cdot \cos\theta \sin\phi , \qquad \frac{\partial y}{\partial \phi} = +r \cdot \sin\theta \cos\phi \tag{2a,b}$$

$$\frac{\partial z}{\partial \theta} = -r \cdot \sin\theta , \qquad \frac{\partial z}{\partial \phi} = 0 . \tag{3a,b}$$

Inserted into (7•5-18):

$$\frac{\partial}{\partial \theta} = +r \cdot \cos\theta \cos\phi \cdot \frac{\partial}{\partial x} + r \cdot \cos\theta \sin\phi \cdot \frac{\partial}{\partial y} - r \cdot \sin\theta \cdot \frac{\partial}{\partial z} \tag{4}$$

Similarly,

$$\frac{\partial}{\partial \phi} = -r \cdot \sin\theta \sin\phi \cdot \frac{\partial}{\partial x} + r \cdot \sin\theta \cos\phi \cdot \frac{\partial}{\partial y} \tag{5}$$

When these are inserted into (7•5-4a), the $\partial / \partial x$-terms cancel, leaving

$$\begin{aligned} \hat{L}_x &= i\hbar r \cdot \left[\sin\phi \left(\cos\theta \sin\phi \cdot \frac{\partial}{\partial y} - \sin\theta \cdot \frac{\partial}{\partial z} \right) + \cot\theta \cos\phi \left(\sin\theta \cos\phi \cdot \frac{\partial}{\partial y} \right) \right] \\ &= i\hbar r \cdot \left[\left(\cos\theta \sin^2\phi + \cos\theta \cos^2\phi \right) \frac{\partial}{\partial y} + \left(-\sin\theta \sin\phi \right) \frac{\partial}{\partial z} \right] \\ &= i\hbar r \cdot \left[\cos\theta \frac{\partial}{\partial y} - \sin\theta \sin\phi \frac{\partial}{\partial z} \right] = -i\hbar \left[y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right], \end{aligned} \tag{6}$$

which is the same as (7•5-2a)

Similarly, inserting (4) and (5) into (7•5-4b) and canceling $\partial / \partial y$-terms yields

$$\begin{aligned} \hat{L}_y &= i\hbar r \cdot \left[-\cos\phi \left(\cos\theta \cos\phi \cdot \frac{\partial}{\partial x} - \sin\theta \cdot \frac{\partial}{\partial z} \right) + \cot\theta \sin\phi \left(-\sin\theta \sin\phi \cdot \frac{\partial}{\partial x} \right) \right] \\ &= i\hbar r \cdot \left[-\left(\cos\theta \cos^2\phi + \cos\theta \sin^2\phi \right) \frac{\partial}{\partial x} + \sin\theta \cos\phi \frac{\partial}{\partial z} \right] \\ &= i\hbar r \cdot \left[-\cos\theta \frac{\partial}{\partial x} + \sin\theta \cos\phi \frac{\partial}{\partial z} \right] = -i\hbar \left[z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right], \end{aligned} \tag{7}$$

which is the same as (7•5-2b)

Finally, inserting (5) into (7•5-4c) yields

$$\hat{L}_z = -i\hbar r\left(-\sin\theta\sin\phi\cdot\frac{\partial}{\partial x}+\sin\theta\cos\phi\cdot\frac{\partial}{\partial y}\right) = -i\hbar\left[x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right] \tag{8}$$

which is the same as (7•5-2c).

QED

#7•6-1: Precessional Motion of a Rotating Charged Sphere in a Magnetic Field

From the Hamiltonian

$$\hat{H} = \frac{1}{2I}\hat{L}^2 - \gamma B_z \hat{L}_z \qquad (\gamma \equiv qQ) \tag{1}$$

and the universal time evolution law (7•6-4), along with the commutation relations (7•5-14), we have

$$\frac{d}{dt}\langle L\rangle = -\frac{i\gamma B_z}{\hbar}\left\langle\left[\hat{L},\hat{L}_z\right]\right\rangle, \tag{2}$$

or, in components,

$$\frac{d}{dt}\langle L_x\rangle = -\gamma B_z\left\langle\hat{L}_y\right\rangle, \qquad \frac{d}{dt}\langle L_y\rangle = +\gamma B_z\left\langle\hat{L}_x\right\rangle, \qquad \frac{d}{dt}\langle L_z\rangle = 0 \tag{3a,b,c}$$

These have the solution

$$\langle L_x\rangle = L_\rho\cos wt\,, \qquad \langle L_y\rangle = L_\rho\sin wt\,, \qquad \langle L_z\rangle = \text{constant} \tag{4a,b,c}$$

where

$$w = \gamma B_z = qQB_z\,, \tag{5}$$

and L_ρ is the time-independent *magnitude* of the angular momentum component perpendicular to **B**, related to Lz and the total angular momentum through

$$L_\rho^{\,2} + L_z^{\,2} = L^2. \tag{6}$$

Equations (4) represent the precessional motion of the vector $\langle L\rangle$ along the surface of a cone.

For a graph, see Fig. 18•3-2 on page 466.